A study on soft rough BCK-algebras in BCK-algebras

Jingru Wang
School of Information and Mathematics
Yangtze University
Jingzhou, 434023
P.R. China
wangjingru@hnu.edu.cn

Kuanyun Zhu*
School of Information and Mathematics
Yangtze University
Jingzhou, 434023
P.R. China
kyzhu@whu.edu.cn

Abstract. In this paper, the notion of soft rough BCK-algebras is introduced, which is an extended notion of subalgebras in BCK-algebras. Moreover, in order to illustrate the roughness in BCK-algebras with respect to MS-approximation spaces in BCK-algebras, we first introduce C-soft sets and CC-soft sets as two special kinds of soft sets in BCK-algebras. Some new soft rough operations in BCK-algebras are explored. In particular, lower and upper soft rough BCK-algebras with respect to another soft set are investigated.
Keywords: soft set, rough set, soft rough set, MSR-set, BCK-algebra.

1. Introduction

It is well known that, classical methods are not always successful in dealing with the problems in economy, engineering and social science, because of various types of uncertainties presented in these problems. As far as known that there are several theories to describe uncertainty, for example, fuzzy set theory [31], rough set theory [27] and other mathematical tools. However, the theories mentioned above have their limitations. In 1999, Molodtsov [24] put forward soft set theory as a new mathematical tool for dealing with uncertainties. Nowadays, research on soft sets is progressing rapidly. In 2003, Maji et al. [22] proposed some basic operations. Further, Ali et al. [1] revised some operations. In 2011, Ali [2] studied another view on reduction of parameters in soft sets. Afterwards, a wide range of applications of soft sets have been studied in many different fields including game theory, probability theory, smoothness of functions, operation researches, Riemann integrations and measurement theory and so on. Recently, there has been a rapid growth of interest in soft set theory and its applications,

∗. Corresponding author

such as [4, 5, 6, 23, 28]. In particular, Zhan and Zhu [34] reviewed on decision making methods based on (fuzzy) soft sets and rough soft sets. At the same time, many researchers applied this theory to algebraic structures [16, 17].

The concept of rough set theory was first proposed by Pawlak [27] as an approach to deal with inexact and uncertain knowledge. It is well known that, an equivalence relation on set into disjoint classes and vice versa. The Pawlak approximation operators are defined by an equivalence relation. However, these equivalence relations in Pawlak rough sets are restrictive for many applied areas. Hence, some more general models have been proposed, such as [35, 36, 37]. In 2010, Herawan et al. [12] studied a rough set approach for selecting clustering attribute. In 2013, Ali et al. [3] investigated some properties of generalized rough sets. Nowadays, this theory has been applied to many fields, such as patter recognition, intelligent systems, machine learning, image processing, cognitive science, signal analysis and so on. On the other hand, many researchers applied this theory to algebraic structures in many papers, such as [7, 8, 15].

As far as known that research of t-norm based on logical systems has become increasingly more important in the field of logic. It is well known that BCK and BCI-algebras are two classes of logic algebras which were introduced by Imai and Iseki [13, 14]. These two classes of logical algebras have been investigated by many researchers, see [20, 21, 30]. Most of the algebras related to the t-norm based logic, such as MTL-algebras, BL-algebras, MV-algebras and Boolean algebras are extensions of BCK-algebras, which shows that BCK/BCI-algebras are considerably general structures, and it means that this is an important topic on these two kinds of logical algebras.

Soft set theory and rough set theory are all mathematical tools to deal with uncertainty. In 2010, Feng et al. [9] provided a framework to combine rough sets with soft sets, which gives rise to some interesting new concepts such as rough soft sets, soft rough sets and soft rough fuzzy sets. In 2014, Li and Xie [18] investigated the relationship among soft sets, soft rough sets and topologies. In [19], Ma and Zhan put forth rough soft BCI-algebras by means of an ideal of BCI-algebra. In recent years, Shabir et al. [29] pointed out that there exist some problems on Feng's soft rough set as follows: (1) An upper approximation of a non-empty set may be empty. (2) The upper approximation of a subset X may not contain the set X. In order to solve these problems, Shabir modified the concept of soft rough set, which is called an MSR-set. The underlying concepts are very similar to Pawlak rough sets.

Based on the above idea, in this paper, we provided a framework to combine rough sets, soft sets with BCK-algebras, the notion of soft rough BCK-algebras is introduced, and we show that this is an extended notion of subalgebras in BCK-algebras. This paper is organized as follows: In Section 2, we recall some concepts and results in BCK-algebras, soft sets and rough sets. In Section 3, we study some operations with respect to MS-approximation spaces and some new soft rough operations in BCK-algebras are explored. Further, lower and upper soft rough BCK-algebras are investigated in Section 4. In particular, in Section 5, we discuss soft rough BCK-algebras based on a soft set.

2. Preliminaries

In this section, we review some basis notions about BCK-algebras, soft sets and rough sets.

An algebra $(X, *, 0)$ of type (2,0) is called a BCK-algebra [14] if it satisfies the following conditions:
(1) $((x * y) * (x * z)) * (z * y) = 0$,
(2) $(x * (x * y)) * y = 0$,
(3) $x * x = 0$,
(4) $x * y = 0$ and $y * x = 0$ imply $x = y$.
(5) $0 * x = 0$,
For all $x, y, z \in X$.

In a BCK-algebra X, we can define a partial order $\leq$ by putting $x \leq y$ if and only if $x * y = 0$. In this paper, X is always a BCK-algebra.

A non-empty subset S of X is called a subalgebra of X if $x*y \in S$ whenever $x, y \in S$. A non-empty subset I of X is called an ideal of X, denoted by $I \lhd X$, if it satisfies: (1) $0 \in I$; (2) $x * y \in I$ and $y \in I$ imply $x \in I$ for all $x, y \in X$.

Definition 2.1 ([24]). *A pair $\mathfrak{S} = (F, A)$ is called a soft set over U, where $A \subseteq E$ and $F : A \to \mathscr{P}(U)$ is a set-valued mapping.*

For a soft set $\mathfrak{S} = (F, A)$, the set $\mathrm{Supp}(F, A) = \{x \in A | F(x) \neq \emptyset\}$ is called a soft support of the soft set (F, A).

Definition 2.2 ([10]). *A soft set $\mathfrak{S} = (F, A)$ over U is called a full soft set if $\bigcup_{a \in A} F(a) = U$.*

Definition 2.3 ([16, 17]). *Let $\mathfrak{S} = (F, A)$ be a soft set over X. Then*

(i) *(F, A) is called a soft BCK-algebra over X if $F(x)$ is a subalgebra of X for all $x \in$ Supp(F, A),*

(ii) *(F, A) is called a soft ideal over X if $F(x)$ is an ideal of X for all $x \in$ Supp(F, A).*

Definition 2.4 ([27]). *Let R be an equivalence relation on the universe U and (U, R) be a Pawlak approximation space. A subset $X \subseteq U$ is called definable if $\underline{R}(X) = \overline{R}(X)$; otherwise, i.e., if $\underline{R}(X) - \overline{R}(X) \neq \emptyset$, X is said to be a rough set, where the two operators are defined as:*

$$\underline{R}(X) = \{x \in U : [x]_R \subseteq X\},$$

$$\overline{R}(X) = \{x \in U : [x]_R \cap X \neq \emptyset\}.$$

Definition 2.5 ([9]). *Let $\mathfrak{S} = (F, A)$ be a soft set over U. Then the pair $P = (U, \mathfrak{S})$ is called a soft approximation space. Based on P, we define the following two operators:*

$$\underline{apr}_P(X) = \{u \in U : \exists a \in A[u \in F(a) \subseteq X]\}$$

$$\overline{apr}_P(X) = \{u \in U : \exists a \in A[u \in F(a), F(a) \cap X \neq \emptyset]\},$$

assigning to every subset $X \subseteq U$.

Two sets $\underline{apr}_P(X)$ and $\overline{apr}_P(X)$ called the lower and upper soft rough approximations of X in P, respectively. If $\underline{apr}_P(X) = \overline{apr}_P(X)$, X is said to be soft definable; otherwise X is called a soft rough set. In what follows, we call it Feng-soft rough set.

In order to resolve theoretical and practical aspects, we usually require the soft set to be full in the above definition. If not, it is often limits the research value by means of Feng-soft rough sets, which can be found in the following example.

Example 2.6. Let $\mathfrak{S} = (F, A)$ be a soft set over U which is given by Table 1.

Table 1 Soft set $\mathfrak{S}$

	u_1	u_2	u_3	u_4	u_5	u_6	u_7
e_1	1	0	1	0	1	0	0
e_2	0	1	0	0	0	1	0
e_3	1	0	1	0	1	0	1

Assume that $P = (U, \mathfrak{S})$ is a soft approximation space, we can see that the $\mathfrak{S}$ is not full.

Then for $X = \{u_1, u_2, u_4, u_6\}$. It follows from Definition 2.5 that $\underline{apr}_P(X) = \{u_2, u_6\}$ and $\overline{apr}_P(X) = \{u_1, u_2, u_3, u_5, u_6, u_7\}$. It's just a shame that $X \not\subseteq \overline{apr}_P(X)$. In order to avoid this situations, in 2013, Shabir discuss another approach to soft rough sets as follows.

Definition 2.7 ([29])**.** *Let (F, A) be a soft set over U and $\xi : U \to \mathscr{P}(A)$ be a mapping defined as $\xi(x) = \{a : x \in F(a)\}$. Then the pair (U, ξ) is called MS-approximation space and for any $X \subseteq U$, the lower MSR-approximation and upper MSR-approximation of X are denoted by $\underline{X}_\xi$ and $\overline{X}_\xi$, respectively, which two operators are defined as*

$$\underline{X}_\xi = \{x \in X | \xi(x) \neq \xi(y) \text{ for all } y \in X^c\}$$

and

$$\overline{X}_\xi = \{x \in U | \xi(x) = \xi(y) \text{ for some } y \in X\}.$$

If $\underline{X}_\xi = \overline{X}_\xi$, then X is said to be MS-definable, otherwise, X is said to be an MSR-set. In what follows, we call it Shabir-soft rough set.

3. Soft rough sets in BCK-algebras

In this section, we investigate some operations and fundamental properties of soft rough sets in BCK-algebras. Meanwhile, some examples are given. Firstly, we give the concept of soft rough sets in BCK-algebras.

Definition 3.1. *Let (F, A) be a soft set over X and $\xi : X \to \mathscr{P}(A)$ be a mapping defined as $\xi(x) = \{a : x \in F(a)\}$. Then the pair (X, ξ) is called MS-approximation space and for any $Y \subseteq X$, the lower MSR-approximation and upper MSR-approximation of Y are denoted by $\underline{Y}_\xi$ and $\overline{Y}_\xi$, respectively, which are two operators are defined as*

$$\underline{Y}_\xi = \{x \in Y | \xi(x) \neq \xi(y) \text{ for all } y \in Y^c\}$$

and

$$\overline{Y}_\xi = \{x \in X | \xi(x) = \xi(y) \text{ for some } y \in Y\}.$$

If $\underline{Y}_\xi = \overline{Y}_\xi$, then Y is said to be MS-definable, otherwise, Y is said to be an MSR-set over X.

Remark 3.2. It follows from Definition 3.1 that for any $Y \subseteq X$, we have $\underline{Y}_\xi \subseteq Y \subseteq \overline{Y}_\xi$.

Now, we study some basic properties of lower and upper MS-approximations of a subset Y of a BCK-algebra X. In order to illustrate the roughness in X w.r.t. MS-approximation spaces in BCK-algebras, we first introduce two special kinds of soft sets in BCK-algebras.

Definition 3.3. *Let $\mathfrak{S} = (F, A)$ be a soft set in X and $\xi : X \to \mathscr{P}(A)$ be a mapping defined as $\xi(x) = \{a : x \in F(a)\}$. Then $\mathfrak{S}$ is called a C-soft set over X if $\xi(a) = \xi(b)$ and $\xi(c) = \xi(d)$ imply $\xi(a * c) = \xi(b * d)$ for all $a, b, c, d \in X$.*

Example 3.4. Let $X = \{0, a, b\}$ be a BCK-algebra with the following Cayley Table 2.

Table 2 BCK-algebra X

$*$	0	a	b
0	0	0	0
a	a	0	a
b	b	b	0

Define a soft set $\mathfrak{S} = (F, A)$ over X which is given by Table 3.

Table 3 Soft set $\mathfrak{S}$

	0	a	b
e_1	1	1	1
e_2	0	1	1
e_3	1	0	0

Then the mapping $\varphi : X \to \mathscr{P}(A)$ in an MS-approximation space (X, φ) is given by $\varphi(0) = \{e_1, e_3\}$, $\varphi(a) = \varphi(b) = \{e_1, e_2\}$. Then we can check that $\mathfrak{S}$ is not a C-soft set over X. In fact, $\varphi(a) = \varphi(b)$ and $\varphi(a) = \varphi(a)$ but $\varphi(a * a) = \varphi(0) \neq \varphi(b) = \varphi(b * a)$.

Example 3.5. Let $X = \{0, a, b, c\}$ be a BCK-algebra with the following Cayley Table 4.

Table 4 BCK-algebra X

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	b
c	c	c	c	0

Define a soft set $\mathfrak{S} = (F, A)$ over X which is given by Table 5.

Table 5 Soft set $\mathfrak{S}$

	0	a	b	c
e_1	1	1	1	1
e_2	1	1	1	1
e_3	0	0	1	1

Then the mapping $\varphi : X \to \mathscr{P}(A)$ in an MS-approximation space (X, φ) is given by $\varphi(0) = \varphi(a) = \{e_1, e_2\}$, $\varphi(b) = \varphi(c) = \{e_1, e_2, e_3\}$. Then we can check that $\mathfrak{S}$ is a C-soft set over X.

Let Y, Z be any two non-empty subsets in X. Denote $Y * Z = \{y * z | \forall y \in Y.z \in Z\}$.

Theorem 3.6. *Let $\mathfrak{S} = (F, A)$ be a C-soft set over X, Y and Z any two non-empty subsets in X. Then*

$$\overline{Y}_\xi * \overline{Z}_\xi \subseteq \overline{Y * Z}_\xi.$$

Proof. Let $d \in \overline{Y}_\xi \cdot \overline{Z}_\xi$. Then $d = a*b$, where $a \in \overline{Y}_\xi$ and $b \in \overline{Z}_\xi$. It follows from Definition 3.1 that $\xi(a) = \xi(y)$ and $\xi(b) = \xi(z)$ for some $y \in Y$, $z \in Z$. Since $\mathfrak{S}$ is a C-soft set, $\xi(a*b) = \xi(y*z)$ for some $y*z \in Y*Z$. Thus $d = a*b \in \overline{Y * Z}_\xi$. That is, $\overline{Y}_\xi * \overline{Z}_\xi \subseteq \overline{Y * Z}_\xi$.. $\square$

The following example shows that the containment in Theorem 3.6 is proper.

Example 3.7. Let $X = \{0, a, b, c\}$ be a BCK-algebra with the following Cayley Table 6.

Table 6 BCK-algebra X

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	a	0	b
c	c	c	c	0

Define a soft set $\mathfrak{S} = (F, A)$ over X which is given by Table 7.

Table 7 Soft set $\mathfrak{S}$

	0	a	b	c
e_1	1	1	1	1
e_2	1	1	1	1
e_3	0	0	0	1

Then the mapping $\varphi : X \to \mathscr{P}(A)$ in an MS-approximation space (X, φ) is given by $\varphi(0) = \varphi(a) = \varphi(b) = \{e_1, e_2\}$, $\varphi(c) = \{e_1, e_2, e_3\}$. Then we can check that $\mathfrak{S}$ is a C-soft set over X.

If we take $Y = \{0, c\}$ and $Z = \{c\}$, then $\overline{Y}_\xi = \{0, a, b, c\}$ and $\overline{Z}_\xi = \{c\}$. So $\overline{Y}_\xi * \overline{Z}_\xi = \{0, c\}$. On the other hand, $\overline{Y * Z}_\xi = \{0, a, b, c\}$. Thus $\overline{Y}_\xi * \overline{Z}_\xi \subsetneq \overline{Y * Z}_\xi$.

Next, we introduce the other kind of soft sets in BCK-algebras.

Definition 3.8. *Let $\mathfrak{S} = (F, A)$ be a C-soft set over X and $\xi : X \to \mathscr{P}(A)$ be a mapping defined as $\xi(x) = \{a : x \in F(a)\}$. Then $\mathfrak{S}$ is called a CC-soft set over X if for all $c \in X$, $\xi(c) = \xi(x * y)$ for $x, y \in R$, there exist $a, b \in X$ such that $\xi(x) = \xi(a)$ and $\xi(y) = \xi(b)$ satisfying $c = a * b$.*

Remark 3.9. (1) $\mathfrak{S}$ in Example 3.7 is a C-soft set over X, however, it is not a CC-soft set.

(2) $\mathfrak{S}$ in Example 3.5 is a CC-soft set over X.

If we strength the condition, we can obtain the following result.

Theorem 3.10. *Let $\mathfrak{S} = (F, A)$ be a CC-soft set over X and Y, Z any two non-empty subsets in X. Then*

$$\overline{Y}_\xi * \overline{Z}_\xi = \overline{Y * Z}_\xi.$$

Proof. It follows from Theorem 3.6 that we only need to show $\overline{Y * Z}_\xi \subseteq \overline{Y}_\xi * \overline{Z}_\xi$. Now let $c \in \overline{Y * Z}_\xi$. Thus $\xi(c) = \xi(y * z)$ for some $y \in Y$ and $z \in Z$. Then there exist $a, b \in X$, such that $\xi(a) = \xi(y)$ and $\xi(b) = \xi(z)$ satisfying $c = a * b$ since $\mathfrak{S}$ is a CC-soft set over X. Thus $a \in \overline{Y}_\xi$ and $b \in \overline{Z}_\xi$. Hence $c = a * b \in \overline{Y}_\xi * \overline{Z}_\xi$. So $\overline{Y}_\xi * \overline{Z}_\xi = \overline{Y * Z}_\xi$. □

Now, we consider lower MSR-approximations over BCK-algebras.

Theorem 3.11. *Let $\mathfrak{S} = (F, A)$ be a CC-soft set over X and Y, Z any two non-empty subsets in X. Then*

$$\underline{X}_\xi * \underline{Y}_\xi \subseteq \underline{X * Y}_\xi.$$

Proof. We suppose that $\underline{X}_\xi * \underline{Y}_\xi \subseteq \underline{X * Y}_\xi$ is false, then there exists $c \in \underline{Y}_\xi * \underline{Z}_\xi$ but $c \notin \underline{X * Y}_\xi$. Then $c = a * b$, where $a \in \underline{Y}_\xi$ and $b \in \underline{Z}_\xi$, and so $\xi(a) \neq \xi(y)$ and $\xi(b) \neq \xi(z)$ for all $y \in Y^c$ and $z \in Z^c$. $(\star)$

On the other hand, $c \notin \underline{X * Y}_\xi$, then we may have the following two conditions:

(i) $c \notin Y * Z$, which contradicts with $c \in \underline{Y}_\xi * \underline{Z}_\xi \subseteq Y \cdot Z$;

(ii) $c \in Y * Y$ and $\xi(c) = \xi(y' * z')$ for some $y' * z'_i \in (X * Y)^c$. Thus $y' \in Y^c$ or $z'_i \in Z^c$. In fact, if $y' \notin Y^c$ and $z'_i \notin Z^c$, we have $x' * y' \in Y * Z$, a contradiction. Since $\mathfrak{S} = (F, A)$ is a CC-soft set over X, there exist $a', b' \in X$ such that $\xi(a') = \xi(y')$ and $\xi(b') = \xi(z'_i)$ satisfying $a' * b' = c$, for some $y' \in Y^c$ or $z'_i \in Z^c$. This is contradiction with $(\star)$. Hence $\underline{Y}_\xi * \underline{Z}_\xi \subseteq \underline{Y * Z}_\xi$. □

If $\mathfrak{S}$ is not a CC-soft set over X, then Theorem 3.11 is not true. See the following example.

Example 3.12. Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley Table 8.

Table 8 BCK-algebra X

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	0
c	c	c	c	0	0
d	d	d	d	c	0

Define a soft set $\mathfrak{S} = (F, A)$ over X which is given by Table 7.

Table 7 Soft set $\mathfrak{S}$

	0	a	b	c	d
e_1	0	1	1	1	1
e_2	1	1	0	0	1
e_3	0	0	1	1	1

Then the mapping $\xi : X \to \mathscr{P}(A)$ in MS-approximation space (X, ξ) is given by $\xi(0) = \{e_2\}$, $\xi(a) = \{e_1, e_2\}$ $\xi(b) = \xi(c) = \{e_1, e_3\}$, $\xi(d) = \{e_1, e_2, e_3\}$. Then we can check that $\mathfrak{S}$ is not a CC-soft set over X.

If we take $Y = \{0, b\}$ and $Z = \{b, d\}$, then $\underline{Y}_\xi = \{0\}$ and $\underline{Z}_\xi = \{c\}$. So $\underline{Y}_\xi * \underline{Z}_\xi = \{c\}$. On the other hand, $\underline{Y * Z}_\xi = \{0\}$. Thus $\underline{Y}_\xi * \underline{Z}_\xi \nsubseteq \underline{Y * Z}_\xi$.

The following example shows that the containment in Theorem 3.11 and is proper.

Example 3.13. Consider the BCK-algebra X and the soft set $\mathfrak{S} = (F, A)$ in Example 3.5. Then we know that $\mathfrak{S}$ is a CC-soft set over X. If we take $Y = \{0, a, b\}$ and $Z = \{0, b, c\}$, then $\underline{Y}_\xi = \{0, a\}$ and $\underline{Z}_\xi = \{b, c\}$. So $\underline{Y}_\xi * \underline{Z}_\xi = \{b, c\}$. On the other hand, $\underline{Y * Z}_\xi = \{0, a, b, c\}$. Thus $\underline{Y}_\xi * \underline{Z}_\xi \subsetneqq \underline{Y * Z}_\xi$.

4. Characterizations of MSR-BCK-algebras in BCK-algebras

In this section, we characterize MSR-BCK-algebras in BCK-algebras. First, we give the notion of MSR-BCK-algebras as follows.

Definition 4.1. *In Definition 3.1, if* $\underline{Y}_\xi \neq \overline{Y}_\xi$*, then*

(i) Y is called a lower (upper) MSR-BCK-algebra (ideal) w.r.t. $\mathfrak{S}$ of X if $\underline{Y}_\xi$ ($\overline{Y}_\xi$) is a subalgebra (ideal) of X;

(ii) Y is called a MSR-BCK-algebra (ideal) w.r.t. $\mathfrak{S}$ of X if $\underline{Y}_\xi$ and $\overline{Y}_\xi$ are subalgebras (ideals) of X.

Example 4.2. Let $X = \{0, a, b, c, d\}$ be a BCK-algebra with the following Cayley Table 8.

Table 8 BCK-algebra X

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	0
b	b	b	0	0	b
c	c	b	a	0	b
d	d	a	d	a	0

Define a soft set $\mathfrak{S} = (F, A)$ over X which is given by Table 9.

Table 9 Soft set $\mathfrak{S}$

	0	a	b	c	d
e_1	1	1	0	1	0
e_2	0	1	0	1	0
e_3	1	1	1	1	1

Then the mapping $\xi : X \to \mathscr{P}(A)$ in an MS-approximation space (X, ξ) is given by $\xi(0) = \{e_1, e_3\}$, $\xi(a) = \xi(c) = \{e_1, e_2, e_3\}$, $\xi(b) = \xi(d) = \{e_3\}$. It follows from Definition 3.1 that for a set $Y = \{0, b, c, d\}$, we have

$$\underline{Y}_\xi = \{0, b, d\} \text{ and } \overline{Y}_\xi = \{0, a, b, c, d\}.$$

It is easy to check that $\underline{Y}_\xi$ and $\overline{Y}_\xi$ are subalgebras of X. That is Y is an MSR-BCK-algebra of X.

Example 4.3. Consider the BCK-algebra in Example 3.5. Define a soft set $\mathfrak{S} = (F, A)$ over X which is given by Table 10.

Table 13 Soft set $\mathfrak{S}$

	0	a	b	c
e_1	1	1	1	1
e_2	0	0	1	1
e_3	1	1	1	1

Then the mapping $\xi : X \to \mathscr{P}(A)$ in an MS-approximation space (X, ξ) is given by $\xi(0) = \xi(a) = \{e_1, e_3\}$, $\xi(b) = \xi(c) = \{e_2, e_3\}$. It follows from Definition 3.1 that for a set $Y = \{0, a, b\}$, we have

$$\underline{Y}_\xi = \{0, a\} \lhd X \text{ and } \overline{Y}_\xi = \{0, a, b, c\} \lhd X.$$

It is easy to check that $\underline{Y}_\xi$ and $\overline{Y}_\xi$ are ideals of X. That is Y is a MSR-ideal of X.

Proposition 4.4. *Let (X, ξ) be an MS-approximation space. If Y and Z are lower MSR-BCK-algebras (ideals) over X, then so is $Y \cap Z$.*

Proof. It follows from Definition 4.1 that $\underline{Y}_\xi$ and $\underline{Z}_\xi$ are subalgebras (ideals) over X, so $\underline{Y}_\xi \cap \underline{Z}_\xi$ is a subalgebra (ideal) of X. It follows from Theorem 3 that in [29], we know that $\underline{X \cap Y}_\xi$ is also a subalgebra (ideal) of X. Thus $X \cap Y$ is a lower MSR-BCK-algebra (ideal) over X. □

In general, if Y and Z are upper MSR-BCK-algebras (ideals) over X, $Y \cap Z$ is not an upper MSR-BCK-algebras (ideals) over X. Actually, we have the following example.

Example 4.5. Consider the BCK-algebra X and the soft set $\mathfrak{S} = (F, A)$ in Example 4.3. Let $Y = \{0, c\}$ and $Z = \{a, c\}$. Then $\overline{Y}_\xi = \overline{Z}_\xi = \{0, a, b, c\}$ are subalgebras of X. Thus Y and Z are upper MSR-BCK-algebras over X. However, $\overline{X \cap Y}_\xi = \{b, c\}$ is not a subalgebra over X.

Finally, we study the upper and lower MSR-BCK-algebras.

Theorem 4.6. *Let $\mathfrak{S} = (F, A)$ be a C-soft set over X. If Y is a subalgebra of X, then Y is an upper MSR-BCK-algebra over X.*

Proof. Let $a, b \in \overline{Y}_\xi$. Then there exist $y, z \in Y$ such that $\xi(a) = \xi(y)$ and $\xi(b) = \xi(z)$. Since $\mathfrak{S}$ is a C-soft set over X, $\xi(a * b) = \xi(y * z)$. Thus, $y * z \in Y$ since Y is a subalgebra of X. Hence $a * b \in \overline{Y}_\xi$. This means that $\overline{Y}_\xi$ is a subalgebra over X. So Y is an upper soft rough BCK-algebra over X. □

Theorem 4.7. *Let $\mathfrak{S} = (F, A)$ be a CC-soft set over X. If Y is a subalgebra of X, then Y is a lower MSR-BCK-algebra over X when $\underline{Y}_\xi \neq \emptyset$.*

Proof. It follows from Theorem 3.11 that $\underline{Y}_\xi * \underline{Y}_\xi \subseteq \underline{Y * Y}_\xi$. Further, it follows from Theorem 3 that in [29], we know that $\underline{Y * Y}_\xi \subseteq \underline{Y}_\xi$ since $Y * Y \subseteq Y$. So $\underline{Y}_\xi * \underline{Y}_\xi \subseteq \underline{Y}_\xi$. Thus Y is a lower MSR-BCK-algebra over X. □

Remark 4.8. The above two theorems show that any MSR-BCK-algebra is a generalization of a subalgebra of BCK-algebras.

5. MSR-BCK-algebras with respect to another soft set

In this section, we investigate MSR-BCK-algebras based on another soft set.

Definition 5.1. *Let $\mathfrak{S} = (F, A)$ be a soft set over X and $\xi : X \to \mathscr{P}(A)$ be a mapping defined as $\xi(x) = \{a : x \in F(a)\}$. Let $\mathfrak{T} = (G, B)$ be another soft set defined over X. The lower and upper MSR-approximations of $\mathfrak{T}$ with respect to $\mathfrak{S}$ are denoted by $\underline{(G, B)_\xi} = (\underline{G_\xi}, B)$ and $\overline{(G, B)_\xi} = (\overline{G_\xi}, B)$, respectively, which are two operators defined as*

$$\underline{G(e)_\xi} = \{x \in G(e) | \xi(x) \neq \xi(y) \text{ for all } y \in X - G(e)\}$$

and

$$\overline{G(e)_\xi} = \{x \in X | \xi(x) = \xi(y) \text{ for some } y \in G(e)\}$$

for all $e \in B, x \in X$.

(i) If $\underline{(G, B)_\xi} = \overline{(G, B)_\xi}$, then $\mathfrak{T}$ is called definable.

(ii) If $\underline{(G, B)_\xi} \neq \overline{(G, B)_\xi}$ and $\overline{G(e)_\xi}$ ($\underline{G(e)_\xi}$) is a subalgebra (ideal) of X for all $e \in B$, then $\mathfrak{T}$ is called a lower (upper) MSR-BCK-algebra (ideal) with respect to $\mathfrak{S}$ over X. Moreover, $\mathfrak{T}$ is called a lower (upper) MSR-BCK-algebra (ideal) with respect to $\mathfrak{S}$ over X if $\overline{G(e)_\xi}$ and $\underline{G(e)_\xi}$ are subalgebras (ideals) with respect to $\mathfrak{S}$ over X for all $e \in B$.

Example 5.2. We consider the BCK-algebra X and the soft set $\mathfrak{S} = (F, A)$ over X in Example 3.5. Define another soft set $\mathfrak{T} = (G, B)$ which is given by Table 14.

Table 14 Soft set $\mathfrak{T}$

	0	a	b	c
e_1	1	1	1	1
e_2	1	0	1	0
e_3	1	1	0	0
e_4	1	1	0	0

By calculating, $\underline{G(e_1)}_\xi = \emptyset$, $\overline{G(e_1)}_\xi = \{0, a\}$, $\underline{G(e_2)}_\xi = \{0, a\}$, $\overline{G(e_2)}_\xi = \{0, a\}$, $\underline{G(e_3)}_\xi = \emptyset$, $\overline{G(e_3)}_\xi = \{0, a, b, c\}$, $\underline{G(e_4)}_\xi = \emptyset$, $\overline{G(e_3)}_\xi = \{0, a, b, c\}$.

It is easy to check that $\underline{(G, B)}_\xi$ and $\overline{(G, B)}_\xi$ are subalgebras and ideals of X for all $e \in B$. In other words, $\mathfrak{T} = (G, B)$ is a MSR-BCK-algebra and MSR-BCK-ideal with respect to $\mathfrak{S}$ over X.

Definition 5.3. *Let $\mathfrak{T} = (G, B)$ and $\mathfrak{I} = (H, C)$ be two soft sets over X with $D = B \cap C \neq \emptyset$. The product operation $*$ is defined as $\mathfrak{T} * \mathfrak{I} = (G, B) * (H, C) = (K, D)$, where $K(a) = G(a) * H(a)$ for all $a \in D$.*

Theorem 5.4. *Let $\mathfrak{S} = (F, A)$ be a C-soft set over X and (X, ξ) be an MS-approximation space. Let $\mathfrak{T}_1 = (G_1, B)$ and $\mathfrak{T}_2 = (G_2, C)$ be two soft sets over X with $D = B \cap C \neq \emptyset$. Then*

$$\overline{(G_1, B)}_\xi * \overline{(G_2, C)}_\xi \subseteq \overline{(G_1 * G_2, D)}_\xi,$$

where $(G_1 * G_2)(e) = G_1(e) * G_2(e)$ for all $e \in D$.

Proof. The proof is similar to the one of Theorem 3.6. □

The following example shows that the containment in Theorem 5.4 is proper.

Example 5.5. Consider the BCK-algebra X in Example 3.12 and define a soft set $\mathfrak{S} = (F, A)$ over X which is given by Table 15.

Table 14 Soft set $\mathfrak{S}$

	0	a	b	c	d
e_1	1	1	1	0	0
e_2	0	0	0	1	1
e_3	1	1	1	1	1
e_4	0	0	0	1	1

Then the mapping $\xi : X \to \mathscr{P}(A)$ in MS-approximation space (X, ξ) is given by $\xi(0) = \xi(a) = \xi(b) = \{e_1, e_3\}$, $\xi(c) = \xi(d) = \{e_2, e_3, e_4\}$. Then we can check that $\mathfrak{S} = (F, A)$ is a C-soft set over X. However, it is not a CC-soft set over X.

Define two soft sets $\mathfrak{T}_1 = (G_1, B)$ and $\mathfrak{T}_2 = (G_2, C)$ over X, where $B = \{e_1, e_2\}$ and $C = \{e_2, e_3\}$ with $B \cap C = \{e_2\}$ by $G_1(e_2) = \{c\}$ and $G_2(e_2) = \{c\}$. By calculating, $\overline{G_1(e_2)}_\xi = \{c, d\}$ and $\overline{G_2(e_2)}_\xi = \{c, d\}$. Thus, $\overline{G_1(e_2)}_\xi * \overline{G_1(e_2)}_\xi = \{0, a\}$. However, $G_1(e_2) * G_2(e_2) = \{0\}$, $\overline{G_1(e_2) * G_2(e_2)} = \overline{\{0\}} = \{0, a, b\}$. Thus $\overline{(G_1, B)}_\xi * \overline{(G_2, C)}_\xi \subseteq \overline{(G_1 * G_2, D)}_\xi$.

If we strength the condition, we can obtain the following result.

Theorem 5.6. *Let $\mathfrak{S} = (F, A)$ be a CC-soft set over X and (X, ξ) be an MS-approximation space. Let $\mathfrak{T}_1 = (G_1, B)$ and $\mathfrak{T}_2 = (G_2, C)$ be two soft sets over X with $D = B \cap C \neq \emptyset$. Then*

$$\overline{(G_1, B)}_\xi * \overline{(G_2, C)}_\xi = \overline{(G_1 * G_2, D)}_\xi.$$

Proof. The proof is similar to the one of Theorem 3.10. □

Theorem 5.7. *Let $\mathfrak{S} = (F, A)$ be a CC-soft set over X and (X, ξ) be an MS-approximation space. Let $\mathfrak{T}_1 = (G_1, B)$ and $\mathfrak{T}_2 = (G_2, C)$ be two soft sets over X with $D = B \cap C \neq \emptyset$. Then $\underline{(G_1, B)}_\xi * \underline{(G_2, C)}_\xi \subseteq \underline{(G_1 * G_2, D)}_\xi$.*

Proof. The proof is similar to the one of Theorem 3.11. □

Finally, we investigate the lower and upper MSR-BCK-algebras with respect to another soft set.

Theorem 5.8. *Let $\mathfrak{S} = (F, A)$ be a C-soft set over X and and (X, ξ) be an MS-approximation space. If $\mathfrak{T} = (G, B)$ is a soft BCK-algebra over X. then $\mathfrak{T}$ is an upper MSR-BCK-algebra over X w.r.t. $\mathfrak{S}$.*

Proof. The proof is similar to the one of Theorem 4.6. □

Theorem 5.9. *Let $\mathfrak{S} = (F, A)$ be a CC-soft set over X and and (X, ξ) be an MS-approximation space. If $\mathfrak{T} = (G, B)$ is a soft BCK-algebra over X. then $\mathfrak{T}$ is a lower MSR-BCK-algebra over X w.r.t. $\mathfrak{S}$ when $\underline{\mathfrak{T}}_\xi \neq \emptyset$.*

Proof. The proof is similar to the one of Theorem 4.7. □

6. Conclusion

This paper is intended to apply soft rough set theory to BCK-algebras and proposes the notion of MSR-BCK-algebras. We discuss some operational properties and algebraic structures of lower and upper soft rough approximations in BCK-algebras. In particular, we discuss MSR-BCK-algebras based on another soft set.

We hope that our results given in this paper would constitute a foundation of studying other more complicated logic algebra structure in different areas.

As an extension of this work, the following topics maybe considered:

(1) Constructing soft rough sets to other algebras, such as groups, hyperrings, BL-algebras and so on;

(2) Investigating decision making methods based on soft rough sets;

(3) Establishing soft rough sets to some applied some areas of applications, such as information sciences, intelligent systems and so on.

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