On groups acting on trees of ends >1

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Abstract. Call a group G to possess the property P if G is finitely generated and G is of end greater than one. That is e(G) >1. The main result of this paper is the following. A group G to possess the property P if and only if there exists a tree X such that G acts on X without inversions, the stabilize G_e for each edge e of X is finite, for each vertex v of X, $G_v \neq G$ and the quotient graph G/X for the action of G on X is finite. We prove the following:

(1) If the group G possesses the property P and H is a subgroup of G, then

  (i) if H is of finite index in G, then H possesses the property P,
  (ii) if H is finite and normal subgroup of G, then the quotient group of G over H possesses the property P.

(2) If G is a group acting on a tree X without inversions such that the stabilize G_v for each vertex v of X possesses the property P, $G_v \neq G$, the stabilizer of each edge e of X is finite, and the quotient graph G/X for the action of G on X is finite, then G to possesses the property P.

As an application, we show that if $A = \prod^*_{i\in I}(A_i; U_{ij} = U_{ji})$ is a tree product of the groups $A_i i$ ∈I with amalgamation subgroups U_{ij}, i, j ∈I such that each A_i has the property P, U_{ij} is finite, i, j ∈I and I is finite, then A has the property P. Furthermore, if G^* is the HNN group

$$G^* =< gen(G), t_i/rel(G), t_iat_i^{-1} = \phi(a), a \in A_i, i \in I >$$

of basis G and associated pairs $(A_i, B_i), i \in I$ of isomorphic subgroups of G such that G has the property P, $A_i, i \in I$ is finite, and I is finite, then G^* has the property P.

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1. Introduction

the structures of group acting on trees with out inversions we refer the readers to [2], [11] and [12]. For the basic concepts of the ends of groups we refer the reader to [1], [2], [3], [13], [14], and [15]. In this paper we discuss and prove theorems connecting groups acting on trees without inversions and ends of groups. The main result of this paper is showing that if the stabilizers of the vertices of a tree are of ends greater than 1, then the group has an end greater than 1 providing that the stabilizers of the edges of the tree are finite, and the quotient graph of the action of group on a tree is finite. Then, we apply such result to tree product of groups and HNN extensions of groups. First, we begin a general background of groups acting on trees without inversions. A graph X consists of two disjoint sets V(x) (the set of vertices of X) and E(X) (the set of edges of X) with V(X) non-empty, together with three functions $\partial_0 : E(X) \to V(X)$, $\partial_1 : E(X) \to V(X)$ and $\eta : E(X) \to E(X)$ is an involution satisfying the conditions $\partial_0\eta = \partial_1$ and $\partial_1\eta = \partial_0$ For simplicity, if $e \in E(X)$, then then we write $\partial_0(e) = O(e)$, $\partial_1(e) = t(e)$ and $\eta(e) = \overline{\mathrm{e}}$. This implies that $O(\overline{\mathrm{e}}) = t(e)$, $t(\overline{\mathrm{e}}) = O(e)$ and $(\overline{\overline{\mathrm{e}}} = e)$. We say that a group G acts on a graph X without inversions if there is a group homomorphism $\phi : G \to Aut(X)$. In this case, if $x \in X$ (vertex or edge) and $g \in G$ then we write g(x) for $(\phi(g(x))$ Thus, if $g \in G$ and $y \in E(x)$, then $g(O(y)) = O(g(y)), (g(t(y)) = t(g(y))), g(\overline{\mathrm{y}}) = \overline{\mathrm{g(y)}})$. If the group G acts on the graph X and $x \in X$ (x is a vertex or edge), then 1. The stabilizer of x, denoted G_x is defined to be the set $G_x = [g \in G : g(x) = x]$. It is clear that $G_x \leqslant G$ and if $x \in E(x)$ and $u \in [o(x), t(t)]$ then $G_{\overline{\mathrm{x}}} = G_x$ and $G_x \leqslant G_u$ 2. the orbit of x denoted G(X) is defined to be the set $G_x = [g \in G : g(x) \in x]$. It is clear that G acts on the graph X without inversions if and only if $G(\overline{\mathrm{e}}) \neq G(e)$. For any $e \in E(x)$. 3. The set of the orbits G/X of the action of G on X is defined as $G/X = [G(x) : x \in X]$ G/X forms a graph called the quotient graph for the action of G on X, where $V(G/X) = [G(V) : v \in V(x)]$, $E(G/X) = [G(e) : e \in E(x)]$ and if $e \in E(x)$ then $O(G(e)) = G(o(e)), t(G(e)) = G(t(e))$ and $\overline{\mathrm{G(e)}} = G(\overline{\mathrm{e}})$.

Definition 1 ([5]). *Let G be a group acting on a tree X without inversions, and let T and Y be two subtrees of X such that $T \subseteq Y$ and each edge of Y has at least one end in T. Assume that T and Y are satisfying the following:*

(i) *T contains exactly one vertex from each vertex orbit.*

(ii) *Y contains exactly one edge y(say) from edge orbit.*

The pair $(T; Y)$ is called a fundamental domain for the action of G on X. For the existence of fundamental domains.

Proposition 2.1. *The set of vertices V(T) of T is in one to one correspondence with the set of vertices V(X/G) of X/G and the set of edges E(Y) of Y is in* *one to one correspondence with the set of edges $E\ (X/G)$ of X/G. For the rest of this section, G is a group acting on a tree X without inversions, and $(T;Y)$ is the fundamental domain for the action of G on X. We have the following definitions.*

Definition 2. *For any vertex $v \in V(x)$ there exist a unique vertex denoted v^* of T and an element g (not necessarily unique) of G such that $g(v^*) = v$ That is $G(v^*) = v$, Moreover, if $v \in V(T)$ then $v^* = v$, and for each edge $y \in E(x)$, let $[y]$ be any element of G satisfying the following:*

(a) *if $o(y) \in V(T)$ then $[y]((t(y))^*)$ and $[y] = 1$ in case $y \in E(T)$.*

(b) *if $t(y) \in V(T)$ then $[y](o(y)) = (o(y)^*)$ and $[y] = [y]^{-1}$. Furthermore, let $+y$ be the edge if $o(y \in V(T))$ and $+y = [y](y)$ if $t(y) \in V(T)$ It is clear that $o(+y) = (o(y))^*$ and $G_{+y} \leqslant G_{(o(y))^*}$. If $y \in E(T)$,then $G_{+y} = G_y$.*

Definition 3. *If $g \in G$ is an element of G and $e \in E(Y)$ is an edge of Y, define $[g,e]$ to the pair $[g,e] = (gG_{+e}, e)$ Define $\hat{X}$ to be the set $\hat{X} = [[g,e] : g \in G, e \in E(Y)]$.*

2. The main result

There are a lot of ideas of introducing the concept of the ends of groups as shown in [1, p.17],[2, p. 124], [3], [13], [14], and [15, p. 171]. For example, in [2, p. 124], the concept of the ends of groups is illustrated as follows. The number of the ends of a group G, denoted $e(G)$ is defined as following. Let $Z_2 = [0,1]$ be the ring of integers modulo 2 and $(G;Z_2)$ be the set of functions from G to Z_2. $(G;Z_2)$ forms a ring where for each $\alpha : G \to Z_2$, $(\alpha+\beta)(g) = \alpha(g) + \beta(g)$ and $(\alpha\beta)(g) = \alpha(g)\beta(g)$ for all $g \in G$. Let IG be the subring of the ring $(G;Z_2)$ consisting of all $\theta \in (G;Z2)$ defined $\theta(g(x)) = \theta(xg^{-1})$ for all $g, x \in G$. Let JG be the set of all complements of all prime ideals of the subring IG. the number of the elements of JG is called the number of the ends of G, and is denoted by $e(G)$.

In [3], the concept of the ends of groups is defined by means of the actions of groups on simplicial complexes. For further background information see [13].

The following are examples of ends of groups.

1. For any finitely generated group G, $e(G) \in [0,1,2,\infty]$ ([15], Corollary 5.9, p.176).

2. For a finite group G, $e(G) = 0$. ([2], Proposition 6.5, p.126).

3. If Z is the infinite cyclic group, and $G = Z^n$, $n \geqslant 2$, then $e(G) = 1$, ([15], Corollary 5.5, p.174).

4. $e(Z) = 2$, ([15], Theorem 5.12, p. 178).

5. If Z_2 is a cyclic group of order 2, and G is the free product $G = Z_2 * Z_2$, then $e(G) = 2$, ([1], Corollary, p. 21).

6. For a free group $F(X)$ of base X where $1 < |X| < \infty$ we have $e(F(X)) = \infty$, ([2], Example, p. 127).

In [1, Th. 3.1] and [14, Th. 4.A.6.5, and Th. 4.A.9], Cohen and Stallings showed that for a finitely generated G, $e(G) \geqslant 1$ if and only if G either G decomposes as a free product with amalgamation $G = *_H A_i, i \in I$ of the groups A_i, $i \in I$ with amalgamated subgroup H such that H and I are finite, $H \neq A_i$ for all $i \in I$, or is an HNN group, $G =< gen(K), t_1...t_n | rel(k), t_i L_i t^{-1} = M_i, i = 1...n >$ of basis K and associated pair (L_i, M_i) of isomorphic subgroups of G for all $i \in I$ such that L_i, M_i and I are finite for all $i \in I$. In fact, the free product with amalgamation group $G = *_H A_i, i \in I$ is equivalent to the tree product of groups as

$$A = \prod_{i \in I}^{*}(A_i; U_{ij} = U_{ji})$$

is a tree product of the groups $A_i, i \in$ with amalgamation subgroups $U_{ij}, i, j \in I$.

By working within the framework of groups acting on trees without inversions this theorem can be put in the following form.

Lemma 2.1. *A group G to possess the property P if and only if there exists a tree X such that G acts on X without inversions, the stabilize G_e for each edge e of X is finite, for each vertex v of X, and the quotient graph G/X for the action of G on X is finite.*

Proof. If $A = \prod_{i \in I}^{*}(A_i; U_{ij} = U_j)$ is a tree product of the groups $A_i, i \in$ with amalgamation subgroups $U_{ij}, i, j \in I$ then A acts on the tree X without inversions defined as follow: $V(X) = [(gA_i, i) : g \in A, i \in I]$ and $E(X) = [(gU_{ij}, ij) : g \in A, i, j \in I]$. If y is the edge $y = (gU_{ij}, ij)$, then $o(y) = (gA_i, i)$, $t(y) = (gA_j, j)$ and $\overline{\text{y}} = (gU_{ji}, ji)$. The group A acts on X as follows: let $f \in A$ Then $f(gA_i, i) = (fgA_i, i)$ and $f(gU_{ij}, ij) = (fgU_{ij}, ij)$. If v is the vertex $v = (gA_i, i) \in V(X)$ and y is the edge $(gU_{ij}, ij) \in E(X)$. Then the stabilizer of v is $A_v = gA_i g^{-1} \cong A_i$, a conjugate of A_i and the stabilizer of y is $A_y = gU_{ij} g^{-1} \cong U_{ij}$, a conjugate of U_{ij} The orbit of v is the set $A(v) = [(agA_i), i : a \in A, i \in I)]$ and the orbit of y is $A_y = [(agU_{ij}), ij : a \in A, i, j \in I)]$. If G is an HNN group $G =< gen(K), t_i...t_n(k), t_i L_i t^{-1} = M_i, i = 1...n >$, of base K and of associated pairs (L_i, M_i) of isomorphic subgroups of $K, i \in I$, where $< gen(K) | rel(k)$ stands for any presentation of K, and $t_i L_i t_i^{-1}$ stands for the relations $t_i t_i^{-1} = \phi_i(x), x \in A_i$ Then G acts on the tree X without inversions defined as follow: $V(X) = [gK : g \in G]$ and $E(X) = [(gM_i, t_i), (gL_i, t_i^{-1})]$, where $g \in G$and $i \in I$. For the edges $(gM_i, t_[i])$ and $(gL_i, t_i^{-1}), i \in I$, defined $o(gM_i, t_i) = o(gL_i, t_i^{-1}) = gK$, $t(gM_i, t_i) = gt_i K$, $t(gL_i, t_i^{-1}) = gt_i^{-1} K$, and $\overline{(gM_i, t_i)} = (gt_i L_i, t_i^{-1})$ and $\overline{(gL_i, t_i^{-1})} = (gt_i^{-1} M_i, t_i)$. Let $f \in G$. Then for the vertex gG and the edges (gM_i, t_i) and (gL_i, t_i^{-1}) of X, define $f(gK) = fgK, f(gM_i, t_i) = (fgM_i, t_i)$ and $f(gL_i, t_i) = (fgL_i, t_i)$, The stabilizer of the vertex $v = gK$ is $K_v = gKg^{-1}$ a conjugate of G, the stabilizers of the edges (gB_i, t_i) and (gA_i, t_i) are $gM_i g^{-1}$ a conjugate of M_i and $gL_i g^{-1}$ a conjugate of L_i are finite for all $i \in I$ The orbits of $gG, (gB_i, t_i)$

and (gL_i, t_i^{-1}) are $[fK : f \in G]$ and $[(fM_i, t_i : f \in G)]$ Now if the group G acts on a tree X without inversions such that $G_v \neq G$ for any vertex $v \in V(X)$o f X, then by theorem of [5], there exists a fundamental domain $(T; Y)$ for the action of G on X, and [11, Th. 5.1], G has the presentation $G =< gen(G_v, y|rel(G_v), G_m = G_m, y.[y]^{-1}G_y[y]y.y^{-1} = G_y) >$, where , m and y stand for edges of $E(Y)$ such that $m \in E(T)$, $o(y) \in V(T), t(y), \neq V(T)$. If $Y = T$ then G has the presentation $\prod^*_{v\in V(X)}(gen(G_v; rel(G_v), G_m = G_m)$, where $L_m = [m]^{-1}G_m[m]$ and $M_m = G_m$. It is clear that G is a tree product of the groups G_v with amalgamation subgroups L_m, M_m . If $Y \neq T$, then $G =< gen(G_v, y|rel(G_v), G_m = G_m, y.[y]^{-1}G_y[y]y.y^{-1} = G_y) >$ is HNN extension group of base K, where $\prod^*_{v\in V(X)}(gen(G_v; rel(G_v), G_m = G_m)$ associated isomorphic subgroups L_m, M_m, and stable letters the edges $y \in E(X)$ of X such that $o(y) \in V(T) andt(y) \neq V(T)$. □

As a consequence of Lemma 2.1, we have the following proposition.

Proposition 2.1. *If the group G has the property P and H is a subgroup of G then:*

(1) *If H is of finite index, then H has the property P.*

(2) *If H is finite and normal, then the quotient group G/H has the property P.*

Proof. Since G has the property P, G is finitely generated, and by Lemma 2.1, exists a tree X such that G acts on X without inversions, the stabilize Ge for each edge e of X is finite and the quotient graph G/X for the action of G on X is finite. (1) Since G is finitely generated and H is of finite index in G, by the Reidemeister-Schreier subgroup theorem [6, Corollary 2.8, page 93], H is finitely generated. Then H acts on X by restriction. It is clear that the vertex stabilize He of the edge e of X satisfies $H_e = H \cap G_e$ Since G_e is finite, H_e is finite. Since H is of finite index in G, and G/X is finite , therefore by Lemma 7 of [11], the quotient graph H/X for the action of H on X is finite. Thus, H has the property P.

(2) Since G is finitely generated, it is clear that G/X is finitely generated. Let X^H be the set $X^H = [x \in X : H \leq G_x]$ By Proposition 4.3 of [12], X^H is a subtree of X and G/H acts on X^H without inversions, where if $g \in G$ and $x \in E(X^H)$, then such that $gH(x) = g(x)$. It is clear that the stabilizer of $x \in X^H$ under the action of G/H on X^H is $(G/H)_x = G_x/H$, where G_x, is the stabilizer of x under the action of G on X. Since stabilizer G_y of each edge y of X under the action of G on X is finite, therefore stabilizer $(G/H)_e = G_e/H$ of each edge $e \in E(X)$ under the action of G/H on X^H is finite. If $x \in V(V)$ and $(G/H)_x = G_x/H = G/H$ then $G_x = G$. Contradiction. Hence $(G/H)_x = G_x/H \neq G/H$ for any vertex $x \in V(V)$. It is clear that if $x \in X^H$, where x is a vertex or an edge, then the orbit $(G/H)(x)$ of x under the action of G/H on X^H is given by $(G/H)(x) = G(x)$ where G(x) is the orbit of x under the action of G on X. This implies that the quotient graph $(G/H)/X^H$ for the action of G/H

on X^H is given by $[(G/H)(x) = G(x) : x \in X] \subseteq G/X$. Since G/X is finite, therefore $(G/H)/X^H$ is finite. Consequently, the quotient group G/H has the property P. This completes the proof.

Remark. In Lemma 2.1, we note that for each vertex v of X, the stabilizer G_v acts on a tree v of one vertex and no edges. It is natural to ask if the tree X can be chosen so that the stabilizer G_v of each vertex v of X acts on a tree X_v of more than one vertex. In [10], Mahmood constructed a tree denoted $\tilde{X}$ on which G acts on $\tilde{X}$ without inversions such that the stabilize G_v for each vertex v of X,$G_v \neq G$ acts on a tree X_v such that X_v contains more than one vertex. The following theorem gives more details. Before we prove the theorem we introduce the following concept taken from [2]. Let H be a subgroup of a group G and let H act on a set X. Define $\equiv$ to be the relation on $G \times X$ defined as $(f, u) \equiv (g, v)$ if there exists $h \in H$ such that $f = gh$ and $u = h^{-1}(v)$ It is easy to show that $\equiv$ is an equivalence relation on $G \times X$ The equivalence class containing $(f; u)$ is denoted by $f \otimes_H u$. Thus, $f \otimes_H u = [(fh, h^{-1}) : h \in H]$ Define $G \otimes_H X$ to be the set $G \otimes_H X = [g \otimes_H x, x \in X]$.

The main result of this section is the following theorem.

Theorem 2.1. *If G is a group acting on a tree X without inversions such that the stabilize G_v for each vertex v of X has the property P,$G_v = G$, the stabilizer G_e of each edge e of X is finite, and the quotient graph G/X for the action of G on X is finite, then the group G has the property P.*

Proof. Since for each vertex $v \in V(X)$, G_v has the property P, G_v is finitely generated. Since the quotient graph G/X for the action of G on X is finite, by Lemma 4.4 of [11], G is finitely generated. Furthermore, by Lemma 2.1, exists a tree X_v such that G_v acts on X_v without inversions, the stabilizer $(G_v)_u$ for each edge u of X_v is finite and $(G_v)_w \neq G_v$ for each vertex w of X_v, and the quotient graph G_v/X_v for the action of G_v on X_v is finite. Let $(T; Y)$ be a fundamental domain for the action of G on X. Since the quotient graph G/H for the action of G on X is finite, T and Y are finite. By Lemma 4.4 of [6], G is generated by the generators of $G_v, v \in V(T)$ and by the elements $[y], y \in E(Y)$ By Theorem 3.4 of [11], there exists a tree denoted as $\tilde{X} = \hat{x} \bigcup_{v \in V(x)} (G \otimes_{G_y} X_v)$ where $\hat{x} = ([g : e] : g \in G, e \in E(Y))$ and $[g; e] = (gG_{+e}, e)$, $V(\tilde{X} = \bigcup_{v \in V(x)} (G \otimes_{G_y} V(X_v)))$ and $E(\tilde{X}) = \hat{x} \bigcup_{v \in V(x)} (G \otimes_{G_y} E(X_v))$ The ends and the inverse of the edge $G \otimes_{G_y} e$ are defined as follows: $t(g \otimes_{G_v} e) = fg \otimes_{G_V} t(e)$, $o(g \otimes_{G_v} e) = g \otimes_{G_v} o(e)$ and $\overline{(g \otimes_{G_v} e)} = g \otimes_{G_v} \overline{e}$, where $t(e), o(e)$ and $\overline{e}$ are the ends and the inverse of the edge e in X_v. G acts on $\tilde{X}$ as follows: if $f, g \in G, y \in E(x), v \in V(T), e \in E(X_v)$ and $u \in V(X_v)$, then $f[g; y] = [fg; y]$, $f(g \otimes_{G_v} e) = fg \otimes_{G_V} e$ and $f(g \otimes_{G_u} e) = fg \otimes_{G_u} e$. If $g \in G$ and $e \in E(X)$,such that $g(1 \otimes_{G_v} e) = \overline{1 \otimes_{G_V} e} = 1 \otimes_{G_V} \overline{e}$, then $g \in G_v$ and $e \in E(X_v)$, $g(e) = \overline{e}$. Hence,G_v acts on X_v with inversions. This is a contradiction because G_v has the property P. This implies that G acts on $\tilde{X}$ without inversions. Now for $g \in G$

and $x \in X_v$, it is clear that the stabilizer $G_g \otimes_{G_v} x$ of the vertex $g \otimes_{G_v} x$ is $G_g \otimes_{G_v} x = g(G_v)_x g^{-1}$, where $(G_v)_x$ is the stabilizer of x under the action of G_v on X_v, Since $(G_v)_x$ is finite, therefore, $g \otimes_{G_v} x$ is finite. So the stabilizer of each vertex of $\tilde{X}$ under the action of G on $\tilde{X}$ is finite. Now we show that the quotient graph $G/\tilde{X}$ for the action of G on $\tilde{X}$ is finite. The fundamental domain $(T;Y)$ for the action of G on X induces a fundamental domain $(T_v;Y_v)$ for the action of G_v on X_v for every vertex v of T.For each edge $e \in E(Y)$, letv_e be a vertex $v_e \in V(T_{(o(e))^*})$, such that $G_{+e} \leqslant (G_{(o(e))^*})_{v_e}$, where $(G_{(o(e))^*})_{v_e}$ is the vertex stabilizer of the vertex v_e under the action of $(G_{(o(e))^*})$ on $(X_{(o(e))^*})$. Let $\hat{T} = [[1;e] : e \in E(T)]$, $\hat{Y} = [[1;e],[[e];e] : e \in E(Y)]$, $\tilde{T} = \hat{T} \cup [\bigcup_{v \in V(T)}(1 \otimes_{G_v} T_v)]$, $\tilde{Y} = \hat{Y} \cup [\bigcup_{v \in V(T)}(1 \otimes_{G_v} Y_v)]$, where $[e]$ is the value of the edge e defined as in Definition 2 and $[[e],e]$ is defined as in Definition 3. Then, by Theorem 3.2 of [9], $(\tilde{T};\tilde{Y})$ forms a fundamental domain for the action of G on $(\tilde{X}$. Since for each edge $e \in E(X)$, the stabilizer G_e is finite and $(T;Y)$ and, $(T_v;Y_v)$, $v \in V(T)$ are finite, then the fundamental domain $(\tilde{T};\tilde{Y})$ is finite. This shows that the quotient graph G/X for the action of G on X is finite. Then, Lemma 2.1 implies that the group G has the property P. This completes the proof. □

We have the following corollaries of Theorem 2.1.

Corollary 2.1. *Let $A = \prod^*_{i \in I}(A_i; U_{ij} = U_{ji})$ be a tree product of the groups A_i of the groups $A_i, i \in I$ with amalgamation subgroups $U_{ij}, i, j \in I$ such that the group Ai has the property P for alli $\in I$, I is finite, and U_{ij} is finite for all $i, j \in I$. Then A has the property P.*

Corollary 2.2. *Let $A = *_c A_i, i \in I$ be the free product of the groups $A_i, i \in I$ with amalgamation subgroup C such that the group A_i has the property P for all $i \in I$, I is finite, and C is finite. Then the group A has the property P.*

Corollary 2.3. *Let G be the HNN group $G =< gen(K), t_i | rel(K), t_i L_i ti^{-1} = M_i, i \in I >$ of base K and of associated pairs (L_i, M_i) of isomorphic subgroups of G such that the group K has the property P, L_i and M_i are finite for all $i \in I$, and I is finite. Then the group G has the property P.*

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