Generalization of fuzzy Ostrowski like inequalities for $(m_1, m_2, \alpha, \beta, \gamma, \mu)$-convex functions

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Abstract. In the present paper, we present the very 1st time the generalized notion of $(m_1, m_2, \alpha, \beta, \gamma, \mu)$-convex function in mixed kind, which is the generalization of 22 functions, which are presented in sequel manner.Our aim is to establish generalized Ostrowski like inequalities for $(m_1, m_2, \alpha, \beta, \gamma, \mu)$-convex functions via Fuzzy Riemann Integrals by applying several techniques in which power mean inequality and Hölder's inequality are included. Moreover, we would obtain various results with respect to the convexity of function as special cases and also recapture several established results of different authors of different papers.

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1. Introduction and definitions

About the features of convex functions, we code some lines from [18] "Convex functions appear in many problems in pure and applied mathematics. They play an extremely important role in the study of both linear and non-linear programming problems. The theory of convex functions is part of the general subject of convexity, since a convex function is one whose epigraph is a convex

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set. Nonetheless it is an important theory which touches almost all branches of mathematics. Graphical analysis is one of the first topics in mathematics which requires the concept of convexity. Calculus gives us a powerful tool in recognizing convexity, the second-derivative test".

The theory of convex functions is a crucial area of mathematics that has applications in a wide range of fields, including optimization theory, control theory, operations research, geometry, functional analysis, and information theory. This theory is also highly relevant in other areas of science, such as economics, finance, engineering, and management sciences.

The importance of convex functions for the generalization of integral inequalities due to the variety of their nature the notion have been established. Integral inequalities are satisfied by many convex functions. Among these, the well known is Ostrowski inequality. To generalize the Ostrowski's inequality, we need to generalize the concept of convex functions, in this way we can easily see the generalizations and its particular cases. From the literature, we remind few definitions for various convex(concave) functions [2].

Definition 1.1. Any function $g : K \subseteq \mathbb{R} \to \mathbb{R}$ is known as convex(concave), if

$$g(ty + (1-t)z) \leq (\geq) tg(y) + (1-t)g(z), \tag{1.1}$$

$\forall\ y, z \in K, t \in [0, 1]$.

Here we remind definition of P-convex(concave) function see [5].

Definition 1.2. Any function $g : K \subseteq \mathbb{R} \to \mathbb{R}$ is known as P-convex(concave), if function g is a non-negative, then we have

$$g(ty + (1-t)z) \leq (\geq) g(y) + g(z), \tag{1.2}$$

$\forall\ y, z \in K, t \in [0, 1]$.

The definition of quasi-convex function is extracted from [9].

Definition 1.3. Any function $g : K \subseteq \mathbb{R} \to \mathbb{R}$ is called a quasi-convex(concave), if

$$g(ty + (1-t)z) \leq (\geq) \max\{g(y), g(z)\} \tag{1.3}$$

$\forall\ y, z \in K, t \in [0, 1]$.

We present definition of s-convex(concave) functions in the 1st kind as follows (see [16]).

Definition 1.4. Suppose $s \in (0, 1]$. Any function $g : K \subseteq [0, \infty) \to [0, \infty)$ is known as s-convex(concave) in the 1st kind, if

$$g(ty + (1-t)z) \leq (\geq) t^s g(y) + (1-t^s) g(z), \tag{1.4}$$

$\forall\ y, z \in K, t \in [0, 1]$.

Remark 1.5. Note that in this definition we also included $s = 0$. Further if we put $s = 0$, we get quasi-convexity (see Definition 1.3).

We also present definition of s-convex(concave) functions in the second kind from [16].

Definition 1.6. Suppose $s \in (0, 1]$. Any function $g : K \subseteq [0, \infty) \to [0, \infty)$ is known as s-convex(concave) in the 2nd kind, if

$$g(ty + (1-t)z) \leq (\geq) t^s g(y) + (1-t)^s g(z), \tag{1.5}$$

$\forall\ y, z \in K, t \in [0, 1]$.

Remark 1.7. In the similar manner, we have slightly improved definition of 2nd kind convexity by including $s = 0$. Further if we put $s = 0$, we easily get P-convexity (see Definition 1.2).

The following definition of m-convex(concave) function is extracted from [10]

Definition 1.8. Suppose $m \in [0, 1]$. Any function $g : [0, \infty) \to \mathbb{R}$ is known as m-convex (concave), if

$$g(ty + m(1-t)z) \leq (\geq) tg(y) + m(1-t)g(z) \tag{1.6}$$

$\forall\ y, z \in [0, \infty)$, $t \in [0, 1]$.

Remark 1.9. For $m = 1$ the above definition recaptures the concept of standard convex(concave) functions in the interval K and for $m = 0$ the concept star-shaped functions.

Following definition is extracted from [10]

Definition 1.10. Let $(m_1, m_2) \in (0, 1]^2$. Any function $g : [0, \infty) \to \mathbb{R}$ is known as (m_1, m_2)-convex(concave), if

$$g(m_1 ty + m_2(1-t)z) \leq (\geq) m_1 tg(y) + m_2(1-t)g(z), \tag{1.7}$$

$\forall\ y, z \in K, t \in [0, 1]$.

In [15], Mihesan stated (α, m)-convexity as in the following:

Definition 1.11. Suppose $(\alpha, m) \in [0, 1]^2$. A function $g : [0, \infty) \to \mathbb{R}$ is known as (α, m)-convex(concave), if

$$g(ty + m(1-t)z) \leq (\geq) t^\alpha g(y) + m(1-t^\alpha)g(z) \tag{1.8}$$

$\forall\ y, z \in [0, \infty)$, $t \in [0, 1]$. Above function can also be written as (m, s)-convex(concave) function in the 1st kind.

Firstly, we introduce a new class of (m, s)-convex(concave) function in the 2nd kind that is given below:

Definition 1.12. Let $(m, s) \in (0, 1]^2$. Any function $g : K \subseteq [0, \infty) \to [0, \infty)$ is known as (m, s)-convex(concave) in the 2nd kind, if

$$g(ty + m(1-t)z) \leq (\geq) t^s g(y) + m(1-t)^s g(z) \tag{1.9}$$

$\forall\, y, z \in K,\ t \in [0, 1]$.

A new class of (s, r)-convex(concave) functions in the mixed kind is extracted from [7].

Definition 1.13. Suppose $(s, r) \in [0, 1]^2$. Any function $g : K \subseteq [0, \infty) \to [0, \infty)$ is known as (s, r)-convex(concave) in the mixed kind, if

$$g(ty + (1-t)z) \leq (\geq) t^{rs} g(y) + (1 - t^r)^s g(z), \tag{1.10}$$

$\forall\, y, z \in K, t \in [0, 1]$.

Definition 1.14 ([6])**.** Suppose $(\alpha, \beta) \in [0, 1]^2$. Any function $g : K \subseteq [0, \infty) \to [0, \infty)$ is known as (α, β)-convex(concave) in the 1st kind, if

$$g(ty + (1-t)z) \leq (\geq) t^{\alpha} g(y) + (1 - t^{\beta}) g(z), \tag{1.11}$$

$\forall\, y, z \in K, t \in [0, 1]$.

Definition 1.15 ([6])**.** Suppose $(\alpha, \beta) \in [0, 1]^2$. Any function $g : K \subseteq [0, \infty) \to [0, \infty)$ is known as (α, β)-convex(concave) in the 2nd kind, if

$$g(ty + (1-t)z) \leq (\geq) t^{\alpha} g(y) + (1 - t)^{\beta} g(z), \tag{1.12}$$

$\forall\, y, z \in K, t \in [0, 1]$.

Secondly, we introduce a new class of (m, s, r)-convex(concave) functions in mixed kind which is given below:

Definition 1.16. Let $(m, s, r) \in [0, 1]^3$. A function $g : K \subseteq [0, \infty) \to [0, \infty)$ is known as (m, s, r)-convex(concave) in the mixed kind, if

$$g(ty + m(1-t)z) \leq (\geq) t^{rs} g(y) + m(1 - t^r)^s g(z), \tag{1.13}$$

$\forall\, y, z \in K, t \in [0, 1]$.

Thirdly, we introduce a new class of (m, α, β)-convex(concave) functions in the 1st kind which is given below:

Definition 1.17. Let $(m, \alpha, \beta) \in [0, 1]^3$. A function $g : K \subseteq [0, \infty) \to [0, \infty)$ is known as (m, α, β)-convex(concave) in the 1st kind, if

$$g(ty + m(1-t)z) \leq (\geq) t^{\alpha} g(y) + m(1 - t^{\beta}) g(z), \tag{1.14}$$

$\forall\, y, z \in K, t \in [0, 1]$.

Fourthly, we introduce a new class of (m, α, β)-convex(concave) functions in the 2nd kind which is given below:

Definition 1.18. Let $(m, \alpha, \beta) \in [0,1]^3$. A function $g : K \subseteq [0, \infty) \to [0, \infty)$ is known as (m, α, β)-convex(concave) in the 2nd kind, if

$$g(ty + m(1-t)z) \leq (\geq) t^{\alpha} g(y) + m(1-t)^{\beta} g(z), \tag{1.15}$$

$\forall\ y, z \in K, t \in [0,1]$.

Following definition is extracted from [10]

Definition 1.19. Let $(\alpha, m_1, m_2) \in (0,1]^3$. Any function $g : [0, \infty) \to \mathbb{R}$ is known as (α, m_1, m_2)-convex(concave), if

$$g(m_1 ty + m_2(1-t)z) \leq (\geq) m_1 t^{\alpha} g(y) + m_2(1-t^{\alpha}) g(z), \tag{1.16}$$

$\forall\ y, z \in K, t \in [0,1]$. Above function can also be written as (m_1, m_2, s)-convex(concave) function in the 1st kind.

Fifthly, we introduce a new class of (m_1, m_2, s)-convex(concave) functions in the 2nd kind which is given below:

Definition 1.20. Let $(m_1, m_2, s) \in (0,1]^3$. Any function $g : [0, \infty) \to \mathbb{R}$ is known as (m_1, m_2, s)-convex(concave) in the 2nd kind, if

$$g(m_1 ty + m_2(1-t)z) \leq (\geq) m_1 t^{s} g(y) + m_2(1-t)^{s} g(z), \tag{1.17}$$

$\forall\ y, z \in K, t \in [0,1]$.

Sixthly, we introduce a new class of (m_1, m_2, s, r)-convex(concave) functions in mixed kind which is given below:

Definition 1.21. Let $(m_1, m_2, s, r) \in (0,1]^4$. A function $g : K \subseteq [0, \infty) \to [0, \infty)$ is known as (m_1, m_2, s, r)-convex(concave) in the mixed kind, if

$$g(m_1 ty + m_2(1-t)z) \leq (\geq) m_1 t^{rs} g(y) + m_2(1-t^{r})^{s} g(z), \tag{1.18}$$

$\forall\ y, z \in K, t \in [0,1]$.

Seventhly, we introduce a new class of $(m_1, m_2, \alpha, \beta)$-convex(concave) functions in the 1st kind which is given below:

Definition 1.22. Let $(m_1, m_2, \alpha, \beta) \in (0,1]^4$. A function $g : K \subseteq [0, \infty) \to [0, \infty)$ is known as $(m_1, m_2, \alpha, \beta)$-convex(concave) in the 1st kind, if

$$g(m_1 ty + m_2(1-t)z) \leq (\geq) m_1 t^{\alpha} g(y) + m_2(1-t^{\beta}) g(z), \tag{1.19}$$

$\forall\ y, z \in K, t \in [0,1]$.

Eighthly, we introduce a new class of $(m_1, m_2, \alpha, \beta)$-convex(concave) functions in the 2nd kind which is given below:

Definition 1.23. Suppose $(m_1, m_2, \alpha, \beta) \in (0,1]^4$. A function $g : K \subseteq [0,\infty) \to [0,\infty)$ is known as $(m_1, m_2, \alpha, \beta)$-convex(concave) in the 2nd kind, if

$$g(m_1 ty + m_2(1-t)z) \leq (\geq) m_1 t^{\alpha} g(y) + m_2(1-t)^{\beta} g(z), \tag{1.20}$$

$\forall\ y, z \in K, t \in [0,1]$.

Upcoming definition is $(\alpha, \beta, \gamma, \mu)$-convex(concave) function which is extracted from [7].

Definition 1.24. Let $(\alpha, \beta, \gamma, \mu) \in [0,1]^4$. A function $g : K \subseteq [0,\infty) \to [0,\infty)$ is known as $(\alpha, \beta, \gamma, \mu)$-convex(concave) in the mixed kind, if

$$g(ty + (1-t)z) \leq (\geq) t^{\alpha\gamma} g(y) + (1-t^{\beta})^{\mu} g(z), \tag{1.21}$$

$\forall\ y, z \in K, t \in [0,1]$.

Ninthly, we introduce a new class of $(m, \alpha, \beta, \gamma, \mu)$-convex(concave) functions in mixed kind that is given below:

Definition 1.25. Let $(m, \alpha, \beta, \gamma, \mu) \in [0,1]^5$. A function $g : K \subseteq [0,\infty) \to [0,\infty)$ is known as $(m, \alpha, \beta, \gamma, \mu)$-convex(concave) in the mixed kind, if

$$g(ty + m(1-t)z) \leq (\geq) t^{\alpha\gamma} g(y) + m(1-t^{\beta})^{\mu} g(z), \tag{1.22}$$

$\forall\ y, z \in K, t \in [0,1]$.

Tenthly and Finally we introduce a new class of function which would be called class of $(m_1, m_2, \alpha, \beta, \gamma, \mu)$-convex(concave) function in mixed kind and containing all above classes of functions. This definition is used sequentially in this paper.

Definition 1.26. Let $(m_1, m_2, \alpha, \beta, \gamma, \mu) \in (0,1]^6$. A function $g : K \subseteq [0,\infty) \to [0,\infty)$ is known as $(m_1, m_2, \alpha, \beta, \gamma, \mu)$-convex(concave) in the mixed kind, if

$$g(m_1 ty + m_2(1-t)z) \leq (\geq) m_1 t^{\alpha\gamma} g(y) + m_2(1-t^{\beta})^{\mu} g(z), \tag{1.23}$$

$\forall\ y, z \in K, t \in [0,1]$.

Remark 1.27. In Definition 1.26, we have the following cases.

(i) If we choose $m_1 = 1$, $m_2 = m$ in (1.23), we get $(m, \alpha, \beta, \gamma, \mu)$-convex(concave) function in the mixed kind.

(ii) If we choose $m_1 = m_2 = 1$ in (1.23), we get $(\alpha, \beta, \gamma, \mu)$-convex(concave) function in the mixed kind.

(iii) If we choose $\beta = \gamma = 1$ and $\mu = \beta$ in (1.23), we get $(m_1, m_2, \alpha, \beta)$-convex(concave) function in the 2nd kind.

(iv) If we choose $\gamma = \mu = 1$ in (1.23), we get $(m_1, m_2, \alpha, \beta)$-convex(concave) function in the 1st kind.

(v) If we choose $\gamma = r$, $\alpha = \mu = s$ and $\beta = 1$ in (1.23), we get (m_1, m_2, s, r)-convex(concave) function in mixed kind.

(vi) If we choose $\alpha = \mu = s$ and $\beta = \gamma = 1$ in (1.23), we get (m_1, m_2, s)-convex(concave) function in the 2nd kind.

(vii) If we choose $\gamma = s$ and $\alpha = \beta = \mu = 1$ in (1.23), we get (m_1, m_2, s)-convex(concave) function in the 1st kind.

(viii) If we choose $m_1 = 1$, $m_2 = m$, $\beta = \gamma = 1$ and $\mu = \beta$ in (1.23), we get (m, α, β)-convex(concave) function in the 2nd kind.

(ix) If we choose $m_1 = 1$, $m_2 = m$ and $\gamma = \mu = 1$ in (1.23), we get (m, α, β)-convex(concave) function in the 1st kind.

(x) If we choose $m_1 = 1$, $m_2 = m$, $\gamma = r$, $\alpha = \mu = s$ and $\beta = 1$ in (1.23), we get (m, s, r)-convex(concave) function in the mixed kind.

(xi) If we choose $m_1 = m_2 = 1$, $\beta = \gamma = 1$ and $\mu = \beta$ in (1.23), we get (α, β)-convex(concave) function in the 2nd kind.

(xii) If we choose $m_1 = m_2 = 1$ and $\gamma = \mu = 1$ in (1.23), we get (α, β)-convex(concave) function in the 1st kind.

(xiii) If we choose $m_1 = m_2 = 1$, $\gamma = r$, $\alpha = \mu = s$ and $\beta = 1$ in (1.23), we get (s, r)-convex(concave) function in the mixed kind.

(xiv) If we choose $m_1 = 1$, $m_2 = m$, $\alpha = \mu = s$ and $\beta = \gamma = 1$ in (1.23), we get (m, s)-convex(concave) function in the 2nd kind.

(xv) If we choose $m_1 = 1$, $m_2 = m$, $\gamma = s$ and $\alpha = \beta = \mu = 1$ in (1.23), we get (m, s)-convex(concave) function in the 1st kind.

(xvi) If we choose $\alpha = \beta = \gamma = \mu = 1$ in (1.23), we get (m_1, m_2)-convex(concave) function.

(xvii) If we choose $m_1 = 1$, $m_2 = m$ and $\alpha = \beta = \gamma = \mu = 1$ in (1.23), we get m-convex(concave) function.

(xviii) If we choose $m_1 = m_2 = 1$, $\alpha = \mu = s$ and $\beta = \gamma = 1$ in (1.23), we get s-convex(concave) function in the 2nd kind.

(xix) If we choose $m_1 = m_2 = 1$, $\alpha = \beta = s$ and $\gamma = \mu = 1$ in (1.23), we get s-convex(concave) function in the 1st kind.

(xx) If we choose $m_1 = m_2 = 1$, $\gamma = s$ and $\alpha = \beta = \mu = 1$ in (1.23), we get s-convex(concave) function in the 1st kind.

(xxi) If we choose $m_1 = m_2 = 1$, $\alpha = \beta = 0$, and $\gamma = \mu = 1$ in (1.23), we get quasi-convex(concave) function.

(xxii) If we choose $m_1 = m_2 = 1$, $\alpha = \mu = 0$ and $\beta = \gamma = 1$ in (1.23), we get P-convex(concave) function.

(xxiii) If we choose $m_1 = m_2 = \alpha = \beta = \gamma = \mu = 1$ in (1.23), gives us ordinary convex(concave) function.

In almost each field of science, inequalities act an important role. Although it is very vast discipline but our focus is mainly on Ostrowski's like inequalities.

In 1938, [17] Ostrowski proved the below interesting inequality for differentiable mappings with bounded derivatives. It is well known in the literature as Ostrowski inequality.

Proposition 1.28. *Suppose $g : K \to \mathbb{R}$ is a differentiable mapping in the interior K^o of K, where $j, k \in K^o$ with $j < k$. If $|g'(y)| \leq \mathfrak{M}\ \forall\ y \in [j,k]$ where $\mathfrak{M} > 0$ is constant. Then*

$$\left| g(y) - \frac{1}{k-j} \int_j^k g(\tau) d\tau \right| \leq \mathfrak{M}(k-j) \left[\frac{1}{4} + \frac{\left(y - \frac{j+k}{2} \right)^2}{(k-j)^2} \right]. \tag{1.24}$$

The value $\frac{1}{4}$ is the best possible constant that this can not be replaced by the smallest one.

Since fuzziness is a natural reality different than randomness and determinism, Anastassiou extends Ostrowski like inequalities into the fuzzy setting in 2003 [1]. Congxin and Ming [3] introduced the concepts of fuzzy Riemann integrals. Fuzzy Riemann integral is a closed interval whose end points are the classical Riemann integrals.

2. Preliminaries with notations

Under this heading, we remind few basic definitions and notations that would help us in the sequel manner.

Definition 2.1 ([3]). *$\rho : \mathbb{R} \to [0,1]$ is called a fuzzy number if satisfies the below properties*

1. *ρ is normal (i.e, there exists an $y_0 \in \mathbb{R}$ such that $\rho(y_0) = 1$).*
2. *ρ is a convex fuzzy set, i.e., $yt + (1-t)z) \geq \min\{\rho(y), \rho(z)\}$, $\forall\ y, z \in \mathbb{R}$, $t \in [0,1]$ (ρ is called a convex fuzzy subset).*

3. *ρ is upper semi continuous on $\mathbb{R}$, i.e., $\forall\, y_0 \in \mathbb{R}$ and $\forall\, \epsilon > 0$, $\exists$ neighborhood $V(y_0) : \rho(y) \leq \rho(y_0) + \epsilon$, $\forall\, y \in V(y_0)$.*

4. *The set $[\rho]^0 = \overline{\{y \in \mathbb{R} : \rho(y) > 0\}}$ is compact where $\overline{A}$ denotes the closure of A.*

$\mathbb{R}^F$ denotes the set of all fuzzy numbers. For $\alpha \in (0,1]$ and $\rho \in \mathbb{R}^F$, $[\rho]^\alpha = \{y \in \mathbb{R} : \rho(y) \geq \alpha\}$. Then, from (1) to (4) it follows that the α-level set $[\rho]^\alpha$ is a closed interval $\forall\, \alpha \in [0,1]$. Moreover, $[\rho]^\alpha = [\rho_-^{(\alpha)}, \rho_+^{(\alpha)}]$ $\forall\, \alpha \in [0,1]$, where $\rho_-^{(\alpha)} \leq \rho_+^{(\alpha)}$ and $\rho_-^{(\alpha)}, \rho_+^{(\alpha)} \in \mathbb{R}$, i.e., $\rho_-^{(\alpha)}$ and $\rho_+^{(\alpha)}$ are the endpoints of $[\rho]^\alpha$.

Definition 2.2 ([4])**.** *Let $\rho, \varrho \in \mathbb{R}^F$ and $a \in \mathbb{R}$. Then, the addition and scalar multiplication are defined by the equations, respectively.*

1. *$[\rho \oplus \varrho]^\alpha = [\rho]^\alpha + [\varrho]^\alpha$;*

2. *$[a \odot \rho]^\alpha = a[\rho]^\alpha$;*

$\forall\, \alpha \in [0,1]$ where $[\rho]^\alpha + [\varrho]^\alpha$ means the usual addition of two intervals (as subsets of $\mathbb{R}$) and $a[\rho]^\alpha$ means the usual product between a scalar and a subset of $\mathbb{R}$.

Proposition 2.1 ([11])**.** *Let $\rho, \varrho \in \mathbb{R}^F$ and $a \in \mathbb{R}$. Then, the given properties holds:*

1. *$1 \odot \rho = \rho$.*

2. *$\rho \oplus \varrho = \varrho \oplus \rho$.*

3. *$a \odot \rho = \rho \odot a$.*

4. *$[\rho]^{\alpha_1} \subseteq [\rho]^{\alpha_2}$ whenever $0 \leq \alpha_2 \leq \alpha_1 \leq 1$.*

5. *For any α_n converging increasingly to $\alpha \in (0,1]$, $\bigcap_{n=1}^{\infty} [\rho]^{\alpha_n} = [\rho]^\alpha$.*

Definition 2.3 ([3])**.** *Let $D : \mathbb{R}^F \times \mathbb{R}^F \to \mathbb{R}_+ \cup \{0\}$ be a function, defined as*

$$D(\rho, \varrho) = \sup_{\alpha \in [0,1]} \max\left\{ \left| \rho_-^{(\alpha)}, \varrho_-^{(\alpha)} \right|, \left| \rho_+^{(\alpha)}, \varrho_+^{(\alpha)} \right| \right\}$$

$\forall\, \rho, \varrho \in \mathbb{R}^F$, Then, D is metric on $\mathbb{R}^F$.

Proposition 2.2 ([3])**.** *Let $\rho, \varrho, \sigma, e \in \mathbb{R}^F$ and $a \in \mathbb{R}$, we have*

1. *$(\mathbb{R}^F, D)$ is a complete metric space.*

2. *$D(\rho \oplus \sigma, \varrho \oplus \sigma) = D(\rho, \varrho)$.*

3. *$D(a \odot \rho, a \odot \varrho) = |a| D(\rho, \varrho)$.*

4. *$D(\rho \oplus \varrho, \sigma \oplus e) = D(\rho, \sigma) + D(\varrho, e)$.*

5. $D(\rho \oplus \varrho, \widetilde{0}) \leq D(\rho, \widetilde{0}) + D(\varrho, \widetilde{0})$.

6. $D(\rho \oplus \varrho, \sigma) \leq D(\rho, \sigma) + D(\varrho, \widetilde{0})$,

where $\widetilde{0} \in \mathbb{R}^F$ *is stated as* $\widetilde{0}(y) = 0 \ \forall \ y \in \mathbb{R}$.

Definition 2.4 ([4])**.** *Let* $y, z \in \mathbb{R}^F$ *if* $\exists \ \theta \in \mathbb{R}^F$ *such that* $y = z \oplus \theta$, *then* θ *is* H*-difference of* y *and* z *denoted by* $\theta = y \ominus z$.

Definition 2.5 ([4])**.** *Let* $T := [y_0, y_0 + \gamma] \subseteq \mathbb{R}$, *with* $\gamma > 0$. *A function* $g : T \to \mathbb{R}^F$ *is* H*-differentiable at* $y \in T$ *if* $\exists \ g'(y) \in \mathbb{R}^F$ *i.e., both limits (with respect to the metric* D*)*

$$\lim_{h \to 0^+} \frac{g(y+h) \ominus g(y)}{h}, \quad \lim_{h \to 0^+} \frac{g(y) \ominus g(y-h)}{h}$$

exists and are equal to $g'(y)$. *We call* g' *the derivative or* H*-derivative of* g *at* y. *If* g *is* H*-differentiable at any* $y \in T$, *we call* g *differentiable or* H*-differentiable and it has* H*-derivative over* T *the function* g'.

Definition 2.6 ([8])**.** *Let* $g : [j, k] \to \mathbb{R}^F$ *if* $\forall \ \zeta > 0$, $\exists \ \eta > 0$, *for any partition* $P = \{[\rho, \varrho]; \delta\}$ *of* $[j, k]$ *with norm* $\Delta(P) < \eta$, *we have*

$$D\left(\sum_{P}^{*} (\varrho - \rho) \odot g(\delta, K) \right) < \zeta,$$

then we say that g *is Fuzzy-Riemann integrable to the interval* $K \in \mathbb{R}^F$, *we write it as*

$$K := (FR) \int_j^k g(y) dy.$$

For some recent results connected with Fuzzy-Riemann integrals (see [12, 13]).

The main purpose of this paper is to establish generalized fuzzy Ostrowski like inequalities for $(m_1, m_2, \alpha, \beta, \gamma, \mu)$-convex function in mixed kind and we obtain various results with respect to the convexity of function as special cases and also recapture several previous established results of different authors of different papers [19] and [14].

3. Generalized fuzzy Ostrowski like inequalities for $(m_1, m_2, \alpha, \beta, \gamma, \mu)$-convex functions

Regarding to prove our main results, we require the below Lemma.

Lemma 3.1. *Let $g : K \subset \mathbb{R} \to \mathbb{R}^F$ be differentiable mapping on K^o where $m_1, m_2j, m_2k \in K$ with $m_2j < m_2k$. If $g' \in C^F[m_2j, m_2k] \cap L^F[m_2j, m_2k]$, then*

$$\begin{aligned}
&\frac{1}{k-j} \odot (FR) \int_{m_2j}^{m_2k} g(u)du \\
&\oplus \frac{(m_1y - m_2j)^2}{k-j} \odot (FR) \int_0^1 t \odot g'(m_1ty + m_2(1-t)j)dt \\
&= m_2 \odot g(m_1y) \oplus \frac{(m_2k - m_1y)^2}{k-j} \odot (FR) \int_0^1 t \odot g'(m_1ty + m_2(1-t)k)dt,
\end{aligned} \tag{3.1}$$

$\forall\ y \in (m_2j, m_2k)$.

Proof. We obtain the required result by using similar techniques of proof of Lemma 3.1 of [19]. □

Remark 3.1. If we choose $m_1 = 1$, $m_2 = m$, $\alpha = \beta = \mu = 1$ and $\gamma = \alpha$, we recapture Lemma 3.1 of [19].

Remark 3.2. If we choose $m_1 = m_2 = 1$ and $\mu = \delta$ in Theorem 3.1, we recapture Lemma 3.1 of [14].

Theorem 3.1. *Let all the suppositions of Lemma 3.1 be true and assuming that $D(g'(y), \widetilde{0})$ is $(m_1, m_2, \alpha, \beta, \gamma, \mu)$-convex function on $[m_2j, m_2k]$ and $D(g'(y), \widetilde{0}) \leq M$. Then*

$$\begin{aligned}
&D\left(m_2 \odot g(m_1y), \frac{1}{k-j} \odot (FR) \int_{m_2j}^{m_2k} g(u)du\right) \\
&\leq M \left(\frac{m_1}{\alpha\gamma + 2} + \frac{m_2}{\beta} B\left(\frac{2}{\beta}, \mu + 1\right)\right) I(y),
\end{aligned} \tag{3.2}$$

$\forall\ y \in (m_2j, m_2k)$ and $\beta > 0$, where $I(y) = \frac{(m_1y - m_2j)^2 + (m_2k - m_1y)^2}{k-j}$.

Proof. From the Lemma 3.1 and using Proposition 2.2, then we have

$$\begin{aligned}
&D\left(m_2 \odot g(m_1y), \frac{1}{k-j} \odot (FR) \int_{m_2j}^{m_2k} g(u)du\right) \\
&\leq D\left(\frac{(m_1y - m_2j)^2}{k-j} \odot (FR) \int_0^1 tg'(m_1ty + m_2(1-t)j)dt, \widetilde{0}\right) \\
&+ D\left(\frac{(m_2k - m_1y)^2}{k-j} \odot (FR) \int_0^1 tg'(m_1ty + m_2(1-t)k)dt, \widetilde{0}\right) \\
&= \frac{(m_1y - m_2j)^2}{k-j} D\left((FR) \int_0^1 tg'(m_1ty + m_2(1-t)j)dt, \widetilde{0}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(m_2k-m_1y)^2}{k-j}D\left((FR)\int_0^1 tg'(m_1ty+m_2(1-t)k)dt,\widetilde{0}\right)\\
&\leq \frac{(m_1y-m_2j)^2}{k-j}\int_0^1 tD\left(g'(m_1ty+m_2(1-t)j),\widetilde{0}\right)dt\\
&+\frac{(m_2k-m_1y)^2}{k-j}\int_0^1 tD\left(g'(m_1ty+m_2(1-t)k),\widetilde{0}\right)dt. \qquad (3.3)
\end{aligned}
$$

Since $D(g'(y),\widetilde{0})$ be $(m_1,m_2,\alpha,\beta,\gamma,\mu)$-convex function & $D(g'(y),\widetilde{0})\leq M$, we have

$$
\begin{aligned}
D\left(g'(m_1ty+m_2(1-t)j),\widetilde{0}\right) &\leq m_1t^{\alpha\gamma}D\left(g'(y),\widetilde{0}\right)+m_2\left(1-t^{\beta}\right)^{\mu}D\left(g'(j),\widetilde{0}\right)\\
&\leq M\left[m_1t^{\alpha\gamma}+m_2\left(1-t^{\beta}\right)^{\mu}\right], \qquad (3.4)
\end{aligned}
$$

$$
\begin{aligned}
D\left(g'(m_1ty+m_2(1-t)k),\widetilde{0}\right) &\leq m_1t^{\alpha\gamma}D\left(g'(y),\widetilde{0}\right)+m_2\left(1-t^{\beta}\right)^{\mu}D\left(g'(k),\widetilde{0}\right)\\
&\leq M\left[m_1t^{\alpha\gamma}+m_2\left(1-t^{\beta}\right)^{\mu}\right]. \qquad (3.5)
\end{aligned}
$$

Now, using (3.4) and (3.5) in (3.3) we get (3.2). □

Note. Where B is Beta function and it is stated as $B(l,m)=\int_0^1 t^{l-1}(1-t)^{m-1}dt=\frac{\Gamma(l)\Gamma(m)}{\Gamma(l+m)}$. Since $\Gamma(l)=\int_0^\infty e^{-u}u^{l-1}du$.

Remark 3.3. Some remarks about Theorem 3.1 are following as special cases:

(i) If we choose $m_1=1$, $m_2=m$ in Theorem 3.1, we can get inequality for $(m,\alpha,\beta,\gamma,\mu)$-convex function in the mixed kind.

(ii) If we choose $\beta=\gamma=1$ and $\mu=\beta$ in Theorem 3.1, we can get inequality for (m_1,m_2,α,β)-convex function in the 2nd kind.

(iii) If we choose $\gamma=\mu=1$ in Theorem 3.1, we can get inequality for (m_1,m_2,α,β)- convex function in the 1st kind.

(iv) If we choose $\gamma=r$, $\alpha=\mu=s$ and $\beta=1$ in Theorem 3.1, we can get inequality for (m_1,m_2,s,r)-convex function in mixed kind.

(v) If we choose $\alpha=\mu=s$ and $\beta=\gamma=1$ in Theorem 3.1, we can get inequality for (m_1,m_2,s)-convex function in the 2nd kind.

(vi) If we choose $\gamma=s$ and $\alpha=\beta=\mu=1$ in Theorem 3.1, we can get inequality for (m_1,m_2,s)-convex function in the 1st kind.

(vii) If we choose $m_1=1$, $m_2=m$, $\beta=\gamma=1$ and $\mu=\beta$ in Theorem 3.1, we can get inequality for (m,α,β)-convex function in the 2nd kind.

(viii) If we choose $m_1 = 1$, $m_2 = m$ and $\gamma = \mu = 1$ in Theorem 3.1, we can get inequality for (m, α, β)-convex function in the 1st kind.

(ix) If we choose $m_1 = 1$, $m_2 = m$, $\gamma = r$, $\alpha = \mu = s$ and $\beta = 1$ in Theorem 3.1, we can get inequality for (m, s, r)-convex function in the mixed kind.

(x) If we choose $m_1 = m_2 = 1$, $\beta = \gamma = 1$ and $\mu = \beta$ in Theorem 3.1, we can get inequality for (α, β)-convex function in the 2nd kind.

(xi) If we choose $m_1 = m_2 = 1$ and $\gamma = \mu = 1$ in Theorem 3.1, we can get inequality for (α, β)-convex function in the 1st kind.

(xii) If we choose $m_1 = m_2 = 1$, $\gamma = r$, $\alpha = \mu = s$ and $\beta = 1$ in Theorem 3.1, we can get inequality for (s, r)-convex function in the mixed kind.

(xiii) If we choose $m_1 = 1$, $m_2 = m$, $\alpha = \mu = s$ and $\beta = \gamma = 1$ in Theorem 3.1, we can get inequality for (m, s)-convex function in the 2nd kind.

(xiv) If we choose $\alpha = \beta = \gamma = \mu = 1$ in Theorem 3.1, we can get inequality for (m_1, m_2)-convex function.

(xv) If we choose $m_1 = 1$, $m_2 = m$ and $\alpha = \beta = \gamma = \mu = 1$ in Theorem 3.1, we can get inequality for m-convex function.

(xvi) If we choose $m_1 = m_2 = 1$, $\alpha = \mu = s$ and $\beta = \gamma = 1$ in Theorem 3.1, we can get inequality for s-convex function in the 2nd kind.

(xvii) If we choose $m_1 = m_2 = 1$, $\alpha = \beta = s$ and $\gamma = \mu = 1$ in Theorem 3.1, we can get inequality for s-convex function in the 1st kind.

(xviii) If we choose $m_1 = m_2 = 1$, $\gamma = s$ and $\alpha = \beta = \mu = 1$ in Theorem 3.1, we can get inequality for s-convex function in the 1st kind.

(xix) If we choose $m_1 = m_2 = 1$, $\alpha = \beta = 0$, and $\gamma = \mu = 1$ in Theorem 3.1,we can get inequality for quasi-convex function.

(xx) If we choose $m_1 = m_2 = 1$, $\alpha = \mu = 0$ and $\beta = \gamma = 1$ in Theorem 3.1, we can get inequality for P-convex function.

(xxi) If we choose $m_1 = m_2 = \alpha = \beta = \gamma = \mu = 1$ in Theorem 3.1, we can get inequality for ordinary convex function.

Remark 3.4. If we choose $m_1 = 1$, $m_2 = m$, $\alpha = \beta = \mu = 1$ and $\gamma = \alpha$ in Theorem 3.1, we recapture the main Theorem 3.2 of [19].

Remark 3.5. If we choose $m_1 = m_2 = 1$ and $\mu = \delta$ in Theorem 3.1, we recapture the main Theorem 3.1 of [14].

Remark 3.6. By choosing suitable values of $m_1, m_2, \alpha, \beta, \gamma, \mu$ in Theorem 3.1, we recapture all results of Corollary 3.1 of [14].

Theorem 3.2. *Let all the suppositions of Lemma* 3.1 *be true and assuming that* $[D(g'(y),\widetilde{0})]^q$ *is* $(m_1, m_2, \alpha, \beta, \gamma, \mu)$*-convex function on* $[m_2 j, m_2 k]$, $q \geq 1$ *&* $D(g'(y),\widetilde{0}) \leq M$. *Then,*

$$D\left(m_2 \odot g(m_1 y), \frac{1}{k-j} \odot (FR)\int_{m_2 j}^{m_2 k} g(u)du\right) \leq \frac{M}{(2)^{1-\frac{1}{q}}}\left(\frac{m_1}{\alpha\gamma+2} + \frac{m_2}{\beta}B\left(\frac{2}{\beta}, \mu+1\right)\right)^{\frac{1}{q}} I(y). \tag{3.6}$$

$\forall\ y \in (m_2 j, m_2 k)$ *and* $\beta > 0$.

Proof. From the inequality (3.3) & appling power mean inequality, we have

$$\begin{aligned}
&D\left(m_2 \odot g(m_1 y), \frac{1}{k-j} \odot (FR)\int_{m_2 j}^{m_2 k} g(u)du\right) \\
&\leq \frac{(m_1 y - m_2 j)^2}{k-j}\int_0^1 tD\left(g'(m_1 ty + m_2(1-t)j), \widetilde{0}\right)dt \\
&+ \frac{(m_2 k - m_1 y)^2}{k-j}\int_0^1 tD\left(g'(m_1 ty + m_2(1-t)k), \widetilde{0}\right)dt \\
&\leq \frac{(m_1 y - m_2 j)^2}{k-j}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left[D\left(g'(m_1 ty + m_2(1-t)j), \widetilde{0}\right)\right]^q dt\right)^{\frac{1}{q}} \\
&+ \frac{(m_2 k - m_1 y)^2}{k-j}\left(\int_0^1 tdt\right)^{1-\frac{1}{q}}\left(\int_0^1 t\left[D\left(g'(m_1 ty + m_2(1-t)k), \widetilde{0}\right)\right]^q dt\right)^{\frac{1}{q}}.
\end{aligned} \tag{3.7}$$

Since $[D(g'(y),\widetilde{0})]^q$ is $(m_1, m_2, \alpha, \beta, \gamma, \mu)$-convex function & $D(g'(y),\widetilde{0}) \leq M$, we have

$$\begin{aligned}
&\left[D\left(g'(m_1 ty + m_2(1-t)j), \widetilde{0}\right)\right]^q \\
&\leq m_1 t^{\alpha\gamma}\left[D\left(g'(y), \widetilde{0}\right)\right]^q + m_2\left(1 - t^\beta\right)^\mu\left[D\left(g'(j), \widetilde{0}\right)\right]^q \\
&\leq M^q\left[m_1 t^{\alpha\gamma} + m_2\left(1 - t^\beta\right)^\mu\right],
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
&\left[D\left(g'(m_1 ty + m_2(1-t)k), \widetilde{0}\right)\right]^q \\
&\leq m_1 t^{\alpha\gamma}\left[D\left(g'(y), \widetilde{0}\right)\right]^q + m_2\left(1 - t^\beta\right)^\mu\left[D\left(g'(k), \widetilde{0}\right)\right]^q \\
&\leq M^q\left[m_1 t^{\alpha\gamma} + m_2\left(1 - t^\beta\right)^\mu\right].
\end{aligned} \tag{3.9}$$

Now, using (3.8) and (3.9) in (3.7) we get (3.6). □

Remark 3.7. All remarks hold for Theorem 3.2 as we have given remarks (i) to (xxi) for Theorem 3.1.

Remark 3.8. If we choose $q = 1$ in Theorem 3.2, we obtain the our main Theorem 3.1.

Remark 3.9. If we choose $m_1 = 1$, $m_2 = m$, $\alpha = \beta = \mu = 1$ and $\gamma = \alpha$ in Theorem 3.2, we recapture the Theorem 3.4 of [19].

Remark 3.10. If we choose $m_1 = m_2 = 1$ and $\mu = \delta$ in Theorem 3.2, we recapture the Theorem 3.2 of [14].

Remark 3.11. By choosing suitable values of $m_1, m_2, \alpha, \beta, \gamma, \mu$ in Theorem 3.2, we recapture all results of Corollary 3.2 and Remarks 3.1 of [14].

Theorem 3.3. *Let all the suppositions of Lemma* 3.1 *be true and assuming that* $[D(g'(y), \widetilde{0})]^q$ *is* $(m_1, m_2, \alpha, \beta, \gamma, \mu)$*-convex function on* $[m_2 j, m_2 k]$, $p, q > 1$ *&* $D(g'(y), \widetilde{0}) \leq M$. *Then,*

$$
\begin{aligned}
&D\left(m_2 \odot g(m_1 y), \frac{1}{k-j} \odot (FR) \int_{m_2 j}^{m_2 k} g(u)du\right) \\
&\leq \frac{M}{(p+1)^{\frac{1}{p}}} \left(\frac{m_1}{\alpha\gamma + 1} + \frac{m_2}{\beta} B\left(\frac{1}{\beta}, \mu + 1\right)\right)^{\frac{1}{q}} I(y), \qquad (3.10)
\end{aligned}
$$

$\forall\ y \in (m_2 j, m_2 k)$ *and* $\beta > 0$. *Where* $p^{-1} + q^{-1} = 1$.

Proof. From inequality (3.3) & by Hölder's inequality, we have

$$
\begin{aligned}
&D\left(m_2 \odot g(m_1 y), \frac{1}{k-j} \odot (FR) \int_{m_2 j}^{m_2 k} g(u)du\right) \\
&\leq \frac{(m_1 y - m_2 j)^2}{k-j} \int_0^1 tD\left(g'(m_1 ty + m_2(1-t)j), \widetilde{0}\right)dt \\
&+ \frac{(m_2 k - m_1 y)^2}{k-j} \int_0^1 tD\left(g'(m_1 ty + m_2(1-t)k), \widetilde{0}\right)dt \\
&\leq \frac{(m_1 y - m_2 j)^2}{k-j} \left(\int_0^1 t^p dt\right)^{\frac{1}{p}} \left(\int_0^1 \left[D\left(g'(m_1 ty + m_2(1-t)j), \widetilde{0}\right)\right]^q dt\right)^{\frac{1}{q}} \\
&+ \frac{(m_2 k - m_1 y)^2}{k-j} \left(\int_0^1 t^p dt\right)^{\frac{1}{p}} \left(\int_0^1 \left[D\left(g'(m_1 ty + m_2(1-t)k), \widetilde{0}\right)\right]^q dt\right)^{\frac{1}{q}}. \qquad (3.11)
\end{aligned}
$$

Since $[D(g'(y), \widetilde{0})]^q$ is $(m_1, m_2, \alpha, \beta, \gamma, \mu)$-convex function & $D(g'(y), \widetilde{0}) \leq M$, we have

$$
\begin{aligned}
&\left[D\left(g'(m_1 ty + m_2(1-t)j), \widetilde{0}\right)\right]^q \\
&\leq m_1 t^{\alpha\gamma} \left[D\left(g'(y), \widetilde{0}\right)\right]^q + m_2\left(1 - t^\beta\right)^\mu \left[D\left(g'(j), \widetilde{0}\right)\right]^q \\
&\leq M^q \left[m_1 t^{\alpha\gamma} + m_2\left(1 - t^\beta\right)^\mu\right], \qquad (3.12)
\end{aligned}
$$

$$
\begin{aligned}
&\left[D\left(g'(m_1ty+m_2(1-t)k),\widetilde{0}\right)\right]^q \\
&\le m_1t^{\alpha\gamma}\left[D\left(g'(y),\widetilde{0}\right)\right]^q+m_2\left(1-t^\beta\right)^\mu\left[D\left(g'(k),\widetilde{0}\right)\right]^q \\
&\le M^q\left[m_1t^{\alpha\gamma}+m_2\left(1-t^\beta\right)^\mu\right]. \qquad (3.13)
\end{aligned}
$$

Now, using (3.12) and (3.13) in (3.11), we get (3.10). □

Remark 3.12. All remarks hold for Theorem 3.3 as we have given remarks (i) to (xxi) for Theorem 3.1.

Remark 3.13. If we choose $m_1 = 1$, $m_2 = m$, $\alpha = \beta = \mu = 1$ and $\gamma = \alpha$ in Theorem 3.3, we recapture the Theorem 3.3 of [19].

Remark 3.14. If we choose $m_1 = m_2 = 1$ and $\mu = \delta$ in Theorem 3.3, we recapture the Theorem 3.3 of [14].

Remark 3.15. By choosing suitable values of $m_1, m_2, \alpha, \beta, \gamma, \mu$ in Theorem 3.3, we recapture all results of Corollary 3.3 and Remarks 3.2 of [14].

4. Conclusion

As we all know Ostrowski inequality is one of the most celebrated inequalities. In this paper, we presented 1st time the generalized notion of $(m_1, m_2, \alpha, \beta, \gamma, \mu)$-convex function in mixed kind, which contains the generalization of many functions as convex, P-convex, quasi-convex, s-convex in the 1^{st} kind, s-convex in the 2^{nd} kind, m-convex, (m_1, m_2)-convex, (m, s)-convex in the 1st kind, (m, s)-convex in the 2nd kind, (s, r)-convex in mixed kind, (α, β)-convex in the 1^{st} kind, (α, β)-convex in the 2^{nd} kind, (m, s, r)-convex in mixed kind, (m, α, β)-convex in the 1st kind, (m, α, β)-convex in the 2nd kind, (m_1, m_2, s)-convex function in the 1st kind, (m_1, m_2, s)-convex function in the 2nd kind, (m_1, m_2, s, r)-convex in mixed kind, $(m_1, m_2, \alpha, \beta)$-convex in the 1st kind, $(m_1, m_2, \alpha, \beta)$-convex in the 2nd kind, $(\alpha, \beta, \gamma, \mu)$-convex in mixed kind, $(m, \alpha, \beta, \gamma, \mu)$-convex in mixed kind. We proved the generalized Ostrowski like inequalities for $(m_1, m_2, \alpha, \beta, \gamma, \mu)$-convex functions via Fuzzy Riemann Integrals by using Hölder's and power mean inequality. Further that we obtained several results with respect to the convexity of function as special cases and recaptured various established results of different authors of different papers [19] and [14].

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