On a class of half-discrete Hilbert-type inequalities in the whole plane involving some classical special constants

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Abstract. In this paper, we first define a new half-discrete kernel function in the whole plane, which involves some exponent functions and unifies some homogeneous and non-homogeneous kernels. By employing some techniques of real analysis, a new half-discrete Hilbert-type inequality with the newly defined kernel function, as well as its equivalent forms are established. Furthermore, the constant factors of the newly obtained inequalities are proved to be optimal. At last, assigning special values to the parameters, we get some interesting Hilbert-type inequalities involving hyperbolic functions, and with the constant factors related to Euler numbers, Bernoulli numbers, and Catalan constant.
Keywords: Hilbert-type inequality, half-discrete, Bernoulli number, Euler number, Catalan constant.

1. Introduction

Suppose that $p > 1$, and $f(x), \mu(x)$ are two non-negative measurable functions defined on a measurable set E. Define

$$L_{p,\mu}(E) := \left\{ f : \|f\|_{p,\mu} := \left[\int_E f^p(x)\mu(x)\mathrm{d}x \right]^{1/p} < \infty \right\}.$$

Specially, if $\mu(x) \equiv 1$, then we have the following abbreviations: $\|f\|_p := \|f\|_{p,\mu}$ and $L_p(E) := L_{p,\mu}(E)$. Additionally, suppose that $p > 1$, $a_n, \nu_n > 0$, $n \in F \subseteq \mathbb{Z}$, $\boldsymbol{a} = \{a_n\}_{n \in F}$. Define

$$l_{p,\nu} := \left\{ \boldsymbol{a} : \|\boldsymbol{a}\|_{p,\nu} := \left(\sum_{n \in F} a_n^p \nu_n \right)^{1/p} < \infty \right\}.$$

Specially, if $\nu_n \equiv 1$, then we have $\|a\|_p := \|a\|_{p,\nu}$ and $l_p := l_{p,\nu}$.

Consider two real-valued sequences: $\boldsymbol{a} = \{a_m\}_{m \in \mathbb{N}^+} \in l_2$ and $\boldsymbol{b} = \{b_n\}_{n \in \mathbb{N}^+} \in l_2$, then

$$\sum_{n \in \mathbb{N}^+} \sum_{m \in \mathbb{N}^+} \frac{a_m b_n}{m+n} < \pi \|\boldsymbol{a}\|_2 \|\boldsymbol{b}\|_2, \tag{1}$$

where the constant factor π is the best possible. Inequality (1) was proposed by D. Hilbert in his lectures on integral equations in 1908, and in 1911, Schur proved the integral analogy of inequality (1) as follows:

$$\int_{y\in\mathbb{R}^+}\int_{x\in\mathbb{R}^+}\frac{f(x)g(y)}{x+y}\mathrm{d}x\mathrm{d}y<\pi\|f\|_2\|g\|_2, \tag{2}$$

where $f,g\geq 0,\ f,g\in L_2(\mathbb{R}^+)$, and the constant factor π is the best possible.

Inequalities (1) and (2) are usually known as Hilbert's inequality [1]. In the past twenty years, by the introduction of some parameters and special functions such as the Beta function, some extended forms of (1) and (2) were established, such as the following[2]:

$$\sum_{n\in\mathbb{N}^+}\sum_{m\in\mathbb{N}^+}\frac{a_mb_n}{(m+n)^\lambda}<B\left(\frac{\lambda}{p},\frac{\lambda}{q}\right)\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu}, \tag{3}$$

where $0<\lambda\leq\min\{p,q\}$, $\mu_m=m^{p-\lambda-1}$, $\nu_n=n^{q-\lambda-1}$, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, and $B(x,y)$ is the Beta function [3, 4], that is,

$$B(x,y):=\int_0^\infty\frac{z^{x-1}}{(1+z)^{x+y}}\mathrm{d}z\ (x,y>0).$$

In addition, Yang [5] proved the following extended form of (2) in 2004:

$$\int_{y\in\mathbb{R}^+}\int_{x\in\mathbb{R}^+}\frac{f(x)g(y)}{x^\lambda+y^\lambda}\mathrm{d}x\mathrm{d}y<\frac{\pi}{\lambda\sin r\pi}\|f\|_{p,\mu}\|g\|_{q,\nu}, \tag{4}$$

where $r,s,\lambda>0$, $r+s=1$, $\mu(x)=x^{p(1-\lambda r)-1}$, $\nu(x)=x^{q(1-\lambda s)-1}$. With regard to some other extensions of (1) and (2), we refer to [6, 7, 8, 9, 10, 11, 12, 13, 14]. Such extended inequalities as (3) and (4) are usually named as Hilbert-type inequality. Furthermore, by constructing new kernel functions, introducing parameters, and considering the reverse form, coefficient refinement and multi-dimensional extension, a great many Hilbert-type inequalities were established in the past 20 years (see, [15, 16, 17, 18, 19, 20, 21, 22, 23]).

It should be noted that, in addition to the discrete and integral forms, Hilbert-type inequality sometimes appears in half-discrete form. The first half-discrete Hilbert-type inequality was put forward by Hardy et al. (see, Theorem 351 of [1]). However, the constant factor was not proved to be the best possible. Until recently, researchers established some new half-discrete Hilbert-type inequalities with the best possible constant factors, such as [24]

$$\int_{x\in\mathbb{R}^+}f(x)\sum_{n\in\mathbb{N}^+}\frac{a_n}{(1+nx)^\lambda}\mathrm{d}x<B\left(\frac{\lambda}{2},\frac{\lambda}{2}\right)\|f\|_{2,\mu}\|a\|_{2,\nu}, \tag{5}$$

where $\mu(x)=x^{\frac{\lambda}{2}-1}$, $\nu_n=n^{\frac{\lambda}{2}-1}$. Regarding some other half-discrete Hilbert-type inequalities, we refer to [25, 26, 27, 28, 29, 30].

The objective of this work is to establish a class of half-discrete Hilbert-type inequalities with the kernel functions related to some hyperbolic functions. Our motivation mainly comes from the following integral Hilbert-type inequalities [31, 32]:

$$\int_{y\in\mathbb{R}^+}\int_{x\in\mathbb{R}^+}\operatorname{csch}(xy)f(x)g(y)\mathrm{d}x\mathrm{d}y<\frac{\pi^2}{4}\|f\|_{p,\mu}\|g\|_{q,\nu}\ , \tag{6}$$

$$\int_{y\in\mathbb{R}}\int_{x\in\mathbb{R}}\frac{f(x)g(y)}{|\mathrm{e}^{pxy}-\mathrm{e}^{-qxy}|}\mathrm{d}x\mathrm{d}y<\left(\frac{\pi}{pq\sin\frac{\pi}{p}}\right)^2\|f\|_{p,\hat{\mu}}\|g\|_{q,\hat{\nu}}, \tag{7}$$

where $\mu(x)=x^{-(p+1)}$, $\nu(y)=y^{-(q+1)}$, $\hat{\mu}(x)=|x|^{-(p+1)}$, $\hat{\nu}(y)=|y|^{-(q+1)}$.

In this work, we will establish the following Hilbert-type inequalities involving hyperbolic secant function and hyperbolic cosecant function:

$$\int_{x\in\mathbb{R}}f(x)\sum_{n\in\mathbb{Z}^0}\operatorname{sech}\left(\sqrt[2m+1]{\frac{n}{x}}\right)a_n\mathrm{d}x<\frac{E_m}{2^{2m}}(2m+1)\pi^{2m+1}\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{8}$$

$$\int_{x\in\mathbb{R}}f(x)\sum_{n\in\mathbb{Z}^0}\left|\operatorname{csch}\left(\sqrt[2m+1]{xn}\right)\right|a_n\mathrm{d}x<\frac{B_m}{m}(2m+1)(2^{2m}-1)\pi^{2m}\|f\|_{p,\hat{\mu}}\|\boldsymbol{a}\|_{q,\hat{\nu}}, \tag{9}$$

where $\mu(x)=|x|^{2p-1}$, $\nu_n=|n|^{-1}$, $\hat{\mu}(x)=|x|^{\frac{p}{2m+1}-1}$, $\hat{\nu}_n=|n|^{\frac{q}{2m+1}-1}$, $E_m(m\in\mathbb{N})$ is the Euler number, and $B_m(m\in\mathbb{N}^+)$ is the Bernoulli number.

More generally, we will construct a new kernel function involving several exponent functions with multiple parameters, which unifies some homogeneous and non-homogeneous kernels, and then a half-discrete Hilbert-type inequality and its equivalent forms are established. Detailed lemmas will be presented in Section 2, and main results and some corollaries will be presented in Section 3 and Section 4, respectively.

2. Some Lemmas

Lemma 2.1. *Let $\tau,\eta\in\{1,-1\}$, and $\tau\neq-1$ when $\eta=1$. Suppose that $c>a\geq b>d>0$, and $ab=cd$ when $\tau\eta=1$. Define*

$$K(z):=\frac{|\ a^z+\tau b^z\ |}{|\ c^z+\eta d^z\ |}\ (z\neq0). \tag{10}$$

Then, $K(z)$ decreases on $\mathbb{R}^+$, and increases on $\mathbb{R}^-$.

Proof. If $\tau=1,\eta=1$, then we have $ab=cd$, and

$$\begin{aligned}\frac{\mathrm{d}K}{\mathrm{d}z}&=\frac{(ac)^z\log\frac{a}{c}+(bd)^z\log\frac{b}{d}+(ad)^z\log\frac{a}{d}+(bc)^z\log\frac{b}{c}}{(c^z+d^z)^2}\\&:=L(z)(c^z+d^z)^{-2}.\end{aligned}$$

Since $c > a \geq b > d > 0$, we have $bc > ad$, and $ac > bd$.
If $z \in \mathbb{R}^+$, we have

$$L(z) < (ac)^z \log \frac{a}{c} + (ac)^z \log \frac{b}{d} + (bc)^z \log \frac{a}{d} + (bc)^z \log \frac{b}{c} = 0.$$

If $z \in \mathbb{R}^-$, we have

$$L(z) > (bd)^z \log \frac{a}{c} + (bd)^z \log \frac{b}{d} + (ad)^z \log \frac{a}{d} + (ad)^z \log \frac{b}{c} = 0.$$

It implies that $\frac{\mathrm{d}K}{\mathrm{d}z} < 0$ for $z \in \mathbb{R}^+$, and $\frac{\mathrm{d}K}{\mathrm{d}z} > 0$ for $z \in \mathbb{R}^-$. Thus, $K(z)$ decreases on $\mathbb{R}^+$ and increases on $\mathbb{R}^-$ for $\tau = 1, \eta = 1$.

If $\tau = 1, \eta = -1$, $z \in \mathbb{R}^+$, then we have

$$\frac{\mathrm{d}K}{\mathrm{d}z} = -\frac{(ac)^z \log \frac{c}{a} + (bd)^z \log \frac{b}{d} + (ad)^z \log \frac{a}{d} + (bc)^z \log \frac{c}{b}}{(c^z - d^z)^2} < 0.$$

If $\tau = 1, \eta = -1$, $z \in \mathbb{R}^-$, then we have

$$\frac{\mathrm{d}K}{\mathrm{d}z} = \frac{(ad)^z \log \frac{a}{d} + (bc)^z \log \frac{c}{b} + (ac)^z \log \frac{c}{a} + (bd)^z \log \frac{b}{d}}{(c^z - d^z)^2} > 0.$$

Therefore, $K(z)$ decreases on $\mathbb{R}^+$ and increases on $\mathbb{R}^-$ for $\tau = 1, \eta = -1$.

If $\tau = -1, \eta = -1$, then $ab = cd$, and we have

$$\begin{aligned} \frac{\mathrm{d}K}{\mathrm{d}z} &= \frac{(ac)^z \log \frac{a}{c} + (bd)^z \log \frac{b}{d} - (ad)^z \log \frac{a}{d} - (bc)^z \log \frac{b}{c}}{(c^z - d^z)^2} \\ &:= g(z) \left[\left(\sqrt{\frac{c}{d}} \right)^z - \left(\sqrt{\frac{d}{c}} \right)^z \right]^{-2}, \end{aligned} \tag{11}$$

where $g(z) = g_1(z) + g_2(z) - g_3(z) - g_4(z)$, and

$$g_1(z) = \left(\frac{a}{d}\right)^z \log \frac{a}{c} = \left(\sqrt{\frac{ac}{bd}}\right)^z \log \sqrt{\frac{ad}{bc}},$$

$$g_2(z) = \left(\frac{b}{c}\right)^z \log \frac{b}{d} = \left(\sqrt{\frac{bd}{ac}}\right)^z \log \sqrt{\frac{bc}{ad}},$$

$$g_3(z) = \left(\frac{a}{c}\right)^z \log \frac{a}{d} = \left(\sqrt{\frac{ad}{bc}}\right)^z \log \sqrt{\frac{ac}{bd}},$$

$$g_4(z) = \left(\frac{b}{d}\right)^z \log \frac{b}{c} = \left(\sqrt{\frac{bc}{ad}}\right)^z \log \sqrt{\frac{bd}{ac}}.$$

It follows that

$$\frac{\mathrm{d}g_1}{\mathrm{d}z}=\left(\sqrt{\frac{ac}{bd}}\right)^z\left[\log^2\sqrt{\frac{a}{b}}-\log^2\sqrt{\frac{c}{d}}\right],$$
$$\frac{\mathrm{d}g_2}{\mathrm{d}z}=\left(\sqrt{\frac{bd}{ac}}\right)^z\left[\log^2\sqrt{\frac{a}{b}}-\log^2\sqrt{\frac{c}{d}}\right],$$
$$\frac{\mathrm{d}g_3}{\mathrm{d}z}=\left(\sqrt{\frac{ad}{bc}}\right)^z\left[\log^2\sqrt{\frac{a}{b}}-\log^2\sqrt{\frac{c}{d}}\right],$$
$$\frac{\mathrm{d}g_4}{\mathrm{d}z}=\left(\sqrt{\frac{bc}{ad}}\right)^z\left[\log^2\sqrt{\frac{a}{b}}-\log^2\sqrt{\frac{c}{d}}\right].$$

Therefore, we get

$$\begin{aligned}\frac{\mathrm{d}g}{\mathrm{d}z}&=\left[\log^2\sqrt{\frac{a}{b}}-\log^2\sqrt{\frac{c}{d}}\right]\\&\quad\times\left[\left(\sqrt{\frac{ac}{bd}}\right)^z+\left(\sqrt{\frac{bd}{ac}}\right)^z-\left(\sqrt{\frac{ad}{bc}}\right)^z-\left(\sqrt{\frac{bc}{ad}}\right)^z\right]\\&=\left[\log^2\sqrt{\frac{a}{b}}-\log^2\sqrt{\frac{c}{d}}\right]\left[\left(\frac{c}{b}\right)^z+\left(\frac{b}{c}\right)^z-\left(\frac{a}{c}\right)^z-\left(\frac{c}{a}\right)^z\right].\end{aligned}$$

Let $h(t):=t^z+t^{-z}$, then it can be shown that $h(t)$ increases on $[1,\infty)$ for arbitrary $z\in\mathbb{R}^+$. Since $\frac{c}{b}\geq\frac{c}{a}>1$, we have $h\left(\frac{c}{b}\right)\geq h\left(\frac{c}{a}\right)$, that is,

$$\left(\frac{c}{b}\right)^z+\left(\frac{b}{c}\right)^z-\left(\frac{a}{c}\right)^z-\left(\frac{c}{a}\right)^z\geq 0.$$

Additionally, in view of $\frac{c}{d}\geq\frac{a}{b}\geq 1$, we have $\log^2\sqrt{\frac{a}{b}}-\log^2\sqrt{\frac{c}{d}}\leq 0$. Thus, we obtain $\frac{\mathrm{d}g}{\mathrm{d}z}\leq 0$ on $\mathbb{R}^+$, which leads to

$$g(z)\leq g(0)=\log\frac{a}{c}+\log\frac{b}{d}-\log\frac{a}{d}-\log\frac{b}{c}=0\ (z\in\mathbb{R}^+).$$

By (11), we have $\frac{\mathrm{d}K}{\mathrm{d}z}\leq 0$ $(z\in\mathbb{R}^+)$, and it implies that $K(z)$ decreases on $\mathbb{R}^+$. Similarly, it can be proved that $K(z)$ increases on $\mathbb{R}^-$. Thus, we proved Lemma 2.1 in the case of $\tau=-1,\eta=-1$. □

Lemma 2.2. *Let $\tau,\eta\in\{1,-1\}$, and $\tau\neq-1$ when $\eta=1$. Suppose that $c>a\geq b>d>0$, and $ab=cd$ when $\tau\eta=1$. Let λ be such that $\lambda\geq 1$, and $\lambda\neq 1$ for $\tau=1,\eta=-1$. $K(z)$ is defined via (10), and*

$$\begin{aligned}\kappa\left(a,b,c,d,\tau,\eta,\lambda\right):=&\sum_{j=0}^{\infty}\left[\frac{(-\eta)^j}{(j\log\frac{c}{d}+\log\frac{c}{a})^\lambda}+\frac{\tau(-\eta)^j}{(j\log\frac{c}{d}+\log\frac{c}{b})^\lambda}\right]\\&+\sum_{j=0}^{\infty}\left[\frac{(-\eta)^j}{(j\log\frac{c}{d}+\log\frac{b}{d})^\lambda}+\frac{\tau(-\eta)^j}{(j\log\frac{c}{d}+\log\frac{a}{d})^\lambda}\right].\end{aligned}\tag{12}$$

Then

$$\int_{z\in\mathbb{R}} K(z)\,|z|^{\lambda-1}\,\mathrm{d}z = \Gamma(\lambda)\kappa\,(a,b,c,d,\tau,\eta,\lambda)\,. \tag{13}$$

Proof. Expanding $\frac{1}{c^z+\eta d^z}$ $(z\in\mathbb{R}^+)$ into power series, and observing that $c > d > 0$, we obtain

$$\frac{1}{c^z+\eta d^z} = \frac{c^{-z}}{1+\eta(c^{-1}d)^z} = c^{-z}\sum_{j=0}^{\infty}(-\eta)^j\left(\frac{d}{c}\right)^{jz}.$$

By Lebesgue term-by-term integration theorem, we get

$$\begin{aligned}\int_0^\infty K(z)z^{\lambda-1}\mathrm{d}z &= \sum_{j=0}^{\infty}(-\eta)^j\Bigg[\int_0^\infty\left(\frac{d}{c}\right)^{jz}\left(\frac{a}{c}\right)^{z}z^{\lambda-1}\mathrm{d}z \\ &\qquad + \tau\int_0^\infty\left(\frac{d}{c}\right)^{jz}\left(\frac{b}{c}\right)^{z}z^{\lambda-1}\mathrm{d}z\Bigg] \\ &:= \sum_{j=0}^{\infty}(-\eta)^j(J_1+\tau J_2).\end{aligned} \tag{14}$$

Let $z = \frac{u}{j\log\frac{c}{d}+\log\frac{c}{a}}$ $(j\in\mathbb{N})$, then we have

$$J_1 = \frac{1}{(j\log\frac{c}{d}+\log\frac{c}{a})^\lambda}\int_0^\infty e^{-u}u^{\lambda-1}\mathrm{d}u = \frac{\Gamma(\lambda)}{(j\log\frac{c}{d}+\log\frac{c}{a})^\lambda}. \tag{15}$$

Similarly, we can obtain

$$J_2 = \frac{1}{(j\log\frac{c}{d}+\log\frac{c}{b})^\lambda}\int_0^\infty e^{-u}u^{\lambda-1}\mathrm{d}u = \frac{\Gamma(\lambda)}{(j\log\frac{c}{d}+\log\frac{c}{b})^\lambda}. \tag{16}$$

Plug (15) and (16) back into (14), then we obtain

$$\int_0^\infty K(z)z^{\lambda-1}\mathrm{d}z = \sum_{j=0}^{\infty}\left[\frac{(-\eta)^j\Gamma(\lambda)}{(j\log\frac{c}{d}+\log\frac{c}{a})^\lambda} + \frac{\tau(-\eta)^j\Gamma(\lambda)}{(j\log\frac{c}{d}+\log\frac{c}{b})^\lambda}\right]. \tag{17}$$

Since $c > a \geq b > d > 0$, we have $\frac{1}{d} > \frac{1}{b} \geq \frac{1}{a} > \frac{1}{c} > 0$. From the above discussion, we get

$$\begin{aligned}\int_{-\infty}^0 K(z)\,|z|^{\lambda-1}\,\mathrm{d}z &= \int_0^\infty K(-z)z^{\lambda-1}\mathrm{d}z \\ &= \sum_{j=0}^{\infty}\left[\frac{(-\eta)^j\Gamma(\lambda)}{(j\log\frac{c}{d}+\log\frac{b}{d})^\lambda} + \frac{\tau(-\eta)^j\Gamma(\lambda)}{(j\log\frac{c}{d}+\log\frac{a}{d})^\lambda}\right].\end{aligned} \tag{18}$$

Combining (17) and (18), and using (12), we get (13). Lemma 2.2 is proved. □

Lemma 2.3. *Let $\tau, \eta \in \{1, -1\}$, and $\tau \neq -1$ when $\eta = 1$. Let*

$$\Omega := \left\{ z : z = \frac{2i+1}{2l+1}, i, l \in \mathbb{Z} \right\},$$

$\beta \in \Omega$, and $\gamma \in \mathbb{R}^+ \cap \Omega$. Suppose that $c > a \geq b > d > 0$, and $ab = cd$ when $\tau\eta = 1$. Let λ be such that $\lambda \geq 1$, $\lambda\gamma \leq 1$, and $\lambda \neq 1$ for $\tau = 1, \eta = -1$. Let $K(z)$ be defined via (10), and for an arbitrary positive natural number s which is large enough, define

$$\tilde{\boldsymbol{a}} := \{\tilde{a}_n\}_{n \in \mathbb{Z}^0} := \left\{ |n|^{\lambda\gamma - 1 - \frac{2\gamma}{qs}} \right\}_{n \in \mathbb{Z}^0},$$

$$\tilde{f}(x) := \begin{cases} |x|^{\lambda\beta - 1 + \frac{2\beta}{ps}}, & x \in E \\ 0, & x \in \mathbb{R} \setminus E \end{cases},$$

where $\mathbb{Z}^0 := \mathbb{Z} \setminus \{0\}$, and $E := \{x : |x|^{\operatorname{sgn} \beta} < 1\}$. Then

$$\tilde{I} := \sum_{n \in \mathbb{Z}^0} \tilde{a}_n \int_{x \in \mathbb{R}} K\left(x^\beta n^\gamma\right) \tilde{f}(x) \mathrm{d}x = \int_{x \in \mathbb{R}} \tilde{f}(x) \sum_{n \in \mathbb{Z}^0} \tilde{a}_n K\left(x^\beta n^\gamma\right) \mathrm{d}x$$
$$> \frac{s}{|\beta\gamma|} \left[\int_{[-1,1]} K(z) \, |z|^{\lambda - 1 + \frac{2}{ps}} \, \mathrm{d}z + \int_{\mathbb{R} \setminus [-1,1]} K(z) \, |z|^{\lambda - 1 - \frac{2}{qs}} \, \mathrm{d}z \right]. \tag{19}$$

Proof. Let

$$E^+ := \{x : x \in E \cap \mathbb{R}^+\}, E^- := \{x : x \in E \cap \mathbb{R}^-\}.$$

Then

$$\tilde{I} = I_1 + I_2 + I_3 + I_4,$$

where

$$I_1 := \int_{x \in E^-} \tilde{f}(x) \sum_{n \in \mathbb{Z}^+} \tilde{a}_n K\left(x^\beta n^\gamma\right) \mathrm{d}x,$$

$$I_2 := \int_{x \in E^-} \tilde{f}(x) \sum_{n \in \mathbb{Z}^-} \tilde{a}_n K\left(x^\beta n^\gamma\right) \mathrm{d}x,$$

$$I_3 := \int_{x \in E^+} \tilde{f}(x) \sum_{n \in \mathbb{Z}^+} \tilde{a}_n K\left(x^\beta n^\gamma\right) \mathrm{d}x,$$

$$I_4 := \int_{x \in E^+} \tilde{f}(x) \sum_{n \in \mathbb{Z}^-} \tilde{a}_n K\left(x^\beta n^\gamma\right) \mathrm{d}x.$$

In view of $\lambda\gamma \le 1$, it follows that $\tilde{a}_n = |n|^{\lambda\gamma-1-\frac{2\gamma}{qs}}$ decreases with respect to n if $n \in \mathbb{Z}^+$. In addition, for $x \in E^-$, $n \in \mathbb{Z}^+$, we have $x^\beta n^\gamma < 0$. By Lemma 2.1, it can be proved that $K\left(x^\beta n^\gamma\right)$ decreases with respect to n if $n \in \mathbb{Z}^+$. Therefore,

$$I_1 > \int_{x\in E^-} |x|^{\lambda\beta-1+\frac{2\beta}{ps}} \int_1^\infty K\left(x^\beta y^\gamma\right) |y|^{\lambda\gamma-1-\frac{2\gamma}{qs}} \mathrm{d}y\mathrm{d}x := W_1.$$

Similarly, we can obtain

$$I_2 > \int_{x\in E^-} |x|^{\lambda\beta-1+\frac{2\beta}{ps}} \int_{-\infty}^{-1} K\left(x^\beta y^\gamma\right) |y|^{\lambda\gamma-1-\frac{2\gamma}{qs}} \mathrm{d}y\mathrm{d}x := W_2,$$
$$I_3 > \int_{x\in E^+} |x|^{\lambda\beta-1+\frac{2\beta}{ps}} \int_1^\infty K\left(x^\beta y^\gamma\right) |y|^{\lambda\gamma-1-\frac{2\gamma}{qs}} \mathrm{d}y\mathrm{d}x := W_3,$$
$$I_4 > \int_{x\in E^+} |x|^{\lambda\beta-1+\frac{2\beta}{ps}} \int_{-\infty}^{-1} K\left(x^\beta y^\gamma\right) |y|^{\lambda\gamma-1-\frac{2\gamma}{qs}} \mathrm{d}y\mathrm{d}x := W_4.$$

We first consider the case where $\beta < 0$, that is, $\beta \in \Omega \cap \mathbb{R}^-$. Letting $x^\beta y^\gamma = z$, and observing that $x^{-\frac{\beta}{\gamma}} = -|x|^{-\frac{\beta}{\gamma}}$ $(x < 0)$ and $z^{\frac{1}{r}-1} = |z|^{\frac{1}{r}-1}$ $(z < 0)$, we get

$$\begin{aligned}
W_1 &= \int_{-\infty}^{-1} |x|^{\lambda\beta-1+\frac{2\beta}{ps}} \int_1^\infty K\left(x^\beta y^\gamma\right) |y|^{\lambda\gamma-1-\frac{2\gamma}{qs}} \mathrm{d}y\mathrm{d}x \\
&= \frac{1}{\gamma} \int_{-\infty}^{-1} |x|^{-1+\frac{2\beta}{s}} \int_{-\infty}^{x^\beta} K(z)\,|z|^{\lambda-1-\frac{2}{qs}} \mathrm{d}z\mathrm{d}x \\
&= \frac{1}{\gamma} \int_{-\infty}^{-1} |x|^{-1+\frac{2\beta}{s}} \int_{-\infty}^{-1} K(z)\,|z|^{\lambda-1-\frac{2}{qs}} \mathrm{d}z\mathrm{d}x \\
&+ \frac{1}{\gamma} \int_{-\infty}^{-1} |x|^{-1+\frac{2\beta}{s}} \int_{-1}^{x^\beta} K(z)\,|z|^{\lambda-1-\frac{2}{qs}} \mathrm{d}z\mathrm{d}x \\
&= \frac{s}{2\,|\beta\gamma|} \int_{-\infty}^{-1} K(z)\,|z|^{\lambda-1-\frac{2}{qs}} \mathrm{d}z \\
&+ \frac{1}{\gamma} \int_{-\infty}^{-1} |x|^{-1+\frac{2\beta}{s}} \int_{-1}^{x^\beta} K(z)\,|z|^{\lambda-1-\frac{2}{qs}} \mathrm{d}z\mathrm{d}x.
\end{aligned} \tag{20}$$

By Fubini's theorem, we have

$$\begin{aligned}
&\int_{-\infty}^{-1} |x|^{-1+\frac{2\beta}{s}} \int_{-1}^{x^\beta} K(z)\,|z|^{\lambda-1-\frac{2}{qs}} \mathrm{d}z\mathrm{d}x \\
&= \int_{-1}^{0} K(z)\,|z|^{\lambda-1-\frac{2}{qs}} \int_{-\infty}^{z^{1/\beta}} |x|^{-1+\frac{2\beta}{s}} \mathrm{d}x\mathrm{d}z \\
&= \frac{s}{2\,|\beta|} \int_{-1}^{0} K(z)\,|z|^{\lambda-1+\frac{2}{ps}} \mathrm{d}z.
\end{aligned} \tag{21}$$

Applying (21) to (20), we get

$$W_1 = \frac{s}{2\,|\beta\gamma|} \left[\int_{-\infty}^{-1} K(z)\,|z|^{\lambda-1-\frac{2}{qs}} \mathrm{d}z + \int_{-1}^{0} K(z)\,|z|^{\lambda-1+\frac{2}{ps}} \mathrm{d}z\right].$$

In addition, it can be proved that $W_1 = W_4$, and

$$W_2 = W_3 = \frac{s}{2\,|\beta\gamma|}\left[\int_1^{\infty} K(z)\,|z|^{\lambda-1-\frac{2}{qs}}\,\mathrm{d}z + \int_0^1 K(z)\,|z|^{\lambda-1+\frac{2}{ps}}\,\mathrm{d}z\right].$$

Therefore, we have

$$\begin{aligned}\tilde{I} &> W_1 + W_2 + W_3 + W_4 \\ &= \frac{s}{|\beta\gamma|}\left[\int_{[-1,1]} K(z)\,|z|^{\lambda-1+\frac{2}{ps}}\,\mathrm{d}z + \int_{\mathbb{R}\backslash[-1,1]} K(z)\,|z|^{\lambda-1-\frac{2}{qs}}\,\mathrm{d}z\right].\end{aligned}$$

Inequality (19) is proved for $\beta < 0$. Similarly, (19) can also be proved to be true for $\beta > 0$, and we complete the proof of Lemma 2.3. □

Lemma 2.4. *Let $s_1, s_2 > 0$, $s_1 + s_2 = 1$, $\psi(z) = \cot z$, $\phi(z) = \csc z$ and $m \in \mathbb{N}$. Then*

$$\sum_{j=0}^{\infty}\left[\frac{1}{(j+s_1)^{2m+1}} - \frac{1}{(j+s_2)^{2m+1}}\right] = \frac{\pi^{2m+1}}{(2m)!}\psi^{(2m)}(s_1\pi), \tag{22}$$

$$\sum_{j=0}^{\infty}\left[\frac{1}{(j+s_1)^{2m+2}} + \frac{1}{(j+s_2)^{2m+2}}\right] = -\frac{\pi^{2m+2}}{(2m+1)!}\psi^{(2m+1)}(s_1\pi), \tag{23}$$

$$\sum_{j=0}^{\infty}\left[\frac{(-1)^j}{(j+s_1)^{2m+1}} + \frac{(-1)^j}{(j+s_2)^{2m+1}}\right] = \frac{\pi^{2m+1}}{(2m)!}\phi^{(2m)}(s_1\pi). \tag{24}$$

Proof. We write the partial fraction expansion of $\psi(z) = \cot z$ $(0 < z < \pi)$ as follows [4]:

$$\psi(z) = \frac{1}{z} + \sum_{j=1}^{\infty}\left(\frac{1}{z+j\pi} + \frac{1}{z-j\pi}\right).$$

Taking the $(2m)$th derivative of $\psi(z)$, we get

$$\begin{aligned}\psi^{(2m)}(z) &= (2m)!\left[\sum_{j=0}^{\infty}\frac{1}{(j\pi+z)^{2m+1}} + \sum_{j=1}^{\infty}\frac{1}{(z-j\pi)^{2m+1}}\right] \\ &= (2m)!\sum_{j=0}^{\infty}\left[\frac{1}{(z+j\pi)^{2m+1}} - \frac{1}{(j\pi+\pi-z)^{2m+1}}\right].\end{aligned} \tag{25}$$

Letting $z = s_1\pi$ in (25), and observing that $s_1 + s_2 = 1$, we obtain (22). Taking the first derivative of (25) and setting $z = s_1\pi$, we arrive at (23). Additionally, owing to the following identity:

$$2\phi(2z) = \psi\left(\frac{\pi}{2} - z\right) + \psi(z)\ \ \left(0 < z < \frac{\pi}{2}\right),$$

we have

$$2^{2m+1}\phi^{(2m)}(2z) = \psi^{(2m)}\left(\frac{\pi}{2}-z\right)+\psi^{(2m)}(z). \tag{26}$$

Let $u = \frac{s_1\pi}{2}$ in (26), and use (22), then we have

$$\begin{aligned}\phi^{(2m)}(s_1\pi) &= \frac{(2m)!}{\pi^{2m+1}}\sum_{j=0}^{\infty}\left[\frac{1}{(2j+s_2)^{2m+1}}-\frac{1}{(2j+1+s_1)^{2m+1}}\right]\\ &+\frac{(2m)!}{\pi^{2m+1}}\sum_{j=0}^{\infty}\left[\frac{1}{(2j+s_1)^{2m+1}}-\frac{1}{(2j+1+s_2)^{2m+1}}\right]\\ &=\frac{(2m)!}{\pi^{2m+1}}\sum_{j=0}^{\infty}\left[\frac{(-1)^j}{(j+s_1)^{2m+1}}+\frac{(-1)^j}{(j+s_2)^{2m+1}}\right].\end{aligned} \tag{27}$$

Equality (27) implies (24) obviously. Lemma 2.4 is proved. □

Remark 2.1. By Lemma 2.4, we have the following identities related to classical special constants:

$$\psi^{(2m)}\left(\frac{\pi}{4}\right) = 2^{2m}E_m, \tag{28}$$

$$\phi^{(2m)}\left(\frac{\pi}{2}\right) = E_m, \tag{29}$$

$$\psi^{(2m+1)}\left(\frac{\pi}{4}\right) = \frac{4^{2m+1}}{m+1}\left(1-2^{2m+2}\right)B_{m+1}, \tag{30}$$

$$\psi^{(2m+1)}\left(\frac{\pi}{2}\right) = \frac{2^{2m+1}}{m+1}\left(1-2^{2m+2}\right)B_{m+1}, \tag{31}$$

where E_m is the Euler number, $E_0 = 1$, $E_1 = 1$, $E_2 = 5$, $\cdots$, and B_{m+1} is Bernoulli number, $B_1 = \frac{1}{6}, B_2 = \frac{1}{30}, B_3 = \frac{1}{42}, \cdots$. In fact, let $s_1 = \frac{1}{4},\ s_2 = \frac{3}{4}$ in (22). In view of [4]

$$\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)^{2m+1}} = \frac{\pi^{2m+1}E_m}{2^{2m+2}(2m)!}, \tag{32}$$

and

$$\sum_{j=0}^{\infty}\left[\frac{1}{(4j+1)^{2m+1}}-\frac{1}{(4j+3)^{2m+1}}\right] = \sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)^{2m+1}},$$

we can get (28). Similarly, let $s_1 = s_2 = \frac{1}{2}$ in (24). By (32), we have (29). Additionally, let $s_1 = \frac{1}{4},\ s_2 = \frac{3}{4}$ in (23), and observe that [4]

$$\sum_{j=0}^{\infty}\frac{1}{(2j+1)^{2m+2}} = \frac{B_{m+1}}{2(2m+2)!}\left(2^{2m+2}-1\right)\pi^{2m+2}\ (m\in\mathbb{N}),$$

then we get (30). At last, letting $s_1 = s_2 = \frac{1}{2}$ in (23), we arrive at (31).

3. Main results

Theorem 3.1. *Let $\tau, \eta \in \{1, -1\}$, and $\tau \neq -1$ when $\eta = 1$. Let*

$$\Omega := \left\{ z : z = \frac{2i+1}{2l+1}, i, l \in \mathbb{Z} \right\}$$

$\beta \in \Omega$, and $\gamma \in \mathbb{R}^+ \cap \Omega$. Suppose that $c > a \geq b > d > 0$, and $ab = cd$ when $\tau\eta = 1$. Let λ be such that $\lambda \geq 1$, $\lambda\gamma \leq 1$, and $\lambda \neq 1$ for $\tau = 1, \eta = -1$. Assume that $\mu(x) = |x|^{p(1-\lambda\beta)-1}$, $\nu_n = |n|^{q(1-\lambda\gamma)-1}$, $n \in \mathbb{Z}^0 := \mathbb{Z} \setminus \{0\}$, $f(x), a_n \geq 0$ with $f(x) \in L_{p,\mu}(\mathbb{R})$ and $\boldsymbol{a} = \{a_n\}_{n\in\mathbb{Z}^0} \in l_{q,\nu}$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Let $K(z)$ and $\kappa(a, b, c, d, \tau, \eta, \lambda)$ be defined via (10) and (12), respectively. Then the following inequalities hold and are equivalent:

$$\begin{aligned}(33)\quad I &:= \sum_{n\in\mathbb{Z}^0} a_n \int_{x\in\mathbb{R}} K\left(x^\beta n^\gamma\right) f(x)\mathrm{d}x = \int_{x\in\mathbb{R}} f(x) \sum_{n\in\mathbb{Z}^0} K\left(x^\beta n^\gamma\right) a_n \mathrm{d}x \\ &< |\beta|^{-\frac{1}{q}} \gamma^{-\frac{1}{p}} \Gamma(\lambda)\kappa(a, b, c, d, \tau, \eta, \lambda) \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu},\end{aligned}$$

$$\begin{aligned}(34)\quad J_1 &:= \sum_{n\in\mathbb{Z}^0} |n|^{p\lambda\gamma-1} \left[\int_{x\in\mathbb{R}} K\left(x^\beta n^\gamma\right) f(x)\mathrm{d}x\right]^p \\ &< \left[|\beta|^{-\frac{1}{q}} \gamma^{-\frac{1}{p}} \Gamma(\lambda)\kappa(a, b, c, d, \tau, \eta, \lambda)\right]^p \|f\|_{p,\mu}^p,\end{aligned}$$

$$\begin{aligned}(35)\quad J_2 &:= \int_{x\in\mathbb{R}} |x|^{q\lambda\beta-1} \left[\sum_{n\in\mathbb{Z}^0} K\left(x^\beta n^\gamma\right) a_n\right]^q \mathrm{d}x \\ &< \left[|\beta|^{-\frac{1}{q}} \gamma^{-\frac{1}{p}} \Gamma(\lambda)\kappa(a, b, c, d, \tau, \eta, \lambda)\right]^q \|\boldsymbol{a}\|_{q,\nu}^q,\end{aligned}$$

where the constant $|\beta|^{-\frac{1}{q}} \gamma^{-\frac{1}{p}} \Gamma(\lambda)\kappa(a, b, c, d, \tau, \eta, \lambda)$ in (33), (34) and (35) is the best possible.

Proof. For $y \in [n-1, n)$, $n \in \mathbb{N}^+$, let $\widetilde{K}\left(x^\beta y^\gamma\right) := K\left(x^\beta n^\gamma\right)$, $g(y) := a_n$, $h(y) := n$. For $y \in [n, n+1)$, $n \in \mathbb{N}^-$, let $\widetilde{K}\left(x^\beta y^\gamma\right) := K\left(x^\beta n^\gamma\right)$, $g(y) := a_n$, $h(y) := |n|$. By Hölder's inequality, we have

$$\begin{aligned}(36)\quad &\sum_{n\in\mathbb{Z}^0} a_n \int_{x\in\mathbb{R}} K\left(x^\beta n^\gamma\right) f(x)\mathrm{d}x = \int_{x\in\mathbb{R}} f(x) \sum_{n\in\mathbb{Z}^0} K\left(x^\beta n^\gamma\right) a_n \mathrm{d}x \\ &= \int_{y\in\mathbb{R}} \int_{x\in\mathbb{R}} \widetilde{K}\left(x^\beta y^\gamma\right) f(x) g(y)\mathrm{d}x\mathrm{d}y \\ &= \int_{y\in\mathbb{R}} \int_{x\in\mathbb{R}} \left[\widetilde{K}\left(x^\beta y^\gamma\right)\right]^{1/p} [h(y)]^{(\lambda\gamma-1)/p} |x|^{(1-\lambda\beta)/q} f(x) \\ &\qquad \times \left[\widetilde{K}\left(x^\beta y^\gamma\right)\right]^{1/q} |x|^{(\lambda\beta-1)/q} [h(y)]^{(1-\lambda\gamma)/p} g(y)\mathrm{d}x\mathrm{d}y \\ &\leq \left\{\int_{x\in\mathbb{R}} \int_{y\in\mathbb{R}} \widetilde{K}\left(x^\beta y^\gamma\right) [h(y)]^{\lambda\gamma-1} |x|^{p(1-\lambda\beta)/q} f^p(x)\mathrm{d}y\mathrm{d}x\right\}^{1/p}\end{aligned}$$

$$\times \left\{\int_{y\in\mathbb{R}}\int_{x\in\mathbb{R}} \widetilde{K}\left(x^{\beta}y^{\gamma}\right)|x|^{\lambda\beta-1}\left[h(y)\right]^{q(1-\lambda\gamma)/p}g^{q}(y)\mathrm{d}x\mathrm{d}y\right\}^{1/q}$$
$$=\left[\int_{x\in\mathbb{R}}\Psi(x)\,|x|^{p(1-\lambda\beta)/q}\,f^{p}(x)\mathrm{d}x\right]^{1/p}\left[\sum_{n\in\mathbb{Z}^{0}}\Phi(n)\,|n|^{q(1-\lambda\gamma)/p}\,a_{n}^{q}\right]^{1/q},$$

where

$$\Psi(x)=\sum_{n\in\mathbb{Z}^{0}}K\left(x^{\beta}n^{\gamma}\right)|n|^{\lambda\gamma-1}, \tag{37}$$

$$\Phi(n)=\int_{x\in\mathbb{R}}K\left(x^{\beta}n^{\gamma}\right)|x|^{\lambda\beta-1}\,\mathrm{d}x. \tag{38}$$

In view of $\lambda\gamma\leq 1$, it can be easy to show that $|n|^{\lambda\gamma-1}$ decreases if $n\in\mathbb{N}^{+}$ and increases if $n\in\mathbb{N}^{-}$. Additionally, using Lemma 2.1, and observing that $\beta\in\Omega$ and $\gamma\in\mathbb{R}^{+}\cap\Omega$, it can be proved that whether $x>0$ or $x<0$, $K\left(x^{\beta}n^{\gamma}\right)$ decreases with respect to n when $n\in\mathbb{N}^{+}$, and increases with respect to n when $n\in\mathbb{N}^{-}$. Therefore, we get

$$\Psi(x)<\int_{y\in\mathbb{R}}K\left(x^{\beta}y^{\gamma}\right)|y|^{\lambda\gamma-1}\,\mathrm{d}y. \tag{39}$$

We first consider the case where $x<0$. Let $x^{\beta}y^{\gamma}=z$. Observing that $\beta\in\Omega$ and $\gamma\in\mathbb{R}^{+}\cap\Omega$, we have $x^{-\frac{\beta}{\gamma}}=-|x|^{-\frac{\beta}{\gamma}}$ $(x<0)$ and $z^{\frac{1}{r}-1}=|z|^{\frac{1}{r}-1}$. It follows therefore that

$$\int_{y\in\mathbb{R}}K\left(x^{\beta}y^{\gamma}\right)|y|^{\lambda\gamma-1}\,\mathrm{d}y=\frac{|x|^{-\lambda\beta}}{\gamma}\int_{z\in\mathbb{R}}K(z)\,|z|^{\lambda-1}\,\mathrm{d}z. \tag{40}$$

Similarly, it can also be proved that (40) holds when $x>0$. Therefore, for arbitrary x $(x\neq 0)$, combining (39) and (40), and using (13), we have

$$\Psi(x)<\frac{|x|^{-\lambda\beta}}{\gamma}\Gamma(\lambda)\kappa\left(a,b,c,d,\tau,\eta,\lambda\right). \tag{41}$$

Furthermore, by similar discussion, we have

$$\Phi(n)=\frac{|n|^{-\lambda\gamma}}{|\beta|}\Gamma(\lambda)\kappa\left(a,b,c,d,\tau,\eta,\lambda\right). \tag{42}$$

Plugging (41) and (42) back into (36), we get (33). In what follows, we will prove (34) and (35) via (33). In fact, assuming (33) holds, and setting $\boldsymbol{b}=\{b_n\}_{n\in\mathbb{N}^{0}}$, where

$$b_n:=|n|^{p\lambda\gamma-1}\left[\int_{x\in\mathbb{R}}K\left(x^{\beta}n^{\gamma}\right)f(x)\mathrm{d}x\right]^{p-1},$$

we obtain

$$J_1 = \sum_{n\in\mathbb{Z}^0} |n|^{p\lambda\gamma-1} \left[\int_{x\in\mathbb{R}} K\left(x^\beta n^\gamma\right) f(x)\mathrm{d}x\right]^p \tag{43}$$
$$= \sum_{n\in\mathbb{Z}^0} b_n \int_{x\in\mathbb{R}} K\left(x^\beta n^\gamma\right) f(x)\mathrm{d}x$$
$$< |\beta|^{-\frac{1}{q}} \gamma^{-\frac{1}{p}} \Gamma(\lambda)\kappa\left(a,b,c,d,\tau,\eta,\lambda\right) \|f\|_{p,\mu} \|\boldsymbol{b}\|_{q,\nu}$$
$$= |\beta|^{-\frac{1}{q}} \gamma^{-\frac{1}{p}} \Gamma(\lambda)\kappa\left(a,b,c,d,\tau,\eta,\lambda\right) \|f\|_{p,\mu} J_1^{1/q}.$$

Inequality (43) implies (34) obviously. Moreover, let

$$g(x) := |x|^{q\lambda\beta-1} \left[\sum_{n\in\mathbb{Z}^0} K\left(x^\beta n^\gamma\right) a_n\right]^{q-1}.$$

By (33), we get

$$J_2 = \int_{x\in\mathbb{R}} |x|^{q\lambda\beta-1} \left[\sum_{n\in\mathbb{Z}^0} K\left(x^\beta n^\gamma\right) a_n\right]^q \mathrm{d}x \tag{44}$$
$$= \int_{x\in\mathbb{R}} g(x) \sum_{n\in\mathbb{Z}^0} K\left(x^\beta n^\gamma\right) a_n \mathrm{d}x$$
$$< |\beta|^{-\frac{1}{q}} \gamma^{-\frac{1}{p}} \Gamma(\lambda)\kappa\left(a,b,c,d,\tau,\eta,\lambda\right) \|g\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}$$
$$= |\beta|^{-\frac{1}{q}} \gamma^{-\frac{1}{p}} \Gamma(\lambda)\kappa\left(a,b,c,d,\tau,\eta,\lambda\right) \|\boldsymbol{a}\|_{q,\nu} J_2^{1/p}.$$

Thus, we get (35) via (33). Conversely, if (34) or (35) holds, it can also be proved that (33) is valid. In fact, we first suppose that (34) holds. By Hölder's inequality, we obtain

$$I = \sum_{n\in\mathbb{Z}^0} \left[|n|^{\lambda\gamma-1/p} \int_{x\in\mathbb{R}} K\left(x^\beta n^\gamma\right) f(x)\mathrm{d}x\right] \left[a_n |n|^{-\lambda\gamma+1/p}\right] \tag{45}$$
$$\le J_1^{1/p} \left[\sum_{n\in\mathbb{Z}^0} a_n^q |n|^{q(1-\lambda\gamma)-1}\right]^{1/q} = J_1^{1/p} \|\boldsymbol{a}\|_{q,\nu}.$$

Applying (34) to (45), we arrive at (33). Similarly, supposing that (35) holds, we can also get (33). Therefore, Based on the above discussions, inequalities (33), (34) and (35) are equivalent.

Lastly, it will be proved that the constant $|\beta|^{-\frac{1}{q}} \gamma^{-\frac{1}{p}} \Gamma(\lambda)\kappa\left(a,b,c,d,\tau,\eta,\lambda\right)$ in (33), (34) and (35) is the best possible. In fact, assume that there exists a constant C satisfying

$$0 < C \le |\beta|^{-\frac{1}{q}} \gamma^{-\frac{1}{p}} \Gamma(\lambda)\kappa\left(a,b,c,d,\tau,\eta,\lambda\right), \tag{46}$$

and

$$I=\sum_{n\in\mathbb{Z}^0}a_n\int_{x\in\mathbb{R}}K\left(x^\beta n^\gamma\right)f(x)\mathrm{d}x=\int_{x\in\mathbb{R}}f(x)\sum_{n\in\mathbb{Z}^0}K\left(x^\beta n^\gamma\right)a_n\mathrm{d}x$$
$$<C\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}. \tag{47}$$

Replacing a_n and $f(x)$ in (47) by $\tilde{a}_n$ and $\tilde{f}(x)$ defined in Lemma 2.3, repectively, and using (19), we have

$$\int_{[-1,1]}K(z)\,|z|^{\lambda-1+\frac{2}{ps}}\,\mathrm{d}z+\int_{\mathbb{R}\backslash[-1,1]}K(z)\,|z|^{\lambda-1-\frac{2}{qs}}\,\mathrm{d}z \tag{48}$$
$$<\frac{|\beta\gamma|}{s}\tilde{I}<\frac{|\beta\gamma|\,C}{s}\|\tilde{f}\|_{p,\mu}\|\tilde{\boldsymbol{a}}\|_{q,\nu}$$
$$=\frac{|\beta\gamma|\,C}{s}\left(2\int_{E^+}x^{\frac{2\beta}{s}-1}\mathrm{d}x\right)^{\frac{1}{p}}\left(2+2\sum_{n=2}^{\infty}n^{\frac{-2\gamma}{s}-1}\right)^{\frac{1}{q}}$$
$$<\frac{2\,|\beta\gamma|\,C}{s}\left(\int_{E^+}x^{\frac{2\beta}{s}-1}\mathrm{d}x\right)^{\frac{1}{p}}\left(1+\int_1^{\infty}x^{-\frac{2\gamma}{s}-1}\mathrm{d}x\right)^{\frac{1}{q}}$$
$$=2\,|\beta\gamma|\,C\left(\frac{1}{2\,|\beta|}\right)^{\frac{1}{p}}\left(\frac{1}{s}+\frac{1}{2\gamma}\right)^{\frac{1}{q}}.$$

Applying Fatou's lemma to (48), and using (13), it follows that

$$\Gamma(\lambda)\kappa\left(a,b,c,d,\tau,\eta,\lambda\right)=\int_{z\in\mathbb{R}}K(z)\,|z|^{\lambda-1}\,\mathrm{d}z$$
$$=\int_{[-1,1]}\lim_{s\to\infty}K(z)\,|z|^{\lambda-1+\frac{2}{ps}}\,\mathrm{d}z+\int_{\mathbb{R}\backslash[-1,1]}\lim_{s\to\infty}L\left(z\right)|z|^{\lambda-1-\frac{2}{qs}}\,\mathrm{d}z$$
$$\leqslant\lim_{s\to\infty}\left[\int_{[-1,1]}L\left(z\right)|z|^{\lambda-1+\frac{2}{ps}}\,\mathrm{d}z+\int_{\mathbb{R}\backslash[-1,1]}L\left(z\right)|z|^{\lambda-1-\frac{2}{qs}}\,\mathrm{d}z\right]$$
$$\leqslant\lim_{s\to\infty}\left[2\,|\beta\gamma|\,C\left(\frac{1}{2\,|\beta|}\right)^{\frac{1}{p}}\left(\frac{1}{s}+\frac{1}{2\gamma}\right)^{\frac{1}{q}}\right]=C\,|\beta|^{\frac{1}{q}}\,\gamma^{\frac{1}{p}}.$$

It implies that

$$C\geq|\beta|^{-\frac{1}{q}}\,\gamma^{-\frac{1}{p}}\Gamma(\lambda)\kappa\left(a,b,c,d,\tau,\eta,\lambda\right). \tag{49}$$

Combining (46) and (49), we get $C=|\beta|^{-\frac{1}{q}}\,\gamma^{-\frac{1}{p}}\Gamma(\lambda)\kappa\left(a,b,c,d,\tau,\eta,\lambda\right)$. Therefore, the constant factor in inequality (33) is the best possible. Owing to the equivalence of (33), (34) and (35), it can be proved that the constant factors in (34) and (35) are the best possible. Theorem 3.1 is proved. □

4. Corollaries

Let $\tau = \eta = -1$, and $\lambda = 2m+1 (m \in \mathbb{N})$ in Theorem 3.1, then we have $ab = cd$. By (22), we have

$$\kappa\left(a, b, c, d, \tau, \eta, \lambda\right) = \sum_{j=0}^{\infty}\left[\frac{2}{(j\log\frac{c}{d} + \log\frac{b}{d})^{2m+1}} - \frac{2}{(j\log\frac{c}{d} + \log\frac{c}{b})^{2m+1}}\right]$$
$$= \frac{2}{(2m)!}\left(\frac{\pi}{\ln\frac{c}{d}}\right)^{2m+1}\psi^{(2m)}\left(\frac{\pi\ln\frac{b}{d}}{\ln\frac{c}{d}}\right).$$

Thus, we have the following corollary.

Corollary 4.1. *Let $\beta \in \Omega$, and $\gamma \in \mathbb{R}^{+} \cap \Omega$, where*

$$\Omega := \left\{z : z = \frac{2i+1}{2l+1}, i, l \in \mathbb{Z}\right\}.$$

Suppose that $c > a \geq b > d > 0$, and $ab = cd$. Let m be such that $(2m+1)\gamma \leq 1$, $m \in \mathbb{N}$. Assume that $\psi(z) = \cot z$, $\mu(x) = |x|^{p[1-(2m+1)\beta]-1}$, $\nu_n = |n|^{q[1-(2m+1)\gamma]-1}$, $n \in \mathbb{Z}^0 := \mathbb{Z} \setminus \{0\}$. Let $f(x), a_n \geq 0$ with $f(x) \in L_{p,\mu}(\mathbb{R})$ and $\boldsymbol{a} = \{a_n\}_{n\in\mathbb{Z}^0} \in l_{q,\nu}$. Then

$$\int_{x\in\mathbb{R}} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a^{x^\beta n^\gamma} - b^{x^\beta n^\gamma}}{c^{x^\beta n^\gamma} - d^{x^\beta n^\gamma}} a_n \mathrm{d}x$$
$$< 2\left|\beta\right|^{-\frac{1}{q}}\gamma^{-\frac{1}{p}}\left(\frac{\pi}{\ln\frac{c}{d}}\right)^{2m+1}\psi^{(2m)}\left(\frac{\pi\ln\frac{b}{d}}{\ln\frac{c}{d}}\right)\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}. \tag{50}$$

Let $a = \mathrm{e}^{\tau_1}$, $b = \mathrm{e}^{-\tau_1}$, $c = \mathrm{e}^{\tau_2}$, $d = \mathrm{e}^{-\tau_2}$ in (50), where $0 < \tau_1 < \tau_2$. Then

$$\int_{x\in\mathbb{R}} f(x) \sum_{n\in\mathbb{Z}^0} \sinh\left(\tau_1 x^\beta n^\gamma\right) \operatorname{csch}\left(\tau_2 x^\beta n^\gamma\right) a_n \mathrm{d}x$$
$$< 2 \mid \beta \mid^{-\frac{1}{q}}\gamma^{-\frac{1}{p}}\left(\frac{\pi}{2\tau_2}\right)^{2m+1}\psi^{(2m)}\left(\frac{(\tau_2-\tau_1)\pi}{2\tau_2}\right)\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}. \tag{51}$$

Let $\tau_2 = 2\alpha$, $\tau_1 = \alpha$ $(\alpha > 0)$ in (51). By (28), we obtain

$$\int_{x\in\mathbb{R}} f(x) \sum_{n\in\mathbb{Z}^0} \operatorname{sech}(\alpha x^\beta n^\gamma) a_n \mathrm{d}x < |\beta|^{-\frac{1}{q}}\gamma^{-\frac{1}{p}}\frac{E_m}{2^{2m}}\left(\frac{\pi}{\alpha}\right)^{2m+1}\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}. \tag{52}$$

Setting $\beta = \gamma = \frac{1}{2m+1}$ $m \in \mathbb{N}$, and $\alpha = 1$ in (52), we get

$$\int_{x\in\mathbb{R}} f(x) \sum_{n\in\mathbb{Z}^0} \operatorname{sech}\left(\sqrt[2m+1]{xn}\right) a_n \mathrm{d}x < \frac{E_m}{2^{2m}}(2m+1)\pi^{2m+1}\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{53}$$

where $\mu(x) = |x|^{-1}$, $\nu_n = |n|^{-1}$. Setting $\beta = -\frac{1}{2m+1}$, $\gamma = \frac{1}{2m+1}$, and $\alpha = 1$ in (52), we get (8).

Let $\tau_2 = 3\alpha$, $\tau_1 = \alpha$ $(\alpha > 0)$ in (51), then we obtain

$$\int_{x\in\mathbb{R}} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a_n}{2\cosh\left(2\alpha x^{\beta} n^{\gamma}\right)+1} \mathrm{d}x < 2\,|\beta|^{-\frac{1}{q}} \gamma^{-\frac{1}{p}} \left(\frac{\pi}{6\alpha}\right)^{2m+1} \psi^{(2m)}\left(\frac{\pi}{3}\right) \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}. \tag{54}$$

Setting $\beta = \gamma = \frac{1}{2m+1}$, and $\alpha = \frac{1}{2}$ in (54), we get

$$\int_{x\in\mathbb{R}} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a_n}{2\cosh\left(\sqrt[2m+1]{xn}\right)+1} \mathrm{d}x < (4m+2) \left(\frac{\pi}{3}\right)^{2m+1} \psi^{(2m)}\left(\frac{\pi}{3}\right) \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}, \tag{55}$$

where $\mu(x) = |x|^{-1}$, $\nu_n = |n|^{-1}$. Let $m = 0$ in (55), then (55) is transformed into

$$\int_{x\in\mathbb{R}} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a_n}{2\cosh\left(xn\right)+1} \mathrm{d}x < \frac{2\sqrt{3}\pi}{9} \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}. \tag{56}$$

Let $\tau_2 = 4\alpha$, $\tau_1 = \alpha$ $(\alpha > 0)$ in (51), then we have

$$\int_{x\in\mathbb{R}} f(x) \sum_{n\in\mathbb{Z}^0} \operatorname{sech}\left(\alpha x^{\beta} n^{\gamma}\right) \operatorname{sech}\left(2\alpha x^{\beta} n^{\gamma}\right) a_n \mathrm{d}x < \frac{1}{2^{6m}} |\beta|^{-\frac{1}{q}} \gamma^{-\frac{1}{p}} \left(\frac{\pi}{\alpha}\right)^{2m+1} \psi^{(2m)}\left(\frac{3\pi}{8}\right) \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}. \tag{57}$$

Setting $\beta = \gamma = \frac{1}{2m+1}$, and $\alpha = 1$ in (57), we get

$$\int_{x\in\mathbb{R}} f(x) \sum_{n\in\mathbb{Z}^0} \operatorname{sech}\left(\sqrt[2m+1]{xn}\right) \operatorname{sech}\left(2\sqrt[2m+1]{xn}\right) a_n \mathrm{d}x < \frac{2m+1}{2^{6m}} \pi^{2m+1} \psi^{(2m)}\left(\frac{3\pi}{8}\right) \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}.$$

Let $\tau = -1$, $\eta = 1$, and $\lambda = 2m+2$ $(m \in \mathbb{N})$ in Theorem 3.1, By (23), we have

$$\begin{aligned}
&\kappa\left(a,b,c,d,\tau,\eta,\lambda\right) \\
=&\sum_{j=0}^{\infty} \left[\frac{1}{(j\log\frac{c}{d}+\log\frac{c}{a})^{2m+2}} + \frac{1}{(j\log\frac{c}{d}+\log\frac{a}{d})^{2m+2}}\right] \\
+&\sum_{j=0}^{\infty} \left[\frac{1}{(j\log\frac{c}{d}+\log\frac{b}{d})^{2m+2}} + \frac{1}{(j\log\frac{c}{d}+\log\frac{c}{b})^{2m+2}}\right] \\
=&\frac{1}{(2m+1)!} \left(\frac{\pi}{\ln\frac{c}{d}}\right)^{2m+2} \left[\psi^{(2m+1)}\left(\frac{\pi\ln\frac{c}{a}}{\ln\frac{c}{d}}\right) + \psi^{(2m+1)}\left(\frac{\pi\ln\frac{b}{d}}{\ln\frac{c}{d}}\right)\right].
\end{aligned}$$

Thus, we have the following corollary.

Corollary 4.2. *Let* $\beta \in \Omega$, *and* $\gamma \in \mathbb{R}^+ \cap \Omega$, *where*

$$\Omega := \left\{ z : z = \frac{2i+1}{2l+1}, i, l \in \mathbb{Z} \right\}.$$

Suppose that $c > a \geq b > d > 0$, *and* m *satisfies* $(2m+2)\gamma \leq 1$ $(m \in \mathbb{N})$. *Assume that* $\psi(z) = \cot z$, $\mu(x) = |x|^{p[1-(2m+2)\beta]-1}$, $\nu_n = |n|^{q[1-(2m+2)\gamma]-1}$, $n \in \mathbb{Z}^0 := \mathbb{Z}\backslash\{0\}$. *Let* $f(x), a_n \geq 0$ *with* $f(x) \in L_{p,\mu}(\mathbb{R})$ *and* $\boldsymbol{a} = \{a_n\}_{n\in\mathbb{Z}^0} \in l_{q,\nu}$. *Then*

$$\int_{x\in\mathbb{R}} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a^{x^\beta n^\gamma} + b^{x^\beta n^\gamma}}{\left|c^{x^\beta n^\gamma} - d^{x^\beta n^\gamma}\right|} a_n \mathrm{d}x < -|\beta|^{-\frac{1}{q}} \gamma^{-\frac{1}{p}} \left(\frac{\pi}{\ln \frac{c}{d}}\right)^{2m+2} \times \left[\psi^{(2m+1)}\left(\frac{\pi \ln \frac{c}{a}}{\ln \frac{c}{d}}\right) + \psi^{(2m+1)}\left(\frac{\pi \ln \frac{b}{d}}{\ln \frac{c}{d}}\right)\right] \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}. \tag{58}$$

Let $a = b = 1$ in (58), then we get $c > 1 > d > 0$. Since

$$\psi^{(2m+1)}(z) = \psi^{(2m+1)}(\pi - z),\ z \in (0, \pi), \tag{59}$$

inequality (58) is transformed into

$$\int_{x\in\mathbb{R}} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a_n}{\left|c^{x^\beta n^\gamma} - d^{x^\beta n^\gamma}\right|} \mathrm{d}x < -|\beta|^{-\frac{1}{q}} \gamma^{-\frac{1}{p}} \times \left(\frac{\pi}{\ln \frac{c}{d}}\right)^{2m+2} \psi^{(2m+1)}\left(\frac{\pi \ln c}{\ln \frac{c}{d}}\right) \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}. \tag{60}$$

Let $c = e^p$, $d = e^{-q}$ in (60), then (60) reduces to

$$\int_{x\in\mathbb{R}} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a_n}{\left|e^{px^\beta n^\gamma} - e^{-qx^\beta n^\gamma}\right|} \mathrm{d}x < -|\beta|^{-\frac{1}{q}} \gamma^{-\frac{1}{p}} \left(\frac{\pi}{pq}\right)^{2m+2} \psi^{(2m+1)}\left(\frac{\pi}{p}\right) \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}. \tag{61}$$

Let $m = 0$, $\beta = \gamma = \frac{1}{3}$ in (61), then we get

$$\int_{x\in\mathbb{R}} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a_n}{\left|e^{p\sqrt[3]{xn}} - e^{-q\sqrt[3]{xn}}\right|} \mathrm{d}x < \left(\frac{\sqrt{3}\pi}{pq \sin \frac{\pi}{p}}\right)^2 \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}, \tag{62}$$

where $\mu(x) = |x|^{\frac{p}{3}-1}$, $\nu_n = |n|^{\frac{q}{3}-1}$.

Let $c = e^\alpha$, $d = e^{-\alpha}$ $(\alpha > 0)$ in (60). By (31), we get

$$\int_{x\in\mathbb{R}} f(x) \sum_{n\in\mathbb{Z}^0} \left|\operatorname{csch}\left(\alpha x^\beta n^\gamma\right)\right| a_n \mathrm{d}x < |\beta|^{-\frac{1}{q}} \gamma^{-\frac{1}{p}} \frac{B_{m+1}}{m+1} (2^{2m+2} - 1) \left(\frac{\pi}{\alpha}\right)^{2m+2} \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}. \tag{63}$$

Setting $\beta=\gamma=\frac{1}{2m+3}$ $(m\in\mathbb{N})$, $\alpha=1$ in (63), and replacing $m+1$ with m, we get (9). Similarly, setting $\beta=-\frac{1}{2m+3}$, $\gamma=\frac{1}{2m+3}$, $\alpha=1$ in (63), and replacing $m+1$ with m, we get

$$\int_{x\in\mathbb{R}} f(x)\sum_{n\in\mathbb{Z}^0}\left|\operatorname{csch}\left(\sqrt[2m+1]{\frac{n}{x}}\right)\right|a_n\mathrm{d}x<\frac{B_m}{m}(2m+1)(2^{2m}-1)\pi^{2m}\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu},$$

where $\mu(x)=|x|^{\frac{(4m+1)p}{2m+1}-1}$, $\nu_n=|n|^{\frac{q}{2m+1}-1}$ $(m\in\mathbb{N}^+)$.

Let $a=\mathrm{e}^{\tau_1}$, $b=\mathrm{e}^{-\tau_1}$, $c=\mathrm{e}^{\tau_2}$, $d=\mathrm{e}^{-\tau_2}$ in (58), where $0<\tau_1<\tau_2$. Then

$$\begin{aligned}(64)\quad &\int_{x\in\mathbb{R}} f(x)\sum_{n\in\mathbb{Z}^0}\cosh\left(\tau_1x^\beta n^\gamma\right)\left|\operatorname{csch}\left(\tau_2x^\beta n^\gamma\right)\right|a_n\mathrm{d}x\\ &<-2\,|\beta|^{-\frac{1}{q}}\,\gamma^{-\frac{1}{p}}\left(\frac{\pi}{2\tau_2}\right)^{2m+2}\psi^{(2m+1)}\left(\frac{(\tau_2-\tau_1)\pi}{2\tau_2}\right)\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}.\end{aligned}$$

Let $\tau_2=2\alpha$, $\tau_1=\alpha$ $(\alpha>0)$ in (64). By using (30), we can also get (63). Let $\tau_2=4\alpha$, $\tau_1=\alpha$ $(\alpha>0)$ in (64), then we have

$$\begin{aligned}(65)\quad &\int_{x\in\mathbb{R}} f(x)\sum_{n\in\mathbb{Z}^0}\left|\operatorname{csch}\left(\alpha x^\beta n^\gamma\right)\right|\operatorname{sech}\left(2\alpha x^\beta n^\gamma\right)a_n\mathrm{d}x\\ &<-\frac{1}{8^{2m+1}}\,|\beta|^{-\frac{1}{q}}\,\gamma^{-\frac{1}{p}}\left(\frac{\pi}{\alpha}\right)^{2m+2}\psi^{(2m+1)}\left(\frac{3\pi}{8}\right)\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}.\end{aligned}$$

Additionally, let $a=b=\mathrm{e}^{-\alpha}$, $c=\mathrm{e}^{2\alpha}$, $d=\mathrm{e}^{-2\alpha}$ $(\alpha>0)$ in (58). By (59) and (30), we get the following inequality with the same constant factor as (63), that is,

$$\begin{aligned}(66)\quad &\int_{x\in\mathbb{R}} f(x)\sum_{n\in\mathbb{Z}^0}\left|\operatorname{csch}\left(\alpha x^\beta n^\gamma\right)-\operatorname{sech}\left(\alpha x^\beta n^\gamma\right)\right|a_n\mathrm{d}x\\ &<|\beta|^{-\frac{1}{q}}\,\gamma^{-\frac{1}{p}}\frac{B_{m+1}}{m+1}(2^{2m+2}-1)\left(\frac{\pi}{\alpha}\right)^{2m+2}\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}.\end{aligned}$$

Furthermore, let $a=b=\mathrm{e}^{\alpha}$, $c=\mathrm{e}^{2\alpha}$, $d=\mathrm{e}^{-2\alpha}$ $(\alpha>0)$ in (58). Then we get

$$\begin{aligned}(67)\quad &\int_{x\in\mathbb{R}} f(x)\sum_{n\in\mathbb{Z}^0}\left|\operatorname{csch}\left(\alpha x^\beta n^\gamma\right)+\operatorname{sech}\left(\alpha x^\beta n^\gamma\right)\right|a_n\mathrm{d}x\\ &<|\beta|^{-\frac{1}{q}}\,\gamma^{-\frac{1}{p}}\frac{B_{m+1}}{m+1}(2^{2m+2}-1)\left(\frac{\pi}{\alpha}\right)^{2m+2}\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}.\end{aligned}$$

Let $\tau=\eta=1$, and $\lambda=2m+1(m\in\mathbb{N})$ in Theorem 3.1, then we have $ab=cd$. By (24), we get

$$\begin{aligned}\kappa\left(a,b,c,d,\tau,\eta,\lambda\right)&=\sum_{j=0}^{\infty}\left[\frac{2(-1)^j}{(j\log\frac{c}{d}+\log\frac{b}{d})^{2m+1}}+\frac{2(-1)^j}{(j\log\frac{c}{d}+\log\frac{c}{b})^{2m+1}}\right]\\ &=\frac{2}{(2m)!}\left(\frac{\pi}{\ln\frac{c}{d}}\right)^{2m+1}\phi^{(2m)}\left(\frac{\pi\ln\frac{b}{d}}{\ln\frac{c}{d}}\right).\end{aligned}$$

Therefore, Theorem 3.1 is transformed into the following corollary.

Corollary 4.3. *Let $\beta \in \Omega$, and $\gamma \in \mathbb{R}^+ \cap \Omega$, where*

$$\Omega := \left\{ z : z = \frac{2i+1}{2l+1}, i, l \in \mathbb{Z} \right\}.$$

Suppose that $c > a \geq b > d > 0$, and $ab = cd$. Let m be such that $(2m+1)\gamma \leq 1$, $m \in \mathbb{N}$. Assume that $\phi(z) = \csc z$, $\mu(x) = |x|^{p[1-(2m+1)\beta]-1}$, $\nu_n = |n|^{q[1-(2m+1)\gamma]-1}$, $n \in \mathbb{Z}^0 := \mathbb{Z} \setminus \{0\}$. Let $f(x), a_n \geq 0$ with $f(x) \in L_{p,\mu}(\mathbb{R})$ and $\boldsymbol{a} = \{a_n\}_{n\in\mathbb{Z}^0} \in l_{q,\nu}$. Then

$$\begin{aligned}(68)\qquad &\int_{x\in\mathbb{R}} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a^{x^\beta n^\gamma} + b^{x^\beta n^\gamma}}{c^{x^\beta n^\gamma} + d^{x^\beta n^\gamma}} a_n \mathrm{d}x \\ &\quad < 2\,|\beta|^{-\frac{1}{q}} \gamma^{-\frac{1}{p}} \left(\frac{\pi}{\ln\frac{c}{d}}\right)^{2m+1} \phi^{(2m)}\left(\frac{\pi \ln \frac{b}{d}}{\ln \frac{c}{d}}\right) \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}.\end{aligned}$$

Let $a = \mathrm{e}^{\tau_1}$, $b = \mathrm{e}^{-\tau_1}$, $c = \mathrm{e}^{\tau_2}$, $d = \mathrm{e}^{-\tau_2}$ in (68), where $0 < \tau_1 < \tau_2$. Then

$$\begin{aligned}(69)\qquad &\int_{x\in\mathbb{R}} f(x) \sum_{n\in\mathbb{Z}^0} \cosh\left(\tau_1 x^\beta n^\gamma\right) \operatorname{sech}\left(\tau_2 x^\beta n^\gamma\right) a_n \mathrm{d}x \\ &\quad < 2\,|\beta|^{-\frac{1}{q}} \gamma^{-\frac{1}{p}} \left(\frac{\pi}{2\tau_2}\right)^{2m+1} \phi^{(2m)}\left(\frac{(\tau_2-\tau_1)\pi}{2\tau_2}\right) \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}.\end{aligned}$$

Letting $\tau_2 = \alpha$ $(\alpha > 0)$, $\tau_1 = 0$ in (69), and using (29), we can also get (52). Letting $\tau_2 = 2\alpha$, $\tau_1 = \alpha$ $(\alpha > 0)$ in (70), we have

$$\begin{aligned}&\int_{x\in\mathbb{R}} f(x) \sum_{n\in\mathbb{Z}^0} \operatorname{csch}\left(\alpha x^\beta n^\gamma\right) \tanh\left(2\alpha x^\beta n^\gamma\right) a_n \mathrm{d}x \\ &\quad < \frac{1}{2^{4m}} |\beta|^{-\frac{1}{q}} \gamma^{-\frac{1}{p}} \left(\frac{\pi}{\alpha}\right)^{2m+1} \phi^{(2m)}\left(\frac{\pi}{4}\right) \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}.\end{aligned}$$

At last, let $\tau = \eta = 1$, $a = b$, and $\lambda = 2$ in Theorem 3.1. Then, we have $cd = a^2$. Let $\frac{c}{a} = \frac{a}{d} = \mathrm{e}^\alpha$ $(\alpha > 0)$, then

$$\log\frac{c}{a} = \log\frac{c}{b} = \log\frac{b}{d} = \log\frac{a}{d} = \frac{1}{2}\log\frac{c}{d} = \alpha,$$

and

$$\kappa\,(a,b,c,d,\tau,\eta,\lambda) = \frac{4}{\alpha^2}\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)^2} = \frac{4c_0}{\alpha^2},$$

where c_0 is the Catalan constant. Thus, Theorem 3.1 is transformed into the following corollary.

Corollary 4.4. *Let* $\alpha > 0$, $\beta \in \Omega$, $\gamma \in \mathbb{R}^+ \cap \Omega$ *and* $\gamma \leq \frac{1}{2}$, *where*

$$\Omega := \left\{ z : z = \frac{2i+1}{2l+1}, i, l \in \mathbb{Z} \right\}.$$

Suppose that $\mu(x) = |x|^{p(1-2\beta)-1}$, $\nu_n = |n|^{q(1-2\gamma)-1}$, $n \in \mathbb{Z}^0 := \mathbb{Z} \setminus \{0\}$. *Let* $f(x), a_n \geq 0$ *with* $f(x) \in L_{p,\mu}(\mathbb{R})$ *and* $\boldsymbol{a} = \{a_n\}_{n \in \mathbb{Z}^0} \in l_{q,\nu}$. *Then*

$$\int_{x \in \mathbb{R}} f(x) \sum_{n \in \mathbb{Z}^0} \operatorname{sech}\left(\alpha x^\beta n^\gamma\right) a_n \mathrm{d}x < \frac{4c_0}{\alpha^2} |\beta|^{-\frac{1}{q}} \gamma^{-\frac{1}{p}} \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}. \tag{70}$$

Setting $\beta = \gamma = \frac{1}{2m+1}$ $(m \in \mathbb{N}^+)$, $\alpha = 1$ in (70), we have

$$\int_{x \in \mathbb{R}} f(x) \sum_{n \in \mathbb{Z}^0} \operatorname{sech}\left(\sqrt[2m+1]{xn}\right) a_n \mathrm{d}x < 4c_0(2m+1) \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}, \tag{71}$$

where $\mu(x) = |x|^{\frac{2m-1}{2m+1}p-1}$, $\nu_n = |n|^{\frac{2m-1}{2m+1}q-1}$ $(m \in \mathbb{N}^+)$.

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