Refinements of unitary invariant norm inequalities for matrices

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Abstract. In this paper, we first establish an arithmetic-geometric mean inequality of unitary invariant norm for matrices, which is an improvement of the result proposed by Zou and He [Linear Algebra Appl., 436(2012), 3354-3361]. Then, we use it to refine the existing inequality. Moreover, we derive two unitarily invariant norm inequalities for matrices, which refine the result of Cao and Wu.

Keywords: positive semidefinite matrix, convex function, unitarily invariant norm, arithmetic-geometric mean inequality.
MSC 2020: 15A60, 15A45

1. Introduction

Let M_n be the space of $n \times n$ complex matrices. A norm $||\cdot||$ is called unitarily invariant norm, if $||UAV|| = ||A||$ for all $A \in M_n$ and for all unitary matrices $U, V \in M_n$. The singular values $s_j(A)(j = 1, 2, \cdots, n)$ of A are the eigenvalues of $|A| = (A^*A)^{\frac{1}{2}}$, arranged in a decreasing order. The Ky Fan k-norm $||\cdot||_{(k)}$ is defined as $\|A\|_{(k)} = \sum_{j=1}^{k} s_j(A), k = 1, \cdots, n$ and the Schatten p-norm $||\cdot||_p$ is defined as $\|A\|_p = \left(\sum_{j=1}^{n} s_j^p(A)\right)^{\frac{1}{p}} = (\mathrm{tr}|A|^p)^{\frac{1}{p}}, 1 \le p < \infty$.

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In what follows, $||\cdot||$ always denotes unitarily invariant norms including Schatten p-norm $||\cdot||_p$ and Ky Fan k-norm $||\cdot||_{(k)}$.

For $A,B,X\in M_n$ and A,B are positive semidefinite, Bhatia and Davis [1] presented

$$||A^{\frac{1}{2}}XB^{\frac{1}{2}}||\le\left|\left|\frac{A^vXB^{1-v}+A^{1-v}XB^v}{2}\right|\right|\le\left|\left|\frac{AX+XB}{2}\right|\right|, \tag{1.1}$$

where $0\le v\le 1$. Letting

$$\varphi(v)=||A^vXB^{1-v}+A^{1-v}XB^v||,$$

inequality (1.1) can be rewritten as

$$\varphi\left(\frac{1}{2}\right)\le\varphi(v)\le\varphi(0).$$

For $A,B,X\in M_n$ and A,B are positive semidefinite, the function $\varphi(v)$ is a continuous convex function on [0,1], attains its minimum at $v=\frac{1}{2}$ and maximum at $v=0$ and $v=1$. Consequently, it is decreasing on $[0,\frac{1}{2}]$ and increasing on $[\frac{1}{2},1]$, moreover, $\varphi(v)=\varphi(1-v)$ for $v\in[0,1]$ (see[2]). Using the convexity of the function $\varphi(v)$, Zou and He [3] obtained a strengthening of the arithmetic-geometric mean inequality $\varphi\left(\frac{1}{2}\right)\le\varphi(0)$ as follows:

$$\varphi\left(\frac{1}{2}\right)+2\left(\int_0^1\varphi(v)dv-\varphi\left(\frac{1}{2}\right)\right)\le\varphi(0), 0\le v\le 1. \tag{1.2}$$

Bhatia and Kittaneh [4] derived if $A,B\in M_n$ are positive semidefinite, then

$$||AB||\le\frac{1}{4}||(A+B)^2||. \tag{1.3}$$

Zou and He [3] gave a stronger version of inequality (1.3) as follows:

$$||AB||+\left(\int_0^1 g(v)dv-2||AB||\right)\le\frac{1}{4}||(A+B)^2||, \tag{1.4}$$

where $g(v)=||A^{\frac{1}{2}+v}B^{\frac{3}{2}-v}+A^{\frac{3}{2}-v}B^{\frac{1}{2}+v}||$.

Kaur and Singh [5] proved that for $A,B,X\in M_n$, if A and B are positive definite, then for any unitarily invariant norm

$$\frac{1}{2}||A^{\nu}XB^{1-\nu}+A^{1-\nu}XB^{\nu}||\le\left|\left|(1-\alpha)A^{\frac{1}{2}}XB^{\frac{1}{2}}+\alpha\left(\frac{AX+XB}{2}\right)\right|\right|, \tag{1.5}$$

where $\frac{1}{4}\le\nu\le\frac{3}{4}$ and $\alpha\in[\frac{1}{2},\infty)$.

Replacing A,B by A^2,B^2 in (1.5) and taking $u=2\nu$, we can obtain

$$\frac{1}{2}||A^uXB^{2-u}+A^{2-u}XB^u||\le\left|\left|(1-\alpha)AXB+\alpha\left(\frac{A^2X+XB^2}{2}\right)\right|\right|, \tag{1.6}$$

where $\frac{1}{2} \leq u \leq \frac{3}{2}$ and $\alpha \in [\frac{1}{2}, \infty)$.

Let $A, B, X \in M_n$ such that A and B are positive semidefinite. Then, for every unitarily invariant norm, the function

$$\psi(u) = ||A^u X B^{2-u} + A^{2-u} X B^u||$$

is convex on $[\frac{1}{2}, \frac{3}{2}]$ and attains its minimum at $u = 1$. So, it is decreasing on $[\frac{1}{2}, 1]$ and increasing on $[1, \frac{3}{2}]$(see[2]). Using the convexity of the function $\psi(u)$, Cao and Wu [6] obtained a refinement of inequality (1.6)

$$\begin{aligned} ||A^u X B^{2-u} + A^{2-u} X B^u|| &\leq 2(4r_0 - 1)||AXB| \\ &+ 2(1 - 2r_0)||A^{\frac{1}{2}} X B^{\frac{3}{2}} + A^{\frac{3}{2}} X B^{\frac{1}{2}}|| \\ &\leq 2(4r_0 - 1)||AXB|| \\ &+ 4(1 - 2r_0)\left|\left|(1-\alpha)AXB + \alpha\left(\frac{A^2X + XB^2}{2}\right)\right|\right|, \end{aligned} \tag{1.7}$$

where $\frac{1}{2} \leq u \leq \frac{3}{2}$, $\alpha \in [\frac{1}{2}, \infty)$ and $r_0 = \min\{\frac{u}{2}, 1 - \frac{u}{2}\}$.

For more information on this topic, the reader is referred to [7-9] and the references therein. In this paper, we first improve the inequality (1.2). As an application of our result, we refine the inequality (1.4). Finally, we establish improved versions of inequality (1.7) by using the convexity of function $\psi(u)$.

2. Main results

In this section, we show four lemmas which will turn out to be useful in the proof of our results.

Lemma 2.1 ([10])**.** *Let $A, B, X \in M_n$ such that A and B are positive semidefinite, then for every unitarily invariant norm*

$$\varphi(v) \leq 2r_0 \left(\varphi\left(\frac{1}{2}\right) - \varphi(0)\right) + \varphi(0),$$

where $0 \leq v \leq 1$ and $r_0 = \min\{v, 1 - v\}$.

Lemma 2.2 ([10])**.** *Let f be a real valued convex function on the interval $[a, b]$, then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a) + f(b)}{2}.$$

Lemma 2.3 ([4])**.** *Let $A, B \in M_n$ be positive semidefinite, then*

$$||A^{\frac{1}{2}}(A+B)B^{\frac{1}{2}}|| \leq \frac{1}{2}||(A+B)^2||.$$

Lemma 2.4 ([10])**.** *Let f be a real valued convex function on an interval $[a,b]$ which contains (x_1,x_2), then*

$$f(x) \le \frac{f(x_2)-f(x_1)}{x_2-x_1}x - \frac{x_1 f(x_2) - x_2 f(x_1)}{x_2-x_1}, x \in (x_1,x_2).$$

Theorem 2.1. *Let $A,B,X \in M_n$ such that A and B are positive semidefine, then for any unitarily invariant norm*

$$\varphi\left(\frac{1}{2}\right) + 2\left(\int_0^1 \varphi(v)dv - \varphi\left(\frac{1}{2}\right)\right) + 2\left(\int_0^1 \varphi(v)dv - \varphi\left(\frac{1}{4}\right)\right) \le \varphi(0), \tag{2.1}$$

where $\varphi(v) = ||A^v X B^{1-v} + A^{1-v} X B^v||$ and $0 \le v \le 1$.

Proof. For $0 \le v \le \frac{1}{4}$, by Lemma 2.1, we have

$$\varphi(v) \le 4\left(\varphi\left(\frac{1}{4}\right) - \varphi(0)\right)v + \varphi(0).$$

Thus,

$$\int_0^{\frac{1}{4}} \varphi(v)dv \le 4\left(\varphi\left(\frac{1}{4}\right) - \varphi(0)\right)\int_0^{\frac{1}{4}} vdv + \int_0^{\frac{1}{4}} \varphi(0)dv.$$

By a small calculation, we have

$$\int_0^{\frac{1}{4}} \varphi(v)dv \le \frac{1}{8}\left(\varphi\left(\frac{1}{4}\right) + \varphi(0)\right). \tag{2.2}$$

For $\frac{1}{4} \le v \le \frac{1}{2}$, by Lemma 2.1, we obtain

$$\varphi(v) \le 4\left(\varphi\left(\frac{1}{2}\right) - \varphi\left(\frac{1}{4}\right)\right)\left(v - \frac{1}{4}\right) + \varphi\left(\frac{1}{4}\right).$$

Consequently

$$\int_{\frac{1}{4}}^{\frac{1}{2}} \varphi(v)dv \le 4\left(\varphi\left(\frac{1}{2}\right) - \varphi\left(\frac{1}{4}\right)\right)\int_{\frac{1}{4}}^{\frac{1}{2}}\left(v - \frac{1}{4}\right)dv + \int_{\frac{1}{4}}^{\frac{1}{2}} \varphi\left(\frac{1}{4}\right)dv,$$

which implies

$$\int_{\frac{1}{4}}^{\frac{1}{2}} \varphi(v)dv \le \frac{1}{8}\left(\varphi\left(\frac{1}{4}\right) + \varphi\left(\frac{1}{2}\right)\right). \tag{2.3}$$

For $\frac{1}{2} \le v \le \frac{3}{4}$, by Lemma 2.1, we obtain

$$\varphi(v) \le 4\left(\varphi\left(\frac{3}{4}\right) - \varphi\left(\frac{1}{2}\right)\right)\left(v - \frac{1}{2}\right) + \varphi\left(\frac{1}{2}\right).$$

Thus,

$$\int_{\frac{1}{2}}^{\frac{3}{4}} \varphi(v)dv \leq 4\left(\varphi\left(\frac{3}{4}\right) - \varphi\left(\frac{1}{2}\right)\right)\int_{\frac{1}{2}}^{\frac{3}{4}}\left(v - \frac{1}{2}\right)dv + \int_{\frac{1}{2}}^{\frac{3}{4}} \varphi\left(\frac{1}{2}\right)dv,$$

by a small calculation, we have

$$\int_{\frac{1}{2}}^{\frac{3}{4}} \varphi(v)dv \leq \frac{1}{8}\left(\varphi\left(\frac{3}{4}\right) + \varphi\left(\frac{1}{2}\right)\right). \tag{2.4}$$

For $\frac{3}{4} \leq v \leq 1$, by Lemma 2.1, we obtain

$$\varphi(v) \leq 4\left(\varphi(1) - \varphi\left(\frac{3}{4}\right)\right)\left(v - \frac{3}{4}\right) + \varphi\left(\frac{3}{4}\right).$$

Thus,

$$\int_{\frac{3}{4}}^{1} \varphi(v)dv \leq 4\left(\varphi(1) - \varphi\left(\frac{3}{4}\right)\right)\int_{\frac{3}{4}}^{1}\left(v - \frac{3}{4}\right)dv + \int_{\frac{3}{4}}^{1} \varphi\left(\frac{3}{4}\right)dv,$$

which implies

$$\int_{\frac{3}{4}}^{1} \varphi(v)dv \leq \frac{1}{8}\left(\varphi(1) + \varphi\left(\frac{3}{4}\right)\right). \tag{2.5}$$

It follows from (2.2)-(2.5) and $\varphi(0) = \varphi(1), \varphi\left(\frac{1}{4}\right) = \varphi\left(\frac{3}{4}\right)$ that

$$\begin{aligned}\int_0^1 \varphi(v)dv &= \int_0^{\frac{1}{4}} \varphi(v)dv + \int_{\frac{1}{4}}^{\frac{1}{2}} \varphi(v)dv + \int_{\frac{1}{2}}^{\frac{3}{4}} \varphi(v)dv + \int_{\frac{3}{4}}^{1} \varphi(v)dv \\ &\leq \frac{1}{4}\left(\varphi(0) + \varphi\left(\frac{1}{2}\right) + 2\varphi\left(\frac{1}{4}\right)\right),\end{aligned}$$

and so

$$4\int_0^1 \varphi(v)dv \leq \varphi(0) + \varphi\left(\frac{1}{2}\right) + 2\varphi\left(\frac{1}{4}\right),$$

which is equivalent to

$$\varphi\left(\frac{1}{2}\right) + 2\left(\int_0^1 \varphi(v)dv - \varphi\left(\frac{1}{2}\right)\right) + 2\left(\int_0^1 \varphi(v)dv - \varphi\left(\frac{1}{4}\right)\right) \leq \varphi(0).$$

This completes the proof. □

Remark 2.1. Theorem 2.1 is sharper than inequality (1.2).

By Lemma 2.2, we have

$$\varphi\left(\frac{1}{4}\right) \leq 2\int_0^{\frac{1}{2}} \varphi(v)dv \tag{2.6}$$

and

$$\varphi\left(\frac{3}{4}\right) \leq 2\int_{\frac{1}{2}}^{1} \varphi(v)dv. \tag{2.7}$$

It follows from (2.6), (2.7) and $\varphi\left(\frac{1}{4}\right) = \varphi\left(\frac{3}{4}\right)$ that

$$2\varphi\left(\frac{1}{4}\right) \leq 2\int_0^1 \varphi(v)dv,$$

that is

$$\int_0^1 \varphi(v)dv \geq \varphi\left(\frac{1}{4}\right).$$

Thus,

$$\int_0^1 \varphi(v)dv - \varphi\left(\frac{1}{4}\right) \geq 0.$$

Obviously, Theorem 2.1 is also an improvement of arithmetic-geometric mean inequality $\varphi\left(\frac{1}{2}\right) \leq \varphi(0)$.

Theorem 2.2. *Let $A, B \in M_n$ be positive semidefinite, then for any unitarily invariant norm*

$$||AB|| + \left(\int_0^1 g(v)dv - 2||AB||\right) + \left(\int_0^1 g(v)dv - ||A^{\frac{3}{4}}B^{\frac{5}{4}} + A^{\frac{5}{4}}B^{\frac{3}{4}}||\right)$$
$$\leq \frac{1}{4}||(A+B)^2||,$$

where $g(v) = ||A^{\frac{1}{2}+v}B^{\frac{3}{2}-v} + A^{\frac{3}{2}-v}B^{\frac{1}{2}+v}||$.

Proof. By (2.1), taking $X = A^{\frac{1}{2}}B^{\frac{1}{2}}$, we have

$$2||AB||+2\left(\int_0^1 g(v)dv-2||AB||\right)+2\left(\int_0^1 g(v)dv-||A^{\frac{3}{4}}B^{\frac{5}{4}}+A^{\frac{5}{4}}B^{\frac{3}{4}}||\right)$$
$$\leq ||A^{\frac{1}{2}}(A+B)B^{\frac{1}{2}}||. \tag{2.8}$$

By Lemma 2.3, it easily follows from (2.8) that

$$2||AB|| + 2\left(\int_0^1 g(v)dv - 2||AB||\right) + 2\left(\int_0^1 g(v)dv - ||A^{\frac{3}{4}}B^{\frac{5}{4}} + A^{\frac{5}{4}}B^{\frac{3}{4}}||\right)$$
$$\leq \frac{1}{2}||(A+B)^2||.$$

This completes the proof. □

Remark 2.2. Obviously, Theorem 2.2 is a refinement of inequality (1.4).

In the following, we utilize the convexity of the function $\psi(u) = ||A^u XB^{2-u} + A^{2-u}XB^u||$ to present two matrix inequalities for unitarily invariant norms that lead to improved versions of inequality (1.7).

Theorem 2.3. *Let $A, X, B \in M_n$ such that A and B are positive semidefinite, then for any unitarily invariant norm*

$$||A^u XB^{2-u} + A^{2-u}XB^u|| \leq 2(4r_0 - 1)||A^{\frac{3}{4}}XB^{\frac{5}{4}} + A^{\frac{5}{4}}XB^{\frac{3}{4}}|| + 2(3 - 8r_0)\left|\left|(1-\alpha)AXB + \alpha\left(\frac{A^2X + XB^2}{2}\right)\right|\right|, \quad u \in [\frac{1}{2}, \frac{3}{4}] \cup [\frac{5}{4}, \frac{3}{2}]$$

and

$$||A^u XB^{2-u} + A^{2-u}XB^u|| \leq 8(1 - 2r_0)\left|\left|(1-\alpha)AXB + \alpha\left(\frac{A^2X + XB^2}{2}\right)\right|\right| + 2(8r_0 - 3)||AXB||, \quad u \in (\frac{3}{4}, \frac{5}{4}),$$

where $\frac{1}{2} \leq u \leq \frac{3}{2}$, $\alpha \in [\frac{1}{2}, \infty)$ and $r_0 = \min\{\frac{u}{2}, 1 - \frac{u}{2}\}$.

Proof. For $\frac{1}{2} \leq u \leq \frac{3}{4}$, by the convexity of the function $\psi(u) = ||A^u XB^{2-u} + A^{2-u}XB^u||$ and Lemma 2.4, we obtain

$$\psi(u) \leq \frac{\psi(\frac{3}{4}) - \psi(\frac{1}{2})}{\frac{1}{4}}u - \frac{\frac{1}{2}\psi(\frac{3}{4}) - \frac{3}{4}\psi(\frac{1}{2})}{\frac{1}{4}},$$

which is equivalent to

$$\psi(u) \leq (4u-2)\psi\left(\frac{3}{4}\right) + (3-4u)\psi\left(\frac{1}{2}\right). \tag{2.9}$$

Combining (1.6) with (2.9), we get

$$||A^u XB^{2-u} + A^{2-u}XB^u|| \leq (4u-2)||A^{\frac{3}{4}}XB^{\frac{5}{4}} + A^{\frac{5}{4}}XB^{\frac{3}{4}}|| + 2(3-4u)\left|\left|(1-\alpha)AXB + \alpha\left(\frac{A^2X + XB^2}{2}\right)\right|\right|.$$

Hence

$$||A^u XB^{2-u} + A^{2-u}XB^u|| \leq 2(4r_0 - 1)||A^{\frac{3}{4}}XB^{\frac{5}{4}} + A^{\frac{5}{4}}XB^{\frac{3}{4}}|| + 2(3 - 8r_0)\left|\left|(1-\alpha)AXB + \alpha\left(\frac{A^2X + XB^2}{2}\right)\right|\right|. \tag{2.10}$$

For $\frac{3}{4} < u \leq 1$, by the convexity of the function $\psi(u)$ and Lemma 2.4, we have

$$\psi(u) \leq \frac{\psi(1) - \psi(\frac{3}{4})}{1 - \frac{3}{4}} u - \frac{\frac{3}{4}\psi(1) - 1\psi(\frac{3}{4})}{1 - \frac{3}{4}},$$

which is equivalent to

$$\psi(u) \leq (4u-3)\psi(1) + (4-4u)\psi\left(\frac{3}{4}\right). \tag{2.11}$$

Combining (1.6) with (2.11), we get

$$\begin{aligned}&||A^u XB^{2-u} + A^{2-u}XB^u||\\ &\leq 8(1-u)\left|\left|(1-\alpha)AXB + \alpha\left(\frac{A^2X + XB^2}{2}\right)\right|\right| + 2(4u-3)||AXB||.\end{aligned}$$

Hence

$$\begin{aligned}&||A^u XB^{2-u} + A^{2-u}XB^u||\\ &\leq 8(1-2r_0)\left|\left|(1-\alpha)AXB + \alpha\left(\frac{A^2X + XB^2}{2}\right)\right|\right| + 2(8r_0-3)||AXB||.\end{aligned} \tag{2.12}$$

For $1 < u < \frac{5}{4}$, similarly, we have

$$\psi(u) \leq (4u-4)\psi\left(\frac{5}{4}\right) + (5-4u)\psi\left(1\right),$$

that is

$$\begin{aligned}&||A^u XB^{2-u} + A^{2-u}XB^u||\\ &\leq 2(4u-4)\left|\left|(1-\alpha)AXB + \alpha\left(\frac{A^2X + XB^2}{2}\right)\right|\right| + 2(5-4u)||AXB||.\end{aligned}$$

Consequently

$$\begin{aligned}&||A^u XB^{2-u} + A^{2-u}XB^u||\\ &\leq 8(1-2r_0)\left|\left|(1-\alpha)AXB + \alpha\left(\frac{A^2X + XB^2}{2}\right)\right|\right| + 2(8r_0-3)||AXB||.\end{aligned} \tag{2.13}$$

For $\frac{5}{4} \leq u \leq \frac{3}{2}$, we have

$$\psi(u) \leq (4u-5)\psi\left(\frac{3}{2}\right) + (6-4u)\psi\left(\frac{5}{4}\right),$$

that is

$$\begin{aligned}||A^u XB^{2-u} + A^{2-u}XB^u|| \leq 2(4u-5)&\left|\left|(1-\alpha)AXB + \alpha\left(\frac{A^2X + XB^2}{2}\right)\right|\right|\\ &+ (6-4u)||A^{\frac{5}{4}}XB^{\frac{3}{4}} + A^{\frac{3}{4}}XB^{\frac{5}{4}}||.\end{aligned}$$

Consequently

$$\begin{aligned}||A^uXB^{2-u}+A^{2-u}XB^u|| &\leq 2(4r_0-1)||A^{\frac{3}{4}}XB^{\frac{5}{4}}+A^{\frac{5}{4}}XB^{\frac{3}{4}}|| \\ &+2(3-8r_0)\left|\left|(1-\alpha)AXB+\alpha\left(\frac{A^2X+XB^2}{2}\right)\right|\right|. \end{aligned} \tag{2.14}$$

It follows from (2.10),(2.12),(2.13),(2.14) and $\frac{1}{2} \leq u \leq \frac{3}{2}$, $\alpha \in [\frac{1}{2}, \infty)$, $r_0 = \min\{\frac{u}{2}, 1-\frac{u}{2}\}$ that

$$\begin{aligned}||A^uXB^{2-u}+A^{2-u}XB^u|| &\leq 2(4r_0-1)||A^{\frac{3}{4}}XB^{\frac{5}{4}}+A^{\frac{5}{4}}XB^{\frac{3}{4}}|| \\ &+2(3-8r_0)\left|\left|(1-\alpha)AXB+\alpha\left(\frac{A^2X+XB^2}{2}\right)\right|\right|, \\ &u\in[\frac{1}{2},\frac{3}{4}]\cup[\frac{5}{4},\frac{3}{2}]\end{aligned}$$

and

$$\begin{aligned}||A^uXB^{2-u}+A^{2-u}XB^u|| &\leq 8(1-2r_0)\left|\left|(1-\alpha)AXB+\alpha\left(\frac{A^2X+XB^2}{2}\right)\right|\right| \\ &+2(8r_0-3)||AXB||,\ u\in(\frac{3}{4},\frac{5}{4}).\end{aligned}$$

This completes the proof. □

Remark 2.3. Theorem 2.3 is sharper than inequality (1.7).

Note that, inequality (1.7) is equivalent to

$$\psi(u) \leq 2(1-u)\psi\left(\frac{1}{2}\right)+(2u-1)\psi(1),\ \frac{1}{2}\leq u\leq 1 \tag{2.15}$$

and

$$\psi(u) \leq (3-2u)\psi(1)+2(u-1)\psi\left(\frac{3}{2}\right),\ 1\leq u\leq \frac{3}{2}. \tag{2.16}$$

For $\frac{1}{2} \leq u \leq \frac{3}{4}$, compared with inequality (2.15), then

$$\begin{aligned}&2(1-u)\psi\left(\frac{1}{2}\right)+(2u-1)\psi(1)-\left((4u-2)\psi\left(\frac{3}{4}\right)+(3-4u)\psi\left(\frac{1}{2}\right)\right) \\ &=(2u-1)\left(\psi\left(\frac{1}{2}\right)-2\psi\left(\frac{3}{4}\right)+\psi(1)\right).\end{aligned}$$

Since $\psi(u) = ||A^uXB^{2-u}+A^{2-u}XB^u||$ is convex on $[\frac{1}{2}, \frac{3}{2}]$, it follows by a slope argument that

$$\frac{\psi(1)-\psi\frac{3}{4})}{1-\frac{3}{4}} \geq \frac{\psi(\frac{3}{4})-\psi(\frac{1}{2})}{\frac{3}{4}-\frac{1}{2}},$$

that is
$$\psi\left(\frac{1}{2}\right)-2\psi\left(\frac{3}{4}\right)+\psi(1)\geq 0.$$

So,

$$2(1-u)\psi\left(\frac{1}{2}\right)+(2u-1)\psi(1)\geq(4u-2)\psi\left(\frac{3}{4}\right)+(3-4u)\psi\left(\frac{1}{2}\right).\tag{2.17}$$

For $\frac{3}{4}<u\leq 1$,

$$\begin{aligned}&2(1-u)\psi\left(\frac{1}{2}\right)+(2u-1)\psi(1)-\left((4u-3)\psi(1)+(4-4u)\psi\left(\frac{3}{4}\right)\right)\\&=(2-2u)\left(\psi\left(\frac{1}{2}\right)-2\psi\left(\frac{3}{4}\right)+\psi(1)\right)\geq 0.\end{aligned}$$

So,

$$2(1-u)\psi\left(\frac{1}{2}\right)+(2u-1)\psi(1)\geq(4u-3)\psi(1)+(4-4u)\psi\left(\frac{3}{4}\right).\tag{2.18}$$

For $1<u<\frac{5}{4}$, compared with inequality (2.16), then we have

$$\begin{aligned}&(3-2u)\psi(1)+2(u-1)\psi\left(\frac{3}{2}\right)-\left((4u-4)\psi\left(\frac{5}{4}\right)+(5-4u)\psi(1)\right)\\&=2(u-1)\left(\psi\left(\frac{3}{2}\right)-2\psi\left(\frac{5}{4}\right)+\psi(1)\right).\end{aligned}$$

Since $\psi(u)=||A^uXB^{2-u}+A^{2-u}XB^u||$ is convex on $[\frac{1}{2},\frac{3}{2}]$, it follows by a slope argument that
$$\frac{\psi(\frac{3}{2})-\psi(\frac{5}{4})}{\frac{3}{2}-\frac{5}{4}}\geq\frac{\psi(\frac{5}{4})-\psi(1)}{\frac{5}{4}-1},$$
that is
$$\psi\left(\frac{3}{2}\right)-2\psi\left(\frac{5}{4}\right)+\psi(1)\geq 0.$$

So,

$$(3-2u)\psi(1)+2(u-1)\psi\left(\frac{3}{2}\right)\geq(4u-4)\psi\left(\frac{5}{4}\right)+(5-4u)\psi(1).\tag{2.19}$$

For $\frac{5}{4}\leq u\leq\frac{3}{2}$, we have

$$\begin{aligned}&(3-2u)\psi(1)+2(u-1)\psi\left(\frac{3}{2}\right)-\left((4u-5)\psi\left(\frac{3}{2}\right)+(6-4u)\psi\left(\frac{5}{4}\right)\right)\\&=(3-2u)\left(\psi\left(\frac{3}{2}\right)-2\psi\left(\frac{5}{4}\right)+\psi(1)\right)\geq 0.\end{aligned}$$

So,

$$(3-2u)\psi(1)+2(u-1)\psi\left(\frac{3}{2}\right)\geq(4u-5)\psi\left(\frac{3}{2}\right)+(6-4u)\psi\left(\frac{5}{4}\right). \tag{2.20}$$

By (2.17)-(2.20), we can conclude that Theorem 2.3 is better than inequality (1.7).

Acknowledgements

This research is supported by the Fund for Fostering Talents in Kunming University of Science and Technology (No. KKZ3202007048).

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Accepted: November 15, 2023