A study on co-intersection graphs of rings

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Abstract. Let R be a ring and $\mathcal{I}^\star(R)$ be the set of all nontrivial left ideals of R. The Co-intersection graph of ideals of R, denoted by $\Omega(R)$, is an undirected simple graph with the vertex set $\mathcal{I}^\star(R)$, and two distinct vertices I and J are adjacent if and only if $I + J \neq R$.

This paper derives a sufficient and necessary condition for $\Omega(R)$ to be a connected graph. We characterize the values of n for which the graph $\Omega(\mathbb{Z}_n)$ is Eulerian and Hamiltonian. Furthermore, the bad (and nice) decision number of $\Omega(\mathbb{Z}_n)$ are studied in the paper.
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1. Introduction

The idea to associate a graph to a ring first appeared in [5]. He let all elements of the ring be vertices of the graph and was interested mainly in coloring. In [4], Anderson and Livingston introduced and studied the zero-divisor graph whose vertices are the nonzero zero-divisors. There are many papers on assigning a graph to a ring R, for instance, see [4, 3, 12, 11, 2, 1]. Also, the intersection graphs and co-intersection graphs of some algebraic structures such as groups, rings, and modules have been studied by several authors, see [2, 9, 7, 10]. The co-intersection graph of submodules is introduced in [9].

The paper is organized as follows. Some definitions and preliminaries are introduced in Section 2. We devote Section 3 to study for connectivity of the co-intersection graph. Also, we characterize all the values of n for which $\Omega(\mathbb{Z}_n)$

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is Eulerian and characterize some values of n for which $\Omega(\mathbb{Z}_n)$ is Hamiltonian in this section. Finally, the bad decision number, and the nice decision number of $\Omega(\mathbb{Z}_n)$ are studied in Section 4.

2. Preliminaries

This section gives some definitions of ring theory and graph theory. Also, we introduce the Co-intersection graph of a ring R and give some basic concepts about rings and maximal left ideals.

We mean from a nontrivial ideal of R is a nonzero proper left ideal of R. The set $\mathcal{I}^\star(R)$ is a set of all nontrivial left ideals of R. A nonzero ring R is called *simple* if it has no nontrivial two-sided ideal. The term *null ring* is used to refer a ring R, in which $x \times y = 0$, for all $x, y \in R$.

By $\mathrm{Max}(R)$ and $\mathrm{Min}(R)$, we denote the set of all nonzero maximal left ideals of R and all nonzero minimal left ideals of R respectively.

A graph G is an ordered pair $G = (V, E)$, consisting of a nonempty set V of vertices, and a set $E \subseteq [V]^2$ of edges, where $[V]^2$ is the set of all 2-element subsets of V. Two vertices $u, v \in V$ are *adjacent* if $uv \in E$ (for simplicity we use uv instead of subset $\{u, v\}$). The *neighbourhood* of a vertex $u \in V$ is $N(u) = \{v \in V | uv \in E\}$, and the *closed neighbourhood* of u is $N[u] = N(u) \cup \{u\}$. The degree of a vertex u in a graph G is the size of set $N(u)$, which is denoted by $deg(u)$. We denote by $\delta(G)$ the minimum degree of the vertices of G. The *complete* graph with n vertices is denoted by K_n, which is a graph with n vertices in which any two distinct vertices are adjacent. A *null graph* is a graph containing no edges. Let G be a graph, suppose that $x, y \in V(G)$, a *walk* between u and v is a sequence $u = v_0 - v_1 - \cdots - v_k = v$ of vertices of G such that for every i with $1 \leq i \leq k$, the vertices v_{i-1} and v_i are adjacent. A (u, v)*-path* between u and v is a walk between u and v without repeated vertices. Two vertices u and v of G are said to be *connected* if there is a (u, v)-path in G. A graph G is called *connected* if every pair of its vertices are connected. If vertices u and v are connected in G, the distance between u and v in G, denoted by $d(u, v)$, is the length of a shortest (u, v)-path in G. In graph G, a *tour* is a closed walk that traverses each edge of G at least once. A graph is *Eulerian* if it contains a tour which traverses each edge exactly once [6].

A *cycle* in a graph is a path of length at least 3 through distinct vertices which begins and ends at the same vertex. A *Hamilton cycle* is a spanning cycle, and a graph which contains such a cycle is said to be *Hamiltonian*.

If $G = (V, E)$ is a finite graph, define $f(U) = \sum_{u \in U} f(u)$, for a function $f : V \to \{-1, 1\}$ and $U \subseteq V$. A function $f : V \to \{-1, 1\}$ is called a *bad function* of G, if $f(N(v)) \leq 1$ for each $v \in V$ [13]. The maximum value of $f(V)$, taken over all bad functions f, is called the *bad decision number* of G, which is denoted by $\beta_D(G)$. The function f is called a *nice function*, if $f(N[v])) \leq 1$ for each $v \in V$. The maximum value of $f(V)$, taken over all nice functions f is called the *nice decision number* of G, and denoted by $\overline{\beta_D(G)}$.

Definition 2.1. *Let R be a ring. The Co-intersection graph $\Omega(R)$ of R, is an undirected simple graph whose the vertex set $V(\Omega(R)) = \mathcal{I}^\star(R)$ is a set of all nontrivial ideals of R and two distinct vertices I, J are adjacent if and only if $I + J \neq R$.*

Remark 2.1. Let $\mathbb{Z}_n$ be the ring of integers modulo n. Suppose that m_1, m_2 are two factor of n. Then, $\langle m_1\rangle + \langle m_2\rangle = \langle (m_1, m_2)\rangle$, where (m_1, m_2) is the greatest common divisor of m_1, m_2.

Example 2.1. Suppose that $R = \mathbb{Z}_{50}$. Then, $\mathcal{I}^\star(R) = \{\langle 2\rangle, \langle 5\rangle, \langle 10\rangle, \langle 25\rangle\}$ and the co-intersection graph $\Omega(R)$ is as follow:

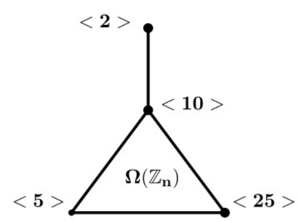

Figure 1: The Co-intersection Graph $\Omega(\mathbb{Z}_{50})$.

3. Connectivity, eulerianity and hamiltonicity

This section derives a sufficient and necessary condition for $\Omega(R)$ to be a connected graph. Also, we determine the values of n for which $\Omega(\mathbb{Z}_n)$ is a connected graph. Further, we characterize the values of n for which the graph $\Omega(\mathbb{Z}_n)$ is Eulerian and Hamiltonian. Before presenting and proving results, we give the following lemma.

Lemma 3.1. *Let R be a ring and I, J be two distinct maximal left ideals of R. Then, I and J are not adjacent.*

Proof. Since I and J are two distinct maximal left ideals of R, therefore $I+J = R$. So I and J are not adjacent. $\square$

Lemma 3.2. *Let R be a ring with co-intersection $\Omega(R)$ and J be a nontrivial left ideal of R. If $deg(J)$ is finite, then R is a left Artinian ring.*

Proof. Since $deg(J) < \infty$, so J is a left Artinian R-modules. Otherwise, there exists a descending chain $J \supset I_1 \supset \cdots \supset I_n \supset \cdots$ of left ideals of R belong to J. Thus, $J + I_i = J \neq R$ for each i and this is a contradiction. Also, R/J is a left Artinian R-modules. Otherwise, there exists a descending chain $R/J \supset I_1/J \supset \cdots \supset I_n/J \supset \cdots$ of left submodules of R/J. Thus, $J + I_i = I_i \neq R$ for each i and this is a contradiction. Hence, according to [8, Proposition 4.5], R is a left Artinian R-module and the proof is complete. $\square$

The following proposition can be obtained in a similar way in [9, Theorem 2.1] about the connectivity.

Proposition 3.1. *Let R be a ring and $\mathcal{I}^\star(R) \neq \emptyset$. Then, $\Omega(R)$ is disconnected if and only if R has at least two maximal left ideals, and every nontrivial left ideal is a maximal left ideal.*

Corollary 3.1. *The graph $\Omega(\mathbb{Z}_n)$ is disconnected if and only if $n = pq$, where p and q are distinct primes.*

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ for some $k \in \mathbb{N}$. According to Remark 2.1, $\mathbb{Z}_n$ has at least two maximal ideals and every nontrivial ideal is a maximal ideal if and only if $k = 2$ and $\alpha_i = 1$. Then, by Proposition 3.1, $\Omega(\mathbb{Z}_n)$ is disconnected if and only if $\mathbb{Z}_n$ has at least two maximal ideals and every nontrivial ideal is a maximal ideal if and only if $k = 2$ and $\alpha_i = 1$.

Corollary 3.2. *Let R be a ring and $\mathcal{I}^\star(R) \neq \emptyset$. If $\Omega(R)$ is disconnected then* $\mathrm{Max}(R) = \mathrm{Min}(R)$.

Proof. By Proposition 3.1, as $\Omega(R)$ is disconnected thus $\mathcal{I}^\star(R) = \mathrm{Max}(R)$. If $I \in \mathrm{Max}(R) = \mathcal{I}^\star(R)$, there is no nontrivial left ideal $J \subsetneq I$, then $I \in \mathrm{Min}(R)$. Also, $\mathrm{Min}(R) \subseteq \mathcal{I}^\star(R) = \mathrm{Max}(R)$. □

Corollary 3.3. *Let R be a ring. If $\Omega(R)$ is disconnected then, $\Omega(R)$ is a null graph.*

Proof. By Lemma 3.1 and Proposition 3.1, the proof is complete. □

Lemma 3.3. *Let R be a ring. If $\mathrm{Max}(R) \cap \mathrm{Min}(R) \neq \emptyset$, then $\mathcal{I}^\star(R) = \mathrm{Max}(R) = \mathrm{Min}(R)$ and thus $\Omega(R)$ is a null graph.*

Proof. Suppose that $\mathfrak{m} \in \mathrm{Max}(R) \cap \mathrm{Min}(R)$, then for each $I \in \mathcal{I}^\star(R)$, $I + \mathfrak{m} = R$. So, $\Omega(R)$ is disconnected and according to Corollary 3.2, $\mathrm{Max}(R) = \mathrm{Min}(R)$. Also, by Corollary 3.3 $\Omega(R)$ is a null graph. □

Proposition 3.2. *Let R be a commutative ring. Then, the graph $\Omega(R)$ is disconnected if and only if $R = R_1 \times R_2$ where each $R_i (i = 1, 2)$ is either a field or a null ring with prime number of elements.*

Proof. For the proof of the necessity part, suppose that, the graph $\Omega(R)$ is disconnected. Then, according to Proposition 3.1 and its proof, there are two maximal ideals I and J of R such that $I + J = R$ and $I \cap J =< 0 >$, as they are minimal ideal too, from Corollary 3.2. Then, $R = I \oplus J \cong \frac{R}{J} \times \frac{R}{I}$ where $\frac{R}{J}$ and $\frac{R}{I}$ are simple commutative rings, as I and J are maximal ideal.

Conversely, let $R = R_1 \times R_2$ where R_1, R_2 are simple commutative rings. If both R_1 and R_2 are two fields, then R has only two nontrivial ideals, $I = R_1 \times \{0_{R_2}\}$ and $\{0_{R_1}\} \times R_2$ and they are maximal ideals and hence according to Lemma 3.1, $\Omega(R)$ is disconnected. If both R_1 and R_2 are two null rings with prime number of elements, let $(R_1,+) \cong (\mathbb{Z}_p,+)$ and $(R_2,+) \cong (\mathbb{Z}_q,+)$, where p , q are prime numbers. If $p \neq q$, then $(R,+) \cong (\mathbb{Z}_p \times \mathbb{Z}_q,+) \cong (\mathbb{Z}_{pq},+)$, which has only two nontrivial subgroups $(\bar{p}),(\bar{q})$. These two subsets are also only nontrivial ideals of the null ring R, and then $\Omega(R)$ is disconnected.

If R_1 is a field and R_2 is a null rings with prime number of elements, such that $(R_2,+) \cong (\mathbb{Z}_p,+)$. In this case $\{(0_{R_1}) \times R_2, R_1 \times (\bar{0})\} = \mathcal{I}^\star(R)$. As $1 \in (0_{R_1}) \times R_2 + R_1 \times (\bar{0})$, then $\Omega(R)$ is disconnected. □

In the following, we characterize all the values of n for which the graph $\Omega(\mathbb{Z}_n)$ is Eulerian; further, some values of n for which $\Omega(\mathbb{Z}_n)$ is Hamiltonian are characterized.

At the first, we give a lemma about the number of vertices of $\Omega(\mathbb{Z}_n)$, and characterize $deg(I)$ for each $I \in \mathcal{I}^\star(\mathbb{Z}_n)$ and also minimum degree $\delta(\Omega(\mathbb{Z}_n))$.

Lemma 3.4. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}, a = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$, where p_i's are all distinct prime numbers, and also $0 \leq \beta_i \leq \alpha_i$. If $I =< a >$ is a nontrivial ideal of $\mathbb{Z}_n$, and suppose that $\mathfrak{B}_a = \{t_j | 1 \leq t_j \leq k, \beta_{t_j} \neq 0\}$ is the ordered set of all indices t_j, such that $\beta_{t_j} \neq 0$, then*

$$deg(I) = \sum_{j=1}^{|\mathfrak{B}_a|} \left(\alpha_{t_j} \prod_{\substack{i=1 \\ i \notin \{t_1,\cdots,t_j\}}}^{k} (\alpha_i + 1) \right) - 2. \tag{1}$$

Also, the number of vertices of $G = \Omega(\mathbb{Z}_n)$ is $|\mathcal{I}^\star(\mathbb{Z}_n)| = \prod_{i=1}^{k} (\alpha_i + 1) - 2$ and

$$\delta(G) = \alpha_{t_0} \prod_{\substack{i=1 \\ i \neq t_0}}^{k} (\alpha_i + 1) - 2$$

wherein $\alpha_{t_0} = \min\{\alpha_i | 1 \leq i \leq k\}$.

Proof. Assume that $b|n$ and $b \neq n$. Then, $J =< b > \neq I$ and I are adjacent if there exist some $t_j \in \mathfrak{B}_a$ such that $p_{t_j}|b$. But there are

$$\alpha_{t_1} \prod_{\substack{i=1 \\ i \neq t_1}}^{k} (\alpha_i + 1)$$

factors of n in the form $b = p_{t_1} b'$ (two of them are n and a), and there are

$$\alpha_{t_2} \prod_{\substack{i=1 \\ i \notin \{t_1,t_2\}}}^{k} (\alpha_i + 1)$$

factors of n in the form $b = p_{t_2} b'$ such that $p_{t_1} \nmid b'$ and so on. It is obvious that these factors of n are distinct. As $< n >, < a >$ are not adjacent to $I =< a >$, thus 2 units are deducted from the total. The proof of other statements are obvious. □

Example 3.1. Let $n = 2^{10} \times 3^5 \times 5^2 \times 7$, $a = 2^5 \times 3^2 \times 5$ and $G = \Omega(\mathbb{Z}_n)$. Thus, $I =< a >$ is a nontrivial ideal of $\mathbb{Z}_n$. Then, according to the Lemma 3.4,
$deg(< a >) = (10 \times 6 \times 3 \times 2) + (5 \times 3 \times 2) + (2 \times 2) - 2 = 392$.
Also, $\delta(G) = (1 \times 11 \times 6 \times 3) - 2 = 196$.

Proposition 3.3 ([6, Theorem 3.7])**.** *A connected graph is Eulerian if and only if all of its vertices have even degree.*

In the next proposition, we characterize all the values of n for which graphs of $\mathbb{Z}_n$ are Eulerian.

Proposition 3.4. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where p_i's are all distinct prime numbers. Then, $\Omega(\mathbb{Z}_n)$ is Eulerian if and only if $\alpha_i = 1$ for each $1 \leq i \leq k$, or each α_i is even ($1 \leq i \leq k$).*

Proof. According to Proposition 3.3, it is enough to show that all vertices of $\Omega(\mathbb{Z}_n)$ have even degree if and only if $\alpha_i = 1$ for each $1 \leq i \leq k$, or each α_i is even $(1 \leq i \leq k)$.

With the same notation in Lemma 3.4, if $\alpha_i = 1$ for each $1 \leq i \leq k$, then for each factor $a \neq n$ of n, there is some $i_0 \in \{1, 2, \cdots, k\} \setminus \mathfrak{B}_a$. Thus

$$2 = (\alpha_{i_0} + 1) | \prod_{\substack{i=1 \\ i \notin \{t_1, \cdots, t_j\}}}^{k} (\alpha_i + 1)$$

for each $1 \leq j \leq |\mathfrak{B}_a|$, and hence $deg(I)$ is even. Also, it is obvious that $deg(I)$ is even if α_i is even for each $1 \leq i \leq k$. Conversely, if there exist an α_i greater than 1 and also $\{s_1, \cdots, s_m | \alpha_{s_i} \text{ is odd}\}$ is the nonempty set of all s_i such that α_{s_i} is odd, then the ideal $I =< a >=< p_{s_1} \cdots p_{s_m} >$ is a nontrivial ideal of $\mathbb{Z}_n$. We show that $deg(I)$ is odd. In this case, it is obvious that the summand

$$\alpha_{s_j} \prod_{\substack{i=1 \\ i \notin \{s_1, \cdots, s_j\}}}^{k} (\alpha_i + 1)$$

in Equation 1 is even for each $1 \leq j \leq m - 1$ and is odd for $j = m$ and thus $deg(I)$ is odd. The proof is complete. □

Proposition 3.5 ([6], Theorem 18.4)**.** *Let G be a simple graph of minimum degree δ, where $\delta \geq \frac{n}{2}$ and $n \geq 3$. Then, G is Hamiltonian.*

Proposition 3.6. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where p_i's are all distinct prime numbers. If $k = 1, \alpha_1 \geq 4$ or $k \geq 2, \alpha_i \geq 3$ for each $1 \leq i \leq k$, then $\Omega(\mathbb{Z}_n)$ is Hamiltonian.*

Proof. If $k = 1, \alpha_1 \geq 4$, then $\Omega(\mathbb{Z}_n)$ is a complete graph with at least 3 vertices ([9, Example 2.14]) and thus is Hamiltonian.

Now, assume that $k \geq 2, \alpha_i \geq 3$ for each $1 \leq i \leq k$. Let $\alpha_{t_0} = \min\{\alpha_i | 1 \leq i \leq k\}$. Therefore, $\Omega(\mathbb{Z}_n)$ has $n \geq 3$ vertices and also

$$(2\alpha_{t_0} - (\alpha_{t_0}+1)) \prod_{\substack{i=1 \\ i \neq t_0}}^{k} (\alpha_i + 1) \geq 2.$$

Hence,

$$\delta(\mathbb{Z}_n) = \alpha_{t_0} \prod_{\substack{i=1 \\ i \neq t_0}}^{k} (\alpha_i + 1) - 2 \geq \frac{\prod_{i=1}^{k} (\alpha_i + 1)}{2} - 1 = \frac{n}{2}.$$

Therefore, by Proposition 3.5, $\Omega(\mathbb{Z}_n)$ is Hamiltonian. □

4. The decision number of $\Omega(\mathbb{Z}_n)$

In this section, the bad decision number and the nice decision number of $G = \Omega(\mathbb{Z}_n)$ are investigated for each n. Some lemma's are presented in the following, and the results are combined to a single theorem at the end of the section.

Lemma 4.1. *Let $n = p^{\alpha}$, $\alpha \geq 3$, and also $G = \Omega(\mathbb{Z}_n)$. Thus,*

$$\beta_D(G) = \begin{cases} 0, & \text{for odd} \quad \alpha \geq 5 \ , \\ 2, & \text{for} \quad \alpha = 3 \ , \\ -1, & \text{for even} \quad \alpha. \end{cases} \qquad \overline{\beta_D(G)} = \begin{cases} 0, & \text{for odd} \quad \alpha \ , \\ 1, & \text{for even} \quad \alpha. \end{cases}$$

Proof. We know $G = \Omega(\mathbb{Z}_n)$ is the complete graph $K_{\alpha-1}$ for $n = p^{\alpha}$. Thus, at least $\lceil \frac{\alpha-1}{2} \rceil$ of the vertices must be signed by -1, for any bad function f and $\alpha > 3$. In the other side, there is a bad function f over G, such that exactly $\lceil \frac{\alpha-1}{2} \rceil$ of the vertices are signed by -1. Further, it is obvious that $\beta_D(\Omega(\mathbb{Z}_3)) = \beta_D(K_2) = 2$. Similarly, at least $\lfloor \frac{\alpha-1}{2} \rfloor$ of the vertices must be signed by -1, for any nice function f. □

Lemma 4.2. *Let $k \geq 2$, α_k be an odd number, $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where p_i's are all distinct prime numbers, and also $G = \Omega(\mathbb{Z}_n)$. The bad decision number and the nice decision number of G are lower than or equal to* 2.

Proof. If $\alpha_i = 1$, for all $1 \leq i \leq k$, then let $v_0 = p_2 \cdots p_k$. Note that, $\mid V(G) \mid = \prod_{i=1}^{k} (\alpha_i + 1) - 2$ is an even number, and $N(< v_0 >) = V(G) \setminus \{< v_0 >, < p_1 >\}$. If f is a bad function, then $f(N(< v_0 >))$ is at most equal to 0, because of $\mid N(< v_0 >) \mid$ is even. Also, $f(N[< v_0 >])$ is at most equal to 1 for a nice function f. Thus, $f(V(G))$ is at most equal to 2 for any bad or nice function f.

If there is an $\alpha_i \geq 2$, then let $v_0 = p_1 p_2 \cdots p_k$. If f is a bad function then $f(N(< v_0 >))$ is at most equal to 1. If f is a nice function then, $f(N[< v_0 >]) \leq 0$ because of $N[< v_0 >] = V(G)$ and $\mid V(G) \mid$ is even. Hence, for any bad or nice function f, $f(V(G)) \leq 2$. □

The upper bound presented in Lemma 4.2 is sharp. As $\overline{\beta_D(\Omega(\mathbb{Z}_{pq}))} = \beta_D(\Omega(\mathbb{Z}_{pq})) = 2$.

Lemma 4.3. *Let $k \geq 2$, α_k be an odd number, $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where p_i's are all distinct prime numbers, and $G = \Omega(\mathbb{Z}_n)$. The bad decision number and the nice decision number of G are greater than or equal to 0.*

Proof. Let $m = \frac{\alpha_k - 1}{2}$. Define the function $f : V \to \{-1, 1\}$ as:

$$f(< a >) = \begin{cases} -1, & \text{if} \quad p_k^{m+1} | a \ , \\ 1, & \text{otherwise.} \end{cases}$$

Suppose that $< a >$ is a nontrivial ideal of $\mathbb{Z}_n$, and $a = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$. We show that $f(N(< a >)) \leq 1$ and then the function f is a bad function. Let $A = \{i | a_i \neq 0\}$.

- If $a_k = 0$:
  There are $X = \left(\prod_{i \in A}(\alpha_i + 1) - 1\right) \prod_{i \notin A, i \neq k}(\alpha_i + 1)(m + 1) - 1$ elements in $N(< a >)$, such that have value 1 under the function f. Also, There are X elements in $N(< a >)$, such that have value -1 under the function f. Hence, $f(N(< a >)) = 0$.

- If $m \neq 0$ and $a_k \neq 0$:
  In this case, $Y = \left(\prod_{i \in A, i \neq k}(\alpha_i + 1)(m + 1) - 1\right) \prod_{i \notin A}(\alpha_i + 1)$ elements of $N(< a >)$ have value 1 under the function f, and Y elements of $N(< a >)$ have value -1 under the function f. Therefore, $f(N(< a >)) = 0$.

- If $m = 0$ and $a_k \neq 0$: In this case, there are $\prod_{i \neq k}(\alpha_i + 1) - 2$ elements of $N(< a >)$ with value -1 and $\prod_{i \in A, i \neq k}(\alpha_i + 1) - 1$ elements of $N(< a >)$ with value 1 under f. Thus, $f(N(< a >)) = \prod_{i \in A, i \neq k}(\alpha_i + 1)\left(1 - \prod_{i \notin A}(\alpha_i + 1)\right) + 1 \leq 1$. Also, $f(N[a]) \leq 0$, as $f(< a >) = -1$.

On the other side, $f(V) = 0$, as exactly the half of the vertices of G have value 1 under the f. Hence, $\beta_D(G) \geq 0$. Furthermore, it is obvious that $f(N[a]) \leq 1$ in all 3 cases, hence f is a nice function and $\overline{\beta_D(G)} \geq 0$. $\square$

The following Lemma, present an upper bound for decision numbers in the case of all of the prime factors of n have even exponent in the prime decomposition of n.

Lemma 4.4. *Let $k \geq 2$, $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where p_i's are all distinct prime numbers, α_i's are all even numbers, and also $G = \Omega(\mathbb{Z}_n)$. The bad decision number and the nice decision number of G are lower than or equal to 1.*

Proof. Let $v = p_1 p_2 \cdots p_k$, and f be a bad function. Note that, $| V(G) | = \prod_{i=1}^{k}(\alpha_i + 1) - 2$ is an odd number. We have, $N(< v >) = V(G) \setminus \{< v >\}$ and $| N(< v >) |$ is even, thus $f(N(< v >)) \leq 0$ and $f(V(G)) \leq 1$. Further, If f is a nice function then, $f(V(G)) = f(N[< v >]) \leq 1$. Hence, $f(V(G))$ is at most equal to 1. $\square$

In the next example we show that the upper bound presented in Lemma 4.4 is sharp.

Example 4.1. Let $n = p_1^2p_2^2p_3^2$, and $G = \Omega(\mathbb{Z}_n)$. Define the function f over $V(G)$ as: $f(p_i^{a_i}) = f(p_1^{a_1}p_2^{a_2}p_3^{a_3}) = 1$, where $1 \leq i \leq 3$ and $a_1a_2a_3 \neq 0$. Otherwise, $f(v) = -1$.

It is easy to check that the function f is a bad (and nice) function, and $f(V(G)) = +13 - 12 = 1$. Hence, $\beta_D(G), \overline{\beta_D(G)} = 1$.

Lemma 4.5. *Let $k \geq 2$, $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$, where p_i's are all distinct prime numbers, α_i's are all even numbers, and also $G = \Omega(\mathbb{Z}_n)$. The bad decision number and the nice decision number of G are greater than or equal to -1.*

Proof. Let $m_i = \frac{\alpha_i}{2}$ for each $1 \leq i \leq k$. Define the function $f : V \to \{-1, 1\}$ as:

$$f(< a >) = \begin{cases} 1, & \text{if} \quad p_1^{\alpha_1}\cdots p_i^{\alpha_i}|a \quad \text{and} \quad p_{i+1}^{m_{i+1}} \nmid a \text{ for some } 0 \leq i \leq k-1 \ , \\ -1, & \text{otherwise.} \end{cases}$$

Suppose that $< a >$ is a nontrivial ideal of $\mathbb{Z}_n$, and $a = p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}$. We show that $f(N(< a >)) \leq 1$ and then the function f is a bad function. Let $A = \{i|a_i \neq 0\}$, and $t = \min\{i|a_i \neq 0\}$.

According to the definition of f,

$$X = \sum_{i=t}^{k} m_i \prod_{j=i+1}^{k}(\alpha_j + 1) + \sum_{i=1}^{t-1} m_i \left(\prod_{j\in A, j>i}(\alpha_j+1) - 1\right) \prod_{j\notin A, j>i}(\alpha_j + 1)$$

elements of $N[< a >]$ have value -1 under the function f, and

$$\begin{aligned} Y = &\sum_{i=t}^{k} m_i \prod_{j=i+1}^{k}(\alpha_j + 1) - \prod_{j\notin A, j>t}(\alpha_j+1) \\ &+ \sum_{i=1}^{t-1} m_i \left(\prod_{j\in A, j>i}(\alpha_j+1) - 1\right) \prod_{j\notin A, j>i}(\alpha_j + 1) \end{aligned}$$

elements of $N[a]$ have value 1 under the function f. Therefore, if $f(< a >) = 1$, then X elements of $N(< a >)$ have value -1, and $Y - 1$ elements of $N(< a >)$ have value $+1$. If $f(< a >) = -1$, then $X - 1$ elements of $N(< a >)$ have value -1, and Y elements of $N(< a >)$ have value $+1$. Thus,

$$f(N(< a >)) = \begin{cases} Y - 1 - X = -\sum_{i=t}^{k} \prod_{j\notin A, j>t}(\alpha_j + 1) - 1, & \text{if} \quad f(< a >) = 1 \ , \\ Y - X + 1 = -\sum_{i=t}^{k} \prod_{j\notin A, j>t}(\alpha_j + 1) + 1, & \text{if} \quad f(< a >) = -1. \end{cases}$$

Consequently, $f(N(< a >)) \leq 0$ and $f(N[< a >]) \leq -1$. Hence, f is both bad function and nice function.

By the definition of the function f,

$$Z = \sum_{i=1}^{k} m_i \prod_{j=i+1}^{k} (\alpha_j + 1)$$

vertices of G have value -1, and $Z - 1$ elements of $N(< a >)$ have value $+1$. Hence, $\beta_D(G), \overline{\beta_D(G)} \geq -1$. □

Finally, the following theorem can immediately be concluded from the above discussions.

Theorem 4.1. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where p_i's are all distinct prime numbers, and also $G = \Omega(\mathbb{Z}_n)$. We have*

$$\beta_D(G) = \begin{cases} -1, & \text{if } \quad k = 1, \alpha_1 \text{ is an even number}, \\ -1 \text{ or } 1, & \text{if } \quad \alpha_i \text{ is an even number, for all } 1 \leq i \leq k \ , \\ 0 \text{ or } 2, & \text{otherwise}. \end{cases}$$

$$\overline{\beta_D(G)} = \begin{cases} 1, & \text{if } \quad k = 1, \alpha_1 \text{ is an even number}, \\ -1 \text{ or } 1, & \text{if } \quad \alpha_i \text{ is an even number, for all } 1 \leq i \leq k \ , \\ 0 \text{ or } 2, & \text{otherwise}. \end{cases}$$

5. Conclusion

In this paper, we have obtained a sufficient and necessary condition for $\Omega(R)$ to be a connected graph. Likewise, we characterized the values of n for which the graph $\Omega(\mathbb{Z}_n)$ is Eulerian and Hamiltonian. Finally, the bad (and nice) decision number of $\Omega(\mathbb{Z}_n)$ has been presented. In our future work, we will introduce new results of connected graphs that are very useful in networks and computer sciences.

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