On solving some classes of second order ODEs

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Abstract. In this paper, we introduce some analytical techniques to solve some classes of second order differential equations. Such classes of differential equations arise in describing some mathematical problems in Physics and Engineering. Moreover, these classes of differential equations generalize some very well known differential equations. For example, Chebyshev's Equation, Cauchy-Euler's Equation, and some of Hypergeometric differential Equations. Using one of these techniques we solve a special case of Lane-Emden Type of Differential Equations.

Keywords: Chebyshev's differential equation, hypergeometric differential equation, Cauchy-Euler's differential equation, exact second order differential equations, nonlinear second order differential equations, Lane-Emden type of differential equations.

1. Introduction

Solving differential equations has important impacts on Mathematics, sciences, and engineering. In [1]-[9], several techniques for solving differential equations are introduced. One of the most important applications in the calculus of variation is to maximize (minimize) the functional

$$Q[y] = \int_a^b \left(\sqrt{p(x)}(y'(x))^2 + \frac{h(y(x))}{\sqrt{p(x)}} \right) dx, \tag{1}$$

where $p(x)$ is a positive and differentiable function on some open interval $I \subset \mathbb{R}$, and $h(x)$ is a differentiable function. In fact, the functional

$$Q[y] = \int_a^b \left(\sqrt{p(x)}(y'(x))^2 + \frac{h(y(x))}{\sqrt{p(x)}} \right) dx \tag{2}$$

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attains its extreme values at a function $y(x) \in C^2(I)$, where I is an open interval in $\mathbb{R}$, when $y(x)$ satisfies the Euler's-Lagrange differential equation [12, 15]

$$\frac{\partial F}{\partial y}(x, y, y') - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}(x, y, y')\right) = 0, \tag{3}$$

where

$$F(x, y, y') := \sqrt{p(x)}(y'(x))^2 + \frac{h(y(x))}{\sqrt{p(x)}}. \tag{4}$$

Therefore, the problem of maximizing (minimizing) $Q[y]$ is reduced to solve the differential equation (3). i.e., to solve

$$p(x)y''(x) + \frac{1}{2}p'(x)y'(x) = \frac{1}{2}h'(y(x)). \tag{5}$$

Hence, it is a matter to solve such differential equations. In the first part of this paper, we solve the following class of second order differential equation:

$$p(x)y''(x) + \frac{1}{2}p'(x)y'(x) + f(\sqrt{p(x)}\; y'(x), y(x)) = 0 \tag{6}$$

which generalizes the differential equation (5). The function $p(x)$ is positive and differentiable function on some open interval $I \subset \mathbb{R}$, and $f(\sqrt{p(x)}\; y'(x), y(x))$ is continuous function on some domain $D \subset \mathbb{R}^2$. In fact, equation (6) not only generalizes equation (5), but also it generalizes many of well known differential equation. For example,

1. the Chebyshev's Differential Equation, e.g., see, [12, 18],

$$(1 - x^2)y''(x) - x\, y'(x) + n^2 y(x) = 0, \quad |x| < 1,$$

2. the Cauchy-Euler's Differential Equation, e.g., see, [13, 16],

$$ax^2 y''(x) + a\, x\, y'(x) + b\, y(x) = 0, \quad x > 0,$$

3. the Nonlinear Chebyshev's Equation,

$$(1 - x^2)y''(x) + \left(\alpha\sqrt{1 - x^2} - x\right) y'(x) + f(y(x)) = 0, \qquad |x| < 1,$$

4. the Hypergeometric Differential Equation, e.g., see, [11, 18],

$$x(1 - x)y''(x) + [c - (a + b + 1)x]\, y'(x) - a\, b\, y(x) = 0, \; 0 < x < 1$$

with c=1/2 and a=-b, and

5. the Nonlinear Hypergeometric Differential Equation,

$$x(1-x)y''(x) + \left(\frac{1}{2} - x + \alpha\sqrt{x(1-x)}\right) y'(x) + f(y(x)) = 0,\ 0 < x < 1.$$

Throughout the paper, the class of differential equations given in (6) is called by Chebyshev's-type of differential equations.

In the second part of this paper, we introduce an approach to solve the class of second order differential equations

$$a_2\left(f'(y)y'' + (y')^2 f''(y)\right) + a_1 f'(y)y' + a_0 f(y) = g(x), \tag{7}$$

where $a_0,\ a_1$ and a_2 are constants, and $f \in C^2(J)$, for some open interval $J \subset \mathbb{R}$. We also give an approach to solve the class of differential equation

$$p(x)\left(f'(y)y'' + (y')^2 f''(y)\right) + \frac{1}{2}p'(x)f'(y)y' + a_0 f(y) = 0, \tag{8}$$

where $p(x)$ is positive and differentiable function on some open interval $I \subset \mathbb{R}$, and $f \in C^2(J)$, for some open interval $J \subset \mathbb{R}$. Throughout the paper, these two classes of second order differential equations are called by $f-$types of second order differential equations.

In the third part of this paper, we introduce an approach to solve the following class of second order differential equations:

$$a_2(x,y,y')\left(f'(y)y'' + f''(y)(y')^2\right) + a_1(x,y,y')(f'(y)y') + a_0(x,y,y') = 0, \tag{9}$$

where $f(y)$ is an invertible function, and $f \in C^2(J)$ for some open interval $J \subset \mathbb{R}$. To solve this class of differential equations, we assume that

$$a_2\left(x, f^{-1}(z), \frac{z'}{f'\left(f^{-1}(z)\right)}\right) z'' + a_1\left(x, f^{-1}(z), \frac{z'}{f'\left(f^{-1}(z)\right)}\right) z' \tag{10}$$

$$+ a_0\left(x, f^{-1}(z), \frac{z'}{f'\left(f^{-1}(z)\right)}\right) = 0 \tag{11}$$

is an exact differential equation. The differential equation (10) is exact if the conditions

$$\frac{\partial a_2}{\partial z} = \frac{\partial a_1}{\partial z'},\ \frac{\partial a_2}{\partial x} = \frac{\partial a_0}{\partial z'},\ \text{and}\ \frac{\partial a_1}{\partial x} = \frac{\partial a_0}{\partial z}. \tag{12}$$

hold, see, [1, 7, 8, 9]. In this case, the first integral of (10) exists and is given by

$$\int_{x_0}^{x} a_0(\alpha, z, z')d\alpha + \int_{z_0}^{z} a_1(x_0, \beta, z')d\beta + \int_{z'_0}^{z'} a_2(x_0, z_0, \gamma)d\gamma = c.$$

Throughout the paper, this class of differential equations is called by $f-$type of second order differential equations that can be transformed to exact differential equations.

The layout of the paper: In the second section, we solve Chebyshev's-type of Second Order Differential Equations. In the third section, we solve the $f-$types of second order differential equations. In the fourth section, we solve the $f-$type of second order differential equation that can be transformed into exact equations. The fifth section has been devoted for the concluding remarks.

2. Solving Chebyshev's-type of second order differential equations

In this section, we present an approach to solve Chebyshev's-type of second order differential equations

$$p(x)y''(x) + \frac{1}{2}p'(x)y'(x) + f(\sqrt{p(x)}\, y'(x), y(x)) = 0, \tag{13}$$

where $p(x)$ is a positive and differentiable function on some open interval $I \in \mathbb{R}$, and $f(\sqrt{p(x)}$
$y'(x), y(x))$ is continuous function on some domain $D \subset \mathbb{R}^2$. The approach is described in the following theorem:

Theorem 2.1. *Assume that $p(x)$ be a positive and differentiable function on the open interval $I \subset \mathbb{R}$. Let $x_0 \in I$. Then*

$$t = \int_{x_0}^{x} \frac{d\xi}{\sqrt{p(\xi)}}$$

transforms the differential equation (13) *into the second order differential equation*

$$y''(t) + f(y(t), y'(t)) = 0. \tag{14}$$

Proof. Let

$$t = \int_{x_0}^{x} \frac{d\xi}{\sqrt{p(\xi)}}.$$

Since,

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}.$$

Hence,

$$\frac{dy}{dt} = \sqrt{p(x)}\, \frac{dy}{dx}. \tag{15}$$

Therefore,

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\sqrt{p(x)}\, \frac{dy}{dx}\right) = \frac{d}{dx}\left(\sqrt{p(x)}\, \frac{dy}{dx}\right)\frac{dx}{dt} = p(x)y'' + \frac{1}{2}p'(x)y'. \tag{16}$$

By substituting (15) and (16) in Equation (6), we get the result. □

Remark 2.1. The differential equation (14) is independent of variable t, and so, it can be solved by setting $\eta(t) = y'(t)$. Hence, it can be reduced to the following first order differential equation:

$$\eta \frac{d\eta}{dy} + f(y, \eta) = 0. \tag{17}$$

Particularly, if $f(\sqrt{p(x)}\, y', y) = f(y)$, then we have

$$\eta^2(t) = -2 \int^y f(\xi) d\xi + c.$$

Therefore,

$$y'(t) = \left(c - 2 \int^y f(\xi) d\xi\right)^{\frac{1}{2}},$$

where c is the integration constant.

Next, we present some illustrative examples.

Example 2.1. Consider the following special case of Lane-Emden type of differential equation:

$$y'' + \frac{n-1}{x} y' + x^{2(1-n)} f(y) = 0, \quad n > 2, \quad x > 0. \tag{18}$$

Generally, the Lane-Emden type of differential equations is given by

$$y'' + \frac{n-1}{x} y' + g(x) f(y) = 0, \quad n > 2, \quad x > 0. \tag{19}$$

This differential equation arises in mathematical physics and astrophysics to describe the temperature variations of a spherical gas cloud under the mutual attraction of its molecules and subject to classical thermodynamics law [17]. To solve (18), we multiply it by $x^{2(n-1)}$ to get

$$x^{2(n-1)} y'' + (n-1) x^{2n-3} y' + f(y) = 0, \quad n > 2, \quad x > 0. \tag{20}$$

Clearly, this equation takes the form of Eq. (13). Therefore it can be reduced to to the following simple differential equation:

$$y'' + f(y) = 0.$$

Example 2.2. Consider the nonlinear Chebyshev's differential equation

$$\begin{cases} (1-x^2) y''(x) - x y'(x) + 4\sqrt{1-x^2}\, y' y(x) = 0, \\ y(0) = \frac{1}{2}, \; y'(0) = -\frac{1}{2}. \end{cases} \tag{21}$$

Then

$$t = \int^x \frac{d\xi}{\sqrt{1-\xi^2}} = \arcsin(x)$$

transforms (21) into

$$y''(t) + 4y'(t)y(t) = 0.$$

Set $\eta(t) = y'(t)$. The above equation becomes

$$\eta\frac{d\eta}{dy} + 4\eta y = 0.$$

The solution of this equation is $y(t) = \frac{1}{2(t+1)}$. Therefore,

$$y(x) = \frac{1}{2(\arcsin(x) + 1)}.$$

Example 2.3. Consider the initial value problem

$$\begin{cases} x^2y'' + xy' - 3y^2 = 0, \ x > 0, \\ y(1) = 2, \ y'(1) = 4. \end{cases} \tag{22}$$

Then

$$t = \int_1^x \frac{d\xi}{\xi} d\xi = \ln(x)$$

transforms the (22) into

$$\begin{cases} y'' - 3y^2 = 0, \\ y(0) = 2, \ y'(0) = 4. \end{cases} \tag{23}$$

The solution of the above differential equation is

$$y(t) = \frac{2}{(1-t)^2}.$$

Hence,

$$y(x) = \frac{2}{(1-\ln(x))^2}\,.$$

Example 2.4. Consider the linear form of (6). i.e.,

$$\phi(x)y'' + \frac{1}{2}\phi'(x)y' + \lambda^2 y = 0, \tag{24}$$

where $\lambda > 0$, and $\phi(x)$ is a positive and differentiable function on some open interval $(a, b) \subset \mathbb{R}$. By applying the transformation

$$t = \int_{x_0}^x \frac{d\xi}{\sqrt{\phi(\xi)}} d\xi$$

to equation (24), then it can be transformed into the following differential equation:

$$\frac{d^2y}{dt^2} + \lambda^2 y = 0.$$

The solution of this differential equation is

$$y(t) = C_1 \sin(\lambda t) + C_2 \cos(\lambda t).$$

Hence, the general solution of equation (24) is

$$y(x) = C_1 \sin\left(\lambda \int_{x_0}^{x} \frac{d\xi}{\sqrt{\phi(\xi)}} d\xi\right) + C_2 \cos\left(\lambda \int_{x_0}^{x} \frac{d\xi}{\sqrt{\phi(\xi)}} d\xi\right).$$

Example 2.5. Consider the following second order linear differential equation [16, eq. 239, p. 335]:

$$4xy'' + 2y' + y = 0. \tag{25}$$

From the previous example, the general solution of this equation is given by

$$y(x) = C_1 \sin\left(\int_{x_0}^{x} \frac{d\xi}{2\sqrt{\xi}}\, d\xi\right) + C_2 \cos\left(\int_{x_0}^{x} \frac{d\xi}{2\sqrt{\xi}}\, d\xi\right)$$

and so,

$$y(x) = C_1 \sin\left(\sqrt{x}\right) + C_2 \cos\left(\sqrt{x}\right).$$

Remark 2.2. Consider the second order linear differential equation

$$(\phi(x))^2 y''(x) + \phi(x)\phi'(x)y'(x) + \lambda y(x) = 0, \quad x \in (a, b), \tag{26}$$

and assume that $\phi(x)$ is a positive and differentiable function on the open interval $(a, b) \subset \mathbb{R}$. Moreover, assume that $\phi(a) = \phi(b) = 0$. Define the linear differential operator

$$L[y] := -\left((\phi(x))^2 y''(x) + \phi(x)\phi'(x)y'(x)\right) = \lambda y(x).$$

Then the boundary value problem

$$\begin{cases} L[y] = -\left((\phi(x))^2 y''(x) + \phi(x)\phi'(x)y'(x)\right) = \lambda y(x), & a < x < b, \\ \phi(a) = \phi(b) = 0, \end{cases} \tag{27}$$

satisfies the Lagrange Identity $\int_a^b \phi L[\psi]dx = \int_a^b \psi L[\phi]dx$, where ϕ and ψ satisfy the above boundary value problem. Therefore, the operator $L[y]$ is self-adjoint. Hence, the boundary value problem (27) has an orthogonal set of eigenfunctions $\{\phi_n(x)\}_{n=1}^{\infty}$ with corresponding eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$. Since the above boundary value problem is a special case of (6). Then, by using the approach described in Theorem 2.1, it is easy to find its orthogonal set of eigenfunctions.

By using the same approach described in Theorem 2.1. We can solve the following class of second order linear differential equations:

$$[P(x)]^2\, y''(x) + P(x)\left[\alpha + P'(x)\right] y'(x) + \beta y(x) = 0, \tag{28}$$

where $P(x) > 0$, $P(x) \in C^1(a,b)$, and α and β are constants. In fact, the transformation

$$t = \int_{x_0}^{x} \frac{d\xi}{P(\xi)}, \tag{29}$$

where x_0, $x \in (a,b)$, transforms Eq. (28) into the following second order differential equation:

$$y''(t) + \alpha y'(t) + \beta y(t) = 0.$$

This differential equation is with constant coefficients can be solved by using the elementary techniques of solving second order differential equations. For illustration, we present the following examples:

Example 2.6. Consider the well-known Cauchy-Euler's Equation

$$x^2 y''(x) + (\alpha+1)xy'(x) + \beta y = 0,\ x > 0.$$

Then $P(x) = x$, and the $t-$transformation is $t = \ln(x)$, which transforms the equation into

$$y''(x) + \alpha y'(x) + \beta y = 0$$

Example 2.7. Consider the Chebyshev's Equation

$$\left[1 - x^2\right] y''(x) - 2xy'(x) + n^2 y = 0,\ |x| < 1.$$

Then $P(x) = \sqrt{1-x^2}$, and the $t-$transformation is $t = \sin^{-1}(x)$, which transforms the equation into

$$y''(x) + n^2 y = 0.$$

Using this transformation, the solution of Chebyshev's Equation is given by

$$y(x) = A\cos(n\sin^{-1}(x)) + B\sin(n\sin^{-1}(x)).$$

Example 2.8. Consider the Hypergeometric Equation

$$x(1-x)y''(x) + \frac{1}{2}(1-2x)y'(x) + a^2 y = 0,\ x \in (0,1). \tag{30}$$

Then $P(x) = \sqrt{x(1-x)}$, and the $t-$transformation is $t = \sin^{-1}(2x-1)$. This transforms the above equation into

$$y''(t) + a^2 y(t) = 0.$$

Hence, the solution of equation (30) is given by

$$y(x) = A\cos(a\sin^{-1}(2x-1)) + B\sin(a\sin^{-1}(2x-1)).$$

For certain functions, $h(x) \in C(a,b)$, for some open interval $(a,b) \in \mathbb{R}$, we can solve the nonhomogeneous second order differential equation

$$[P(x)]^2 y''(x) + P(x)\left[\alpha + P'(x)\right] y'(x) + \beta y = h(x). \tag{31}$$

Particularly, when $h(x)$ can be written in the form $H(t)$, where $t = \int_{x_0}^{x} \frac{d\xi}{P(\xi)}$. The following example shows this idea:

Example 2.9. Consider the nonhomogeneous differential equation

$$x(1-x)y''(x) + \frac{1}{2}(1-2x)y'(x) + a^2 y = 2x,\ x \in (0,1). \tag{32}$$

Then $P(x) = \sqrt{x(1-x)}$. The $t-$transformation is $t = \sin^{-1}(2x-1)$. This transforms the equation into

$$y''(t) + a^2 y(t) = 1 + \sin(t).$$

The solution of equation (32) is given by

$$y(x) = \begin{cases} A\cos(a\sin^{-1}(2x-1)) + B\sin(a\sin^{-1}(2x-1)) + \dfrac{2x-1}{a^2-1} + \dfrac{1}{a^2}, \\ \text{if } \ a \neq \pm 1, \\ A\cos(\sin^{-1}(2x-1)) + B(2x-1) + \dfrac{1}{2}(1-2x)\sin^{-1}(2x-1) + 1, \\ \text{if } \ a = \pm 1. \end{cases}$$

3. Solving the $f-$types of second order differential equations

In this section, we solve the following class second order nonlinear differential equation

$$a_2\left(f'(y)y'' + (y')^2 f''(y)\right) + a_1 f'(y)y' + a_0 f(y) = g(x), \tag{33}$$

where a_2, a_1 and a_0 are constants, and $f \in C^2(J)$, for some open interval $J \subset \mathbb{R}$. In this section, we also solve the following class of second order nonlinear differential equation:

$$p(x)\left(f'(y)y'' + (y')^2 f''(y)\right) + \frac{1}{2}p'(x)f'(y)y' + a_0 f(y) = 0, \tag{34}$$

where $p(x)$ is a positive and differentiable function on the open interval $I \subset \mathbb{R}$, and $f \in C^2(J)$. To solve (33), let $z = f(y)$. Hence, $z' = f'(y)y'$, and $z'' = f'(y)y'' + (y')^2 f''(y)$. Substitute z, z' and z'' in equation (33), we get

$$a_2 z'' + a_1 z' + a_0 z = g(x), \tag{35}$$

Similarly, equation (34) becomes

$$p(x)z'' + \frac{1}{2}p'(x)z' + a_0 z = 0, \tag{36}$$

which is the linear form of (6). Therefore, it can be solved by using the technique described in Example 2.4. To illustrate the procedure of solving (33) and (34), we present the following examples:

Example 3.1. Consider Langumir Equation, with a slightly modification,

$$3yy'' + 3(y')^2 + 4yy' + y^2 = 1. \tag{37}$$

The original Langumir Equation is given by

$$3yy'' + (y')^2 + 4yy' + y^2 = 1$$

which originally appears in connection with the theory of current flow from hot cathode to an anode in a hight vacuum [10, 14]. To solve (37), we let $z = \frac{y^2}{2}$. Then $z' = yy'$ and $z'' = yy'' + (y')^2$. Hence, equation (37) becomes

$$3z'' + 4z' + 2z = 1.$$

The solution of this equation is

$$z(x) = e^{-\frac{2}{3}x}\left(A\cos\left(\frac{\sqrt{2}}{3}x\right) + B\sin\left(\frac{\sqrt{2}}{3}x\right)\right) + \frac{1}{2}.$$

Hence, the solution of (37) is given by

$$y^2 = 2e^{-\frac{2}{3}x}\left(A\cos\left(\frac{\sqrt{2}}{3}x\right) + B\sin\left(\frac{\sqrt{2}}{3}x\right)\right) + 1.$$

Example 3.2. Consider the initial value problem

$$\begin{cases} y'' + (y')^2 + 1 = (\cos\omega x)e^{-y}, \ \ \omega \neq \pm 1, \\ y(0) = y'(0) = 0. \end{cases}$$

This problem is equivalent to

$$\begin{cases} \left(y'' + (y')^2\right)e^y + e^y = (\cos\omega x), \ \ \omega \neq \pm 1, \\ y(0) = y'(0) = 0. \end{cases}$$

Let $z = e^y$. Then $z' = y'e^y$ and $z'' = y''e^y + (y')^2e^y$. By substituting z, z' and z'' in the above initial value problem, we get

$$\begin{cases} z'' + z = \cos\omega x, \ \ \omega \neq \pm 1, \\ z(0) = 1, \ z'(0) = 0. \end{cases}$$

The solution of this problem is $z(x) = \frac{1}{1-\omega^2}\left(\cos\omega x - \omega^2\cos x\right)$, $\omega \neq \pm 1$. Therefore, $y(x) = \ln\left(\frac{1}{1-\omega^2}\left(\cos\omega x - \omega^2\cos x\right)\right)$, $\omega \neq \pm 1$.

Example 3.3. Let $\phi(x)$ be a positive and differentiable function on the open interval $(a,b) \subset \mathbb{R}$, and consider the differential equation

$$\phi(x)\left(y'' + (y')^2\right) + \frac{1}{2}\phi'(x)y' + \lambda = 0.$$

By multiplying this equation by e^y, we get

$$\phi(x)\left(y'' + (y')^2\right)e^y + \frac{1}{2}\phi'(x)y'e^y + \lambda e^y = 0.$$

Let $z = e^y$. Then $z' = y'e^y$ and $z'' = y''e^y + (y')^2e^y$. By substituting z, z' and z'' in the above differential equation, we get

$$\phi(x)z'' + \frac{1}{2}\phi'(x)z' + \lambda z = 0.$$

The solution of this equation is (see Example 2.4)

$$z(x) = C_1 \sin\left(\lambda \int_{x_0}^{x} \frac{d\xi}{\sqrt{\phi(\xi)}} d\xi\right) + C_2 \cos\left(\lambda \int_{x_0}^{x} \frac{d\xi}{\sqrt{\phi(\xi)}} d\xi\right).$$

Therefore,

$$y(x) = \ln\left[C_1 \sin\left(\lambda \int_{x_0}^{x} \frac{d\xi}{\sqrt{\phi(\xi)}} d\xi\right) + C_2 \cos\left(\lambda \int_{x_0}^{x} \frac{d\xi}{\sqrt{\phi(\xi)}} d\xi\right)\right].$$

4. Solving the f-type of second order differential equations that can be transformed into exact differential equations

In this section, we solve the following class of second order differential equations:

$$a_2(x,y,y')\left(f'(y)y'' + f''(y)(y')^2\right) + a_1(x,y,y')(f'(y)y') + a_0(x,y,y') = 0, \tag{38}$$

where $f(y)$ is invertible function and $f \in C^2(J)$, for some open interval $J \in \mathbb{R}$. To solve this class of differential equations, we let $z = f(y)$. Then $z' = f'(y)y'$ and $z'' = f''(y)(y')^2 + f'(y)y''$. Moreover, we let $y = f^{-1}(z)$. Then $y' = \frac{z'}{f'(f^{-1}(z))}$. Hence, equation (38) can be transformed into the following differential equation:

$$\begin{aligned} a_2\left(x, f^{-1}(z), \frac{z'}{f'\left(f^{-1}(z)\right)}\right) z'' \;&+\; a_1\left(x, f^{-1}(z), \frac{z'}{f'\left(f^{-1}(z)\right)}\right) z' \\ &+\; a_0\left(x, f^{-1}(z), \frac{z'}{f'\left(f^{-1}(z)\right)}\right) = 0. \end{aligned} \tag{39}$$

Assuming that (39) is exact, i.e., the conditions (12) hold. Then its first integral exists, and so, it can reduced to a first order differential equation which in turn can be solved by using elementary techniques. To explain this procedure, we present the following illustrative examples:

Example 4.1. Consider the second order nonlinear differential equation

$$\begin{cases} e^y\left[y''+(y')^2\right]+12xe^{4y}y'+\left(3e^{4y}-1\right)=0, \\ y(0)=\ln 2,\ y'(0)=0. \end{cases} \tag{40}$$

Let $z=e^y$. Then $z'=e^yy'$ and $z''=e^yy''+e^y(y')^2$. Hence, Eq. (40) becomes

$$\begin{cases} z''+12xz^3z'+\left(3z^4-1\right)=0, \\ z(0)=2,\ z'(0)=0. \end{cases} \tag{41}$$

Therefore, $a_2(x,z,z')=1$, $a_1(x,z,z')=12xz^3$, and $a_0(x,z,z')=\left(3z^4-1\right)$. In addition, we have

$$\frac{\partial a_2}{\partial z}=\frac{\partial a_1}{\partial z'}=0,\ \ \frac{\partial a_2}{\partial x}=\frac{\partial a_0}{\partial z'}=0,\ \text{and}\ \ \frac{\partial a_1}{\partial x}=\frac{\partial a_0}{\partial z}=12z^3, \tag{42}$$

Hence, equation (41) is exact differential equation. Hence, its first integral is given by

$$z'+3xz^4-x=0.$$

For which an implicit solution can be obtained by separating the variables, and hence, $y(x)=\ln(z(x))$.

Remark 4.1. Assume that (39) is not exact. Then an integrating factor of (39) could be exist. Hence, it can be transformed into an exact differential equation [1]. To explain the procedure of solving (39), we present the following example:

Example 4.2. Consider the second order nonlinear differential equation

$$xe^y\left(2x+e^y\right)\left(y''+(y')^2\right)+x\left(x+e^y\right)y'+\left(3x+e^y\right)=0. \tag{43}$$

By multiplying this equation by e^y, we get

$$xe^{2y}\left(2x+e^y\right)\left(y''+(y')^2\right)+x\left(x+e^y\right)e^yy'+e^y\left(3x+e^y\right)=0. \tag{44}$$

Let $z=e^y$. Then $z'=e^yy'$ and $z''=e^yy''+e^y(y')^2$. Hence, by substituting z, z' and z'' in (44), we get

$$xz(2x+z)z''+x(x+z)z'+z(3x+z)=0. \tag{45}$$

This equation is not exact since $\frac{\partial a_2}{\partial z}=2(x+z)\neq 0=\frac{\partial a_1}{\partial z'}$. An integrating factor of this second order nonlinear differential equation exists, and it is given by $\mu(x,z)=\frac{1}{xz(2x+z)}$. Multiplying (45) by $\mu(x,z)$, we get

$$z''+\frac{(x+z)}{z(2x+z)}z'+\frac{(3x+z)}{x(2x+z)}=0. \tag{46}$$

Clearly,

$$\frac{\partial a_2}{\partial z} = \frac{\partial a_1}{\partial z'} = 0, \ \frac{\partial a_2}{\partial x} = \frac{\partial a_0}{\partial z'} = 0, \text{ and } \frac{\partial a_1}{\partial x} = \frac{\partial a_0}{\partial z} = \frac{-1}{(2x+z)^2}. \tag{47}$$

Therefore, the differential equation (46) is exact, and its first integral is given by

$$c = z' + \ln\left(xz\sqrt{2x+z}\right). \tag{48}$$

This first order differential equation can be solved by using the elementary techniques of solving first order differential equations. Hence, $y(x) = \ln(z(x))$.

Finally, we consider the nonhomogeneous second order linear differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = h(x),$$

where $a_2(x) \neq 0$, $a_1(x)$, and $a_0(x)$ are differentiable functions on an open interval $(a,b) \subset \mathbb{R}$. This equation admits an integrating factor $\mu(x) = \frac{1}{a_2(x)}$ provided that $W(a_2,a_1)(x) = a_0(x)a_2(x)$, where $W(a_2,a_1)(x) = a_2(x)a_1'(x) - a_1(x)a_2'(x)$. For this case, we present the following example:

Example 4.3. consider the second order linear differential equation

$$e^x y'' + \cos xy' - (\cos x + \sin x)y = h(x).$$

By multiplying this equation by the integrating factor e^{-x}, we get

$$y'' + e^{-x}\cos xy' - e^{-x}(\cos x + \sin x)y = h(x)e^{-x}.$$

This equation can be written as

$$\frac{d}{dx}\left[y' + (e^{-x}\cos x)y\right] = h(x)e^{-x}$$

Hence, its first integral is given by

$$y' + (e^{-x}\cos x)y = \int^x h(\xi)e^{-\xi}d\xi + c_1$$

which can be solved by using the elementary techniques of solving first order differential equations.

5. Concluding remarks

In this paper, we solved some classes of second order differential equation. In fact, we solved the following classes of second order differential equations:

1. The Chebyshev's type of second order differential equations which is given by

$$p(x)y''(x) + \frac{1}{2}p'(x)y'(x) + f(\sqrt{p(x)}\, y'(x), y(x)) = 0, \quad x \in (a, b), \tag{49}$$

   where $p(x)$ is a positive and differentiable function on the some open interval $I \subset \mathbb{R}$, and $f(\sqrt{p(x)}\, y'(x), y(x))$ is a continuous function on some domain $D \subset \mathbb{R}^2$.

2. The $f-$types of second order differential equations:

   a) the first type is given by the second order differential equation

$$a_2\left(f'(y)y'' + (y')^2 f''(y)\right) + a_1 f'(y)y' + a_0 f(y) = g(x), \tag{50}$$

   where a_2, a_1 and a_0 are constants, and the function f is a second differentiable function of on some open interval $J \subset \mathbb{R}$, and

   b) the second type is given by the second order differential equation

$$p(x)\left(f'(y)y'' + (y')^2 f''(y)\right) + \frac{1}{2}p'(x)f'(y)y' + a_0 f(y) = 0, \tag{51}$$

   where $p(x)$ is a positive and differentiable function on some open interval $I \subset \mathbb{R}$, and f is a second differentiable function on some open interval $J \subset \mathbb{R}$.

3. f-type of second order differential equations that can be transformed into exact second order Differential Equations. Particularly, the class of differential equations

$$a_2(x, y, y')\left(f'(y)y'' + f''(y)(y')^2\right) + a_1(x, y, y')(f'(y)y') + a_0(x, y, y') = 0, \tag{52}$$

   where the function $f(y)$ is an invertible function and is second differentiable function on some open interval $J \subset \mathbb{R}$.

In the paper, we also presented some illustrative examples.

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