On completeness of fuzzy metric spaces

Changqing Li*
School of Mathematics and Statistics
Minnan Normal University
Zhangzhou, Fujian 363000
China
helen_smile0320@163.com

Yanlan Zhang
College of Computer
Minnan Normal University
Zhangzhou, Fujian 363000
China
zyl_1983_2004@163.com

Abstract. Recently, p-convergence in fuzzy metric spaces, in George and Veeramani's sense, has been explored by Gregori et al. [6]. In this paper, we study consistency of Cauchyness (completeness, respectively) and p-Cauchyness (p-completeness, respectively) in fuzzy metric spaces.

Keywords: fuzzy metric, Cauchy sequence, completeness.

MSC 2020: 54A40, 54D35, 54E50

1. Introduction

Many authors have defined several concepts of fuzzy metric space in different ways [3, 4, 11, 12]. In particular, to make the topology induced by a fuzzy metric to be Hausdorff, George and Veeramani [4] gave the concept of fuzzy metric space with the help of continuous t-norms. Later, Gregori and Romaguera [10] proved that the topological space induced by a fuzzy metric is metrizable. In [13], Mihet introduced the concept of p-convergence in fuzzy metric spaces. Whereafter, some authors studied some aspects relative to p-convergence, p-Cauchy sequence and p-completeness in fuzzy metric spaces in [1, 6, 7, 8]. Specifically, Gregori et al. [6] posed an open problem of characterizing consistency of Cauchyness (completeness, respectively) and p-Cauchyness (p-completeness, respectively) in fuzzy metric spaces. Here, we will study those fuzzy metric spaces, that we call k-unequal, in which the family of p-Cauchy sequences and Cauchy sequences agree, moreover, completeness and p-completeness coincide.

*. Corresponding author

2. Preliminaries

From now on, $\mathbb{N}$ shall denote the set of positive integer numbers . Our basic reference for general topology is [2].

Definition 2.1 ([4])**.** *A binary operation* $* : [0,1]\times[0,1] \to [0,1]$ *is a continuous t-norm if it satisfies the following conditions:*

(*i*) $*$ *is associative and commutative;*

(*ii*) $*$ *is continuous;*

(*iii*) $a * 1 = a$*, for all* $a \in [0,1]$*;*

(*iv*) $a * b \le c * d$ *whenever* $a \le c$ *and* $b \le d$*, and* $a, b, c, d \in [0,1]$.

Observe that $a * b = \min\{a, b\}$ and $a * b = a \cdot b$ are two common examples of continuous t-norms.

Definition 2.2 ([4])**.** *An ordered triple* $(X, M, *)$ *is said to be a fuzzy metric space if* X *is a nonempty set,* $*$ *is a continuous t-norm and* M *is a fuzzy set on* $X \times X \times (0, +\infty)$ *satisfying the following conditions, for all* $x, y, z \in X$ *and* $s, t \in (0, +\infty)$*:*

(*i*) $M(x, y, t) > 0$*;*

(*ii*) $M(x, y, t) = 1$ *if and only if* $x = y$*;*

(*iii*) $M(x, y, t) = M(y, x, t)$*;*

(*iv*) $M(x, y, t) * M(y, z, s) \le M(x, z, t + s)$*;*

(*v*) *the function* $M(x, y, \cdot) : (0, +\infty) \to (0, 1]$ *is continuous.*

If $(X, M, *)$ is a fuzzy metric space, then we will call $(M, *)$, or simply M, a *fuzzy metric on* X*.*

Definition 2.3 ([4])**.** *Let* $(X, M, *)$ *be a fuzzy metric space and let* $r \in (0, 1), t > 0$ *and* $x \in X$*. The set*

$$B_M(x, r, t) = \{y \in X | M(x, y, t) > 1 - r\}$$

is called the open ball with center x *and radius* r *with respect to* t*.*

George and Veeramani [4] proved that $\{B_M(x, r, t) | x \in X, t > 0, r \in (0, 1)\}$ forms a base of a topology τ_M in X.

Proposition 2.1 ([4])**.** *Let* $(X, M, *)$ *be a fuzzy metric space. A sequence* $\{x_n\}$ *in* X *converges to* $x_0 \in X$ *if and only if* $\lim_n M(x_n, x_0, t) = 1$*, for all* $t > 0$.

Definition 2.4 ([4]). *Let (X, d) be a metric space. Define $a * b = a \cdot b$, for all $a, b \in [0, 1]$, and let M_d be the real value mapping on $X \times X \times (0, +\infty)$ defined by*

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}.$$

Then, $(X, M_d, \cdot)$ is a fuzzy metric space and $(M_d, \cdot)$ is called the standard fuzzy metric induced by d.

Definition 2.5 ([5]). *Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\}$ in X is called Cauchy if for each $r \in (0, 1)$ and each $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - r$, for all $n, m \geq n_0$. X is called complete if every Cauchy sequence in X is convergent with respect to τ_M. In such a case M is called complete.*

Definition 2.6 ([9]). *A fuzzy metric M on X is said to be stationary, if M does not depend on t, i.e. if, for all $x, y \in X$ and t, $M(x, y, t)$ is constant. In this case we write $M(x, y)$ instead of $M(x, y, t)$.*

Definition 2.7 ([6]). *We say that the fuzzy metric space $(X, M, *)$ is principal (or simply, M is principal) if $\{B_M(x, r, t) | r \in (0, 1)\}$ is local base at $x \in X$, for each $x \in X$ and each $t > 0$.*

Definition 2.8 ([6, 13]). *Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\}$ in X is said to be point convergent to $x_0 \in X$ if $\lim_n M(x_n, x_0, t_0) = 1$ for some $t_0 > 0$. In such a case we say that $\{x_n\}$ is p-convergent to x_0 for $t_0 > 0$, or, simply, $\{x_n\}$ is p-convergent.*

Remark 2.1 ([6, 13]). Clearly, $\{x_n\}$ is convergent to $x_0 \in X$ if and only if $\{x_n\}$ is p-convergent to x_0, for all $t > 0$.

Definition 2.9 ([6]). *Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\}$ in X is said to be p-Cauchy if for each $r \in (0, 1)$, there are $n_0 \in \mathbb{N}$ and $t_0 > 0$ such that $M(x_n, x_m, t_0) > 1 - r$, for all $n, m \geq n_0$, i.e. $\lim_{m,n} M(x_n, x_0, t_0) = 1$ for some $t_0 > 0$. In such a case we say that $\{x_n\}$ is p-Cauchy for $t_0 > 0$, or, simply, $\{x_n\}$ is p-Cauchy.*

Remark 2.2 ([6]). It is not hard to see that $\{x_n\}$ is a Cauchy sequence if and only if $\{x_n\}$ is p-Cauchy, for all $t > 0$ and, obviously, p-convergent sequences are p-Cauchy.

Definition 2.10 ([6]). *A fuzzy metric space $(X, M, *)$ is called p-complete if every p-Cauchy sequence in X is p-convergent to some point of X. In such a case M is called p-complete.*

3. Main results

We start this section with the following definition.

Definition 3.1. *A fuzzy metric space* $(X, M, *)$ *is said to be* k*-unequal if* $k(1 - M(x, y, kt)) \geq 1 - M(x, y, t)$ *whenever* $x, y \in X, t > 0$ *and* $k > 1$*. In such a case* M *is called* k*-unequal.*

Now, we recall several examples, which were given in [6].

Example 3.1. (a) A stationary fuzzy metric M_1 is principal.

(b) The well-known standard fuzzy metric M_2 is principal.

(c) $M_3(x, y, t) = e^{-\frac{d(x,y)}{t}}$, where d is a metric on X, is principal.

(d) $M_4(x, y, t) = \frac{\min\{x,y\}+t}{\max\{x,y\}+t}$ is a fuzzy metric on $(0, +\infty)$, which is principal.

Example 3.2. Consider the above examples. It is an easy exercise to verify that M_1, M_2 and M_4 are all k-unequal. Now, we only prove that M_3 is also k-unequal. If $x = y$, then it is clear that $k(1 - M_3(x, y, kt)) = 1 - M_3(x, y, t)$. Let $x \neq y$ and $k > 1$. Suppose that $k(1 - M_3(x, y, kt)) < 1 - M_3(x, y, t)$, namely $k(1 - e^{-\frac{d(x,y)}{kt}}) < 1 - e^{-\frac{d(x,y)}{t}}$. Then, $k \cdot \frac{e^{\frac{d(x,y)}{kt}} - 1}{e^{\frac{d(x,y)}{kt}}} < \frac{e^{\frac{d(x,y)}{t}} - 1}{e^{\frac{d(x,y)}{t}}}$, which means that $ke^{\frac{d(x,y)}{t}}(e^{\frac{d(x,y)}{kt}} - 1) < e^{\frac{d(x,y)}{kt}}(e^{\frac{d(x,y)}{t}} - 1)$. Notice that $e^{\frac{d(x,y)}{kt}} > e^0 = 1$. We deduce that $ke^{\frac{d(x,y)}{t}} - ke^{\frac{(k-1)d(x,y)}{kt}} < e^{\frac{d(x,y)}{t}} - 1$, that is $(k-1)z^k - kz^{k-1} + 1 < 0$, where $z = e^{\frac{d(x,y)}{kt}}$. Set $f(z) = (k-1)z^k - kz^{k-1} + 1$, $z \in (1, +\infty)$. Then, $f(z) < 0$, for all $z > 1$. Since $f'(z) = (k-1)kz^{k-1} - k(k-1)z^{k-2} = (k-1)kz^{k-2}(z-1) > 0$, for all $z > 1$, we conclude that f is a strictly increasing function on $(1, +\infty)$. Note that, f is a continuous function on $[1, +\infty)$. We get that $f(z) > f(1) = 0$, for all $z > 1$, which is a contradiction. So, M_3 is k-unequal.

Theorem 3.1. *Let* $(X, M, *)$ *be a fuzzy metric space. If* M *is* k*-unequal, then* M *is principal.*

Proof. Let $B_M(x, \varepsilon, s)$ be an open ball with center x and radius ε with respect to s, where $x \in X$, $\varepsilon \in (0, 1)$ and $s > 0$. Put $t > 0$. In case $0 < t < s$. Take $r = \varepsilon$. Then, $x \in B_M(x, r, t) \subseteq B_M(x, \varepsilon, s)$. In case $t \geq s$. Then, $\frac{t}{s} \geq 1$. Hence, there exists $r = \frac{\varepsilon s}{2t}$ such that $x \in B_M(x, r, t) \subseteq B_M(x, \varepsilon, s)$. In fact, let $y \in B_M(x, r, t)$. Since M is k-unequal, we have that

$$\frac{s}{t}(M(x, y, s) - 1) + 1 \geq M(x, y, \frac{t}{s} \cdot s) = M(x, y, t) > 1 - r.$$

Thus

$$M(x, y, s) > 1 - \frac{rt}{s} = 1 - \frac{\varepsilon}{2} > 1 - \varepsilon,$$

which follows that $y \in B_M(x, \varepsilon, s)$. Consequently, M is principal. □

The converse of the preceding theorem is not true, in general. We illustrate this fact with the next example.

Example 3.3. Let $X = (0,1)$. Denote $a * b = a \cdot b$, for all $a, b \in [0,1]$. Define the function M on $X \times X \times (0,+\infty)$ by

$$M(x,y,t) = \begin{cases} 1, & x = y, \\ xyt, & x \neq y, t \leq 1, \\ xy, & x \neq y, t > 1. \end{cases}$$

Then, $(X, M, *)$ is a principal fuzzy metric space (see [6]). Choose $x_0 = 0.95, y_0 = 0.96, t_0 = 0.875$ and $k_0 = 2$. Then

$$k_0(1-M(x_0,y_0,k_0t_0)) = 2(1-M(0.95,0.96,2\cdot 0.875)) = 2(1-0.95\cdot 0.96) = 0.176,$$

and

$$1 - M(x_0,y_0,t_0) = 1 - M(0.95,0.96,0.875) = 1 - 0.95 \cdot 0.96 \cdot 0.875 = 0.202.$$

So, $k_0(1 - M(x_0,y_0,k_0t_0)) < 1 - M(x_0,y_0,t_0)$, which means that M is not k-unequal.

Due to Example 3.2 and Theorem 3.1, the following chain of implications is fulfilled obviously.

$$\text{stationary} \Rightarrow k\text{-unequal} \Rightarrow \text{principal}$$

At the end of paper [6], Gregori et al. posed an open problem of characterizing those fuzzy metric spaces where the family of p-Cauchy sequences and Cauchy sequences agree, or further, when it is satisfied that completeness is equivalent to p-completeness.

Next, we will solve the above open problem by the following results.

Theorem 3.2. *Let $\{x_n\}$ be a sequence in a k-unequal fuzzy metric space $(X, M, *)$. Then, $\{x_n\}$ is Cauchy if and only if $\{x_n\}$ is p-Cauchy.*

Proof. Suppose that $\{x_n\}$ is Cauchy. Then, by Remark 2.2 we deduce that $\{x_n\}$ is p-Cauchy.

Conversely, suppose that $\{x_n\}$ is p-Cauchy for $t_0 > 0$. Let $\varepsilon \in (0,1)$ and $t > 0$. Pick $\varepsilon_1 = \min\{\frac{t\varepsilon}{t_0}, \varepsilon\}$. Then, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t_0) > 1 - \varepsilon_1$, for all $n, m \geq n_0$. If $t > t_0$, then

$$M(x_n,x_m,t) \geq M(x_n,x_m,t_0) > 1 - \varepsilon_1 \geq 1 - \varepsilon.$$

If $0 < t \leq t_0$, then $\frac{t_0}{t} \geq 1$. Since M is k-unequal, we obtain that

$$\frac{t}{t_0}(M(x_n,x_m,t) - 1) + 1 \geq M(x_n,x_m,\frac{t_0}{t} \cdot t) = M(x_n,x_m,t_0) > 1 - \varepsilon_1.$$

It follows that

$$M(x_n,x_m,t) > 1 - \frac{t_0}{t} \cdot \varepsilon_1 \geq 1 - \frac{t_0}{t} \cdot \frac{t\varepsilon}{t_0} = 1 - \varepsilon.$$

So, $\{x_n\}$ is Cauchy. The proof is finished. □

Proposition 3.1 ([6]). *Let $(X, M, *)$ be a principal fuzzy metric space. If X is p-complete, then X is complete.*

It was shown in [6] that the converse of the above proposition is false, in general. Nevertheless, the next proposition can be obtained.

Proposition 3.2. *Let $(X, M, *)$ be a k-unequal fuzzy metric space. If X is complete, then X is p-complete.*

Proof. Let $\{x_n\}$ be a p-Cauchy sequence. According to Theorem 3.2, we have that $\{x_n\}$ is Cauchy. Hence, $\{x_n\}$ converges to some point $x_0 \in X$. Due to Remark 2.1, we obtain that $\{x_n\}$ is p-convergent to x_0. We are done. □

With Theorem 3.1, Proposition 3.1 and Proposition 3.2, we get the next corollary.

Corollary 3.1. *Let $(X, M, *)$ be a k-unequal fuzzy metric space. Then, X is complete if and only if X is p-complete.*

Since stationary fuzzy metric does not depend on t, obviously "p-Cauchy sequences and Cauchy sequences" and also "p-completeness and completeness" are equivalent concepts in stationary fuzzy metrics.

Acknowledgments

This work was supported by Nation Natural Science Foundation of China (No. 11701258, No. 11871259), Natural Science Foundation of Fujian (No. 2022J01912, No. 2020J01801, No. 2019J01749, No. 2016J01671), the Institute of Meteorological Big Date-Digital Fujian and Fujian Key Laboratory of Data Science and Statistics.

References

[1] G. Baydar, M. Telci, *On completeness and compactness in fuzzy metric spaces,* The Journal of Analysis, 31 (2023), 747-758.

[2] R. Engelking, *General topology*, PWN-Polish Scientific Publishers, Warsaw, 1977.

[3] M. A. Erceg, *Metric spaces in fuzzy set theory,* Journal of Mathematical Analysis and Applications, 69 (1979), 205-230.

[4] A. George, P. Veeramani, *On some resules in fuzzy metric spaces*, Fuzzy Sets and Systems, 64 (1994), 395-399.

[5] A. George, P. Veeramani, *On some resules of analysis for fuzzy metric spaces,* Fuzzy Sets and Systems, 90 (1997) 365-368.

[6] V. Gregori, A. López-Crevillén, S. Morillas, A. Sapena, *On convergence in fuzzy metric spaces,* Topology and its Applications, 156 (2009), 3002-3006.

[7] V. Gregori, J. J. Miñana, S. Morillas, A. Sapena, *On principal fuzzy metric spaces,* Mathematics, 10 (2022), 2860.

[8] V. Gregori, J. J. Miñana, B. Roig, A. Sapena, *A characterization of p-complete fuzzy metric spaces*, Fuzzy Sets and Systems, 444 (2022), 144-155.

[9] V. Gregori, S. Romaguera, *Characterizing completable fuzzy metric spaces*, Fuzzy Sets and Systems, 144 (2004), 411-420.

[10] V. Gregori, S. Romaguera, *Some properties of fuzzy metric spaces*, Fuzzy Sets and Systems, 115 (2000), 485-489.

[11] O. Kaleva, S. Seikkala, *On fuzzy metric spaces*, Fuzzy Sets and Systems, 12 (1984), 215-229.

[12] I. Kramosil, J. Michalek, *Fuzzy metric and statistical metric spaces*, Kybernetika, 11 (1975), 326-334.

[13] D. Mihet, *On fuzzy contractive mapping in fuzzy metric spaces*, Fuzzy Sets and Systems, 158 (2007), 915-921.

Accepted: August 25, 2023