Fixed point results for mappings satisfying Ciric and Hardy Roger type contractions

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Abstract. The aim of this paper is to establish some common fixed point results for generalized Ciric and Hardy Roger type contraction in ordered complete metric space. An example is constructed which shows the novelty of our results. Our results generalize and extend the results of Altun et. al (J. Funct. Spaces, Article ID 6759320, 2016).
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1. Introduction

Fixed point Theory has a wide range of applications in the different fields of analysis. The most important tool in fixed point theory is Banach contraction principle. Many authors obtained fixed point results in various metric spaces under certain contractive conditions (see [1]-[15]).

Let $S : W \to W$ be a mapping. A point $u \in W$ is called a fixed point of S, if $u = Su$. Fixed point theorems are used to find the solution of different mathematical models. Ran and Reurings [11] proved a fixed point theorem in metric space endowed with a partial order and gave applications to matrix equations. Nieto et. al. [10] extended the result in [11] for nondecreasing

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mappings and applied it to obtain a unique solution for a 1st order ordinary differential equation with periodic boundary conditions.

In this paper, we obtained some common fixed point theorems for generalized Ciric and Hardy Roger type contraction endowed with ordered metric space. We start with some basic notions which will be needed in the sequel.

Definition 1.1 ([11])**.** Let (W, d) be a metric space. Then:

(i) A sequence $\{u_n\}$ in (W, d) is called Cauchy sequence if given $\varepsilon > 0$, there corresponds a natural number n_0 such that for all $n, m \geq n_0$ we have $d(u_m, u_n) < \varepsilon$.

(ii) A sequence $\{u_n\}$ converges to u if $\lim_{n\to\infty} d(u_n, u) = 0$.

(iii) (W, d) is called complete if every Cauchy sequence in W converges to a point $u \in W$.

Definition 1.2 ([1])**.** Let $\psi \in \Psi$ and Ψ denotes the set of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the conditions:

(Ψ_1) ψ is non-decreasing.

(Ψ_2) For all $t > 0$, we have $\mu_0(t) = \sum_{k=0}^{\infty} \psi^k(t) < \infty$. Where, ψ^k is the k^{th} iterate of ψ. The function $\psi \in \Psi$ is called comparison function.

Lemma 1.3 ([1])**.** Let $\psi \in \Psi$. Then:

(i) $\psi(t) < t$, for all $t > 0$,

(ii) $\psi(0) = 0$.

Definition 1.4 ([1])**.** Let W be a nonempty set. Then $\preceq$ is a partial order on W if:

(i) $u \preceq u$ for all $u \in W$.

(ii) $u \preceq v$ and $v \preceq u \Rightarrow u = v$ for all $(u, v) \in W \times W$.

(iii) $u \preceq v$ and $v \preceq w \Rightarrow u \preceq w$ for all $(u, v, w) \in W \times W \times W$.

Definition 1.5 ([3])**.** Let W be a nonempty set. Then $(W, \preceq, d)$ is called an ordered metric space if:

(i) d is a metric on W and (ii) $\preceq$ is a partial order on W.

Definition 1.6 ([1])**.** Let $S : W \to W$ be a function. Then S is level closed from left, if the set $levS_{\preceq} = \{u \in W : u \preceq Su\}$ is non-empty and closed.

2. Fixed point results for Ciric type contraction

In this section, we will prove fixed point result for generalized Ciric contraction in ordered metric space. Our result extend the result given in [1].

Theorem 2.1 Let $(W, \preceq, d)$ be an ordered complete metric space and $S, T : W \to W$ be the self mappings. Suppose that the following assertions hold:

(i) The operator $S : W \to W$ is level closed from left.

(ii) For every $u \in W$, we have $u \preceq Su \Longrightarrow Tu \succeq STu$,and $u \succeq Su \Longrightarrow Tu \preceq STu$.

(iii) There exists a function $\rho \in \Psi$ such that for every $(u, v) \in W \times W$, we have $d(Tu, Tv) \leq \rho(\max\{d(u, v), d(u, Tu), d(v, Tv)\})$, whenever $u \preceq Su$ and $v \succeq Sv$. Then S and T have a unique common fixed point.

Proof. Suppose that u_0 be the arbitrary element of $levS_{\preceq}$, that is, $u_0 \preceq Su_0$. From condition (ii), we have $u_1 \succeq Su_1$, where $u_1 = Tu_0$. Again from condition (ii), we have $u_2 \preceq Su_2$, where $u_2 = Tu_1$. Now, consider the Picard sequence $\{u_n\} \subset W$ define by $u_{n+1} = Tu_n$ where $n = 0, 1, 2, \cdots$. Continuing in this way, we get for even terms of sequence

$$u_{2n} \preceq Su_{2n}. \tag{2.1}$$

For odd terms of sequence, we have

$$u_{2n+1} \succeq Su_{2n+1}. \tag{2.2}$$

As inequalities (2.1) and (2.2) holds, so condition (iii) can be applied. Now, $d(u_{2n+1}, u_{2n+2}) = d(Tu_{2n}, Tu_{2n+1}) \leq \rho(\max\{d(u_{2n}, u_{2n+1}), d(u_{2n+1}, Tu_{2n+1}), d(u_{2n}, Tu_{2n})\}) = \rho(\max\{d(u_{2n}, u_{2n+1}), d(u_{2n+1}, u_{2n+2})\})$. If $\max\{d(u_{2n+1}, u_{2n}), d(u_{2n+1}, u_{2n+2})\} = d(u_{2n+1}, u_{2n+2})$, then a contradiction arises. Therefore,

$$d(u_{2n+1}, u_{2n+2}) \leq \rho(d(u_{2n}, u_{2n+1})). \tag{2.3}$$

As inequalities (2.1) and (2.2) holds, so $d(u_{2n}, u_{2n+1}) = d(Tu_{2n-1}, Tu_{2n})$ where $n = 0, 1, 2, 3, ... \leq \rho(\max\{d(u_{2n}, u_{2n-1}), d(u_{2n}, Tu_{2n}), d(u_{2n-1}, Tu_{2n-1}))$ $= \rho(\max\{d(u_{2n}, u_{2n-1}), d(u_{2n}, u_{2n+1})\})$. If $\max\{d(u_{2n}, u_{2n-1}), d(u_{2n}, u_{2n+1})\} = d(u_{2n}, u_{2n+1})$, then a contradiction arises. Therefore, $d(u_{2n}, u_{2n+1}) \leq \rho(d(u_{2n-1}, u_{2n}))$. As ρ is non-decreasing, so

$$\rho(d(u_{2n}, u_{2n+1})) \leq \rho(\rho(d(u_{2n-1}, u_{2n}))). \tag{2.4}$$

Using inequality (2.4) in (2.3), so inequality (2.3) becomes $d(u_{2n+1}, u_{2n+2}) \leq \rho^2(d(u_{2n-1}, u_{2n}))$. Continuing in this way, we have

$$d(u_{2n+1}, u_{2n+2}) \leq \rho^{2n+1}(d(u_0, u_1)). \tag{2.5}$$

Similarly, we have

$$d(u_{2n}, u_{2n+1}) \leq \rho^{2n}(d(u_0, u_1)). \tag{2.6}$$

Combining inequalities (2.5) and (2.6) , we have

$$d(u_n, u_{n+1}) \leq \rho^n(d(u_0, u_1) . \tag{2.7}$$

Now, let $d(u_0, \ u_1) = 0$. This implies that $u_0 = u_1$. As $u_1 = Tu_0$, so

$$u_0 = Tu_0. \tag{2.8}$$

So, u_0 is a fixed point for T. Now, $u_0 = u_1 \succeq Su_1 = Su_0$. This implies that $u_0 \succeq Su_0$, which further implies

$$u_0 = Su_0. \tag{2.9}$$

From inequalities (2.8) and (2.9), u_0 is a common fixed point for S and T. Now, if $d(u_0, u_1) \neq 0$. Then we assumed that $\delta = d(u_0, u_1) > 0$. So from inequality (2.7), we have $d(u_n, u_{n+1}) \leq \rho^n(\delta)$, where $n = 0, 1, 2, \cdots$. For $\epsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that $\sum_{k \geq n_0}^{\infty} \rho^k(\delta) < \epsilon$. Let $n, m \in \mathbb{N}$, such that $n + m > n \geq n_0$. Then

$$\begin{aligned} d(u_n, u_{n+m}) &\leq d(u_n, u_{n+1}) + d(u_{n+1}, u_{n+2}) + ... + d(u_{n+m-1}, u_{n+m}). \\ &\leq \rho^n(\delta) + \rho^{n+1}(\delta) + ... + \rho^{n+m-1}(\delta). \\ &= \sum_{i=n}^{n+m-1} \rho^i(\delta) \leq \sum_{k \geq n_0}^{\infty} \rho^k(\delta) < \epsilon. \end{aligned}$$

Therefore, the sequence $\{u_n\}$ is a Cauchy sequence in (W, d). So, there exists some $u^* \in W$ such that $\lim_{n\to\infty} d(u_n, u^*) = 0$. But, we know that $u_{2n} \in levS_{\preceq}$ where $n = 0, 1, 2, \cdots$. As $levS_{\preceq}$ is a closed set, and every closed set in a complete metric space is complete. Therefore $u^* \in levS_{\preceq}$. This implies that $u^* \preceq Su^*$. Now, we have

$$\begin{aligned} &d(u^*, Tu^*) \leq d(u^*, u_{2n+2}) + d(u_{2n+2}, Tu^*) \\ &\leq d(u^*, u_{2n+2}) + \rho(\max\{d(u_{2n+1}, u^*), d(u_{2n+1}, Tu_{2n+1}), d(u^*, Tu^*)\}). \end{aligned}$$

Letting $n \to \infty$, we have $d(u^*, Tu^*) \leq 0 + \rho(d(u^*, Tu^*))$. As $\rho(t) < t$, therefore $d(u^*, Tu^*) = 0$. This implies that $u^* = Tu^*$. As, $u^* \preceq Su^*$, so, from condition (ii) we have $u^* = Tu^* \succeq STu^* = Su^*$. This implies that $u^* = Su^*$. Hence, u^* is a common fixed point for S and T.

Uniqueness. Suppose u be the another fixed point for S and T. Then $Su = Tu = u$. As, $u \preceq u$, then $u \succeq u = Su \Rightarrow u \succeq Su$. Also $u^* \preceq Su^*$. Now,

$$\begin{aligned} &d(u^*, u) = d(Tu^*, Tu) \leq \rho(\max\{d(u^*, u), d(u^*, Tu^*), d(u, Tu)\}) \\ &d(u^*, u) \leq \rho d(u^*, u) \end{aligned}$$

As, $\rho(t) < t$ for all $t > 0$, so, $d(u^*, u) = 0$. Thus, u^* is unique common fixed point for S and T.

Example 2.2. Let $W = [0, \infty]$ and d be the metric on W defined by $d(u, v) = |u - v|, (u, v) \in W \times W$. Then (W, d) is a complete metric space. Let $\Re$ be a binary relation on W defined by $\Re = \{(u, u) : u \in W\} \cup \{(0, 2)\}$. Consider the partial order on W defined by $(u, v) \in W \times W, u \preceq v \Leftrightarrow (u, v) \in \Re$. Let us define the pair of mappings $T, S : W \to W$ by

$$Tu = \begin{cases} u, & \text{if } u \notin \{0, 2\} \\ 2, & \text{otherwise} \end{cases}, \quad Su = \begin{cases} 2, & \text{if } u \in [0, 2] \\ 1, & \text{if } u > 2. \end{cases}.$$

Observe that, $levS_{\preceq} = \{0,2\}$, which is non-empty and closed. Therefore, the operator $S : W \to W$ is level closed from the left. Moreover, we have $\{u \in W : Su \preceq u\} = \{2\}$. In order to check the condition (ii) of Theorem 2.1 , let $u \in W$ be such that $u \preceq Su$; that is, $u \in \{0,2\}$. If $u = 0$, then $Tu = T0 = 2$ and $STu = ST0 = S2 = 2$. Then $STu \preceq Tu$. If $u = 2$ then $Tu = T2$ and $STu = ST2 = S2 = 2$. Then $STu \preceq Tu$. Now, let $u \in W$ be such that $Su \preceq u$; that is $u = 2$. In this case, we have $STu = ST2 = S2 = 2$ and $Tu = T2 = 2$. Then $Tu \preceq STu$. Therefore, condition (ii) of Theorem 2.1 is satisfied. Now, let $(u,v) \in W \times W$ be such that $u \preceq Su$ and $Sv \preceq v$; that is, $u \in \{0,2\}$ and $v = 2$. For $(u,v) = (0,2)$, we have $d(Tu,Tv) = d(T0,T2) = d(2,2) = 0 \leq \rho(2) = \rho(\max\{d(u,v), d(u,Tu), d(v,Tv)\})$. Now, for $(u,v) = (2,2)$, $d(Tu,Tv) \leq d(T2,T2) \leq d(2,2) \leq 0 \leq \rho(0) \leq \rho d(2,2)$, for every $\rho \in \Psi$. Therefore, all conditions of Theorem 2.2 are satisfied and 2 is the common fixed point.

Now, we will prove fixed point results for generalized Hardy Roger contraction. Our result extend the result given in [1].

Theorem 2.3. Let $(W, \preceq, d)$ be an ordered complete metric space and $S,T : W \to W$ be the self mappings. Suppose that the following assertions hold:

(i)′ The operator $S : W \to W$ is level closed from left.

(ii)′ For every $u \in W$, we have $u \preceq Su \Longrightarrow Tu \succeq STu$, and $u \succeq Su \Longrightarrow Tu \preceq STu$.

(iii)′ There exist constants a, b, c such that $0 \leq a + 2b + 2c < 1$

$$d(Tu,Tv) \leq a(d(u,v)) + b[d(u,Tu) + d(v,Tv)] + c[d(u,Tv) + d(v,Tu)],$$

whenever $u \preceq Su$ and $v \succeq Sv$. Then S and T have a unique common fixed point.

Proof. Suppose that u_0 be the arbitrary element of $levS_{\preceq}$, that is, $u_0 \preceq Su_0$. From condition (ii), we have $u_1 \succeq Su_1$, where $u_1 = Tu_0$. Again from condition (ii), we have $u_2 \preceq Su_2$, where $u_2 = Tu_1$. Now, consider the Picard sequence $\{u_n\} \subset W$ define by $u_{n+1} = Tu_n$ where $n = 0, 1, 2, \cdots$. Continuing in this way we get, for even terms of sequence

$$u_{2n} \preceq Su_{2n}. \tag{2.10}$$

For odd terms of sequence, we have

$$u_{2n+1} \succeq Su_{2n+1}. \tag{2.11}$$

As a consequence, we have

$$\begin{aligned}
d(u_{2n}, u_{2n+1}) &= d(Tu_{2n-1}, Tu_{2n}) \\
\leq a(d(u_{2n-1}, u_{2n})) &+ b[d(u_{2n-1}, Tu_{2n-1}) + d(u_{2n}, Tu_{2n})] \\
&+ c[d(u_{2n-1}, Tu_{2n}) + d(u_{2n}, Tu_{2n-1})] \\
&\leq (a+b+c)d(u_{2n-1}, u_{2n}) + (b+c)\, d(u_{2n}, u_{2n+1}) \\
(1-b-c)\, d(u_{2n}, u_{2n+1}) &\leq (a+b+c)d(u_{2n-1}, u_{2n}).
\end{aligned}$$

Dividing by $(1-b-c)$ on both sides, we get

$$d(u_{2n},\ u_{2n+1}) \leq \frac{(a+b+c)}{(1-b-c)} d(u_{2n-1},\ u_{2n}). \tag{2.12}$$

Let

$$\xi = \frac{(a+b+c)}{(1-b-c)}. \tag{2.13}$$

Using inequality (2.13) in (2.12), then inequality (2.12) becomes

$$d(u_{2n}, u_{2n+1}) \leq \xi(d(u_{2n-1}, u_{2n})). \tag{2.14}$$

As, $0 < a + 2b + 2c < 1$ and $\xi \in [0,1)$. So,

$$d(u_{2n-1}, u_{2n}) \leq \xi(d(u_{2n-2}, u_{2n-1})). \tag{2.15}$$

Using inequality (2.15) in (2.14), then inequality (2.14) becomes

$$d(u_{2n}, u_{2n+1}) \leq \xi^2(d(u_{2n-2}, u_{2n-1})).$$

Continuing in this way, we get

$$d(u_{2n}, u_{2n+1}) \leq \xi^{2n}(d(u_0, u_1)). \tag{2.16}$$

Similarly

$$d(u_{2n+1}, u_{2n+2}) \leq \xi^{2n+1}(d(u_0, u_1)). \tag{2.17}$$

Combining inequalities (2.16) and (2.17), we have

$$d(u_n, u_{n+1}) \leq \xi^n(d(u_0, u_1)). \tag{2.18}$$

Now, let $d(u_0, u_1) = 0$. This implies that $u_0 = u_1$. As $u_1 = Tu_0$, so

$$u_0 = Tu_0. \tag{2.19}$$

So u_0 is a fixed point for T. Now, $u_0 = u_1 \succeq Su_1 = Su_0$, therefore

$$u_0 = Su_0. \tag{2.20}$$

From inequalities (2.19) and (2.20), u_0 is a common fixed point for S and T. Now, if $d(u_0, u_1) \neq 0$. Then we suppose that $\eta = d(u_0, u_1) > 0$. So $d(u_n, u_{n+1}) \leq \xi^n(\eta), n = 0, 1, 2, 3, ...$ For $\epsilon > 0$ there exist $n_0 \in \mathbb{N}$ such that $\sum_{k \geq n_0}^{\infty} \xi^k(\eta) < \epsilon$. Let $n, m \in \mathbb{N}$, such that $n + m > n \geq n_0$. Then

$$\begin{aligned} d(u_n, u_{n+m}) &= d(u_n, u_{n+1}) + d(u_{n+1}, u_{n+2}) + ... + d(u_{n+m-1}, u_{n+m}). \\ &\leq \xi^n(\eta) + \xi^{n+1}(\eta) + ... + \xi^{n+m-1}(\eta). \\ &= \sum_{i=n}^{n+m-1} \xi^i(\eta) \leq \sum_{k \geq n}^{\infty} \xi^k(\eta) < \epsilon. \end{aligned}$$

Therefore, the sequence $\{u_n\}$ is a Cauchy sequence in $(W,\ d)$. So, there exist some $u^* \in W$ such that $\lim_{n\to\infty} d(u_n, u^*) = 0$. By following similar steps as in previous theorem, we have $u^* \preceq Su^*$. Now,

$$\begin{aligned} d(u^*, Tu^*) &\leq d(u^*, u_{2n+2}) + d(u_{2n+2}, Tu^*) \\ &\leq d(u^*, u_{2n+2}) + a(d(u_{2n+1}, u^*)) + b[d(u_{2n+1}, Tu_{2n+1}) + d(u^*, Tu^*)] \\ &\quad + c[d(u_{2n+1}, Tu^*) + d(u^*, Tu_{2n+1})] \\ &\leq d(u^*, u_{2n+2}) + a(d(u_{2n+1}, u^*)) + b(d(u_{2n+1}, u_{2n+2})) + b(d(u^*, Tu^*)) \\ &\quad + c(d(u_{2n+1}, Tu^*)) + c(d(u^*, u_{2n+2})). \end{aligned}$$

If we take $\lim_{n\to\infty}$, then we obtain $(1-b-c)\,d(u^*, Tu^*) \leq 0$. This implies that $u^* = Tu^*$. Now, by following similar steps as in previous theorem, we have $u^* = Su^*$. Hence, u^* is a common fixed point for S and T.

Uniqueness. Suppose r be the another common fixed point for S and T. Then $Sr = Tr = r$. As, $r \preceq r$, then $r \succeq r = Sr \Rightarrow r \succeq Sr$. Also $u^* \preceq Su^*$. Now,

$$\begin{aligned} d(u^*, r) &= d(Tu^*, Tr) \\ &\leq a(d(u^*, r)) + b[d(u^*, Tu^*) + d(r, Tr)] \\ &\quad + c[d(u^*, Tr) + d(r, Tu^*)] \\ &\leq a(d(u^*, r)) + b[d(u^*, u^*) + d(r, r)] \\ &\quad + c[d(u^*, r) + d(r, u^*)] \\ (1-a-2c)\,d(u^*, r) &\leq 0. \end{aligned}$$

But, $(1-a-2c) > 0$. So, $d(u^*, r) = 0$ or $u^* = r$. Hence, u^* is unique common fixed point for S and T

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