On the dynamics of the generalized Hènon-Heiles of the galactic potential

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Abstract. The bifurcation of Liouville tori of a generalized Hènon-Heiles System (GHH) are determined. The phase portrait of separation functions of (GHH) are studied, and the classification of the singular points were found. Some figures are presented by using Poincaré surface section.
Keywords: Hamilton-Jacobi's equations, bifurcations of Liouville tori, topology of the level sets, momentum maps, periodic solution, elliptic functions, phase portrait, Poincaré surface-section.

1. Introduction

The integrable Hamiltonian system with n-degree of freedom possessing a periodic solutions and it has an n-invariant torus around these solution. The Hamiltonian Hènon-Heiles (HH) system [22] is one of the famous model systems which in general is not integrable. It represents the motion of a star in the rotating meridian plane of a galaxy in the neighborhood of a circular orbit or in the equatorial plane of a galaxy with axial symmetry. The problem can be integrated in the sense of Liouvill-Arnold [3] if there exist a single-valued, integral of the motion which are functionally independent and involution besides the integral of energy and the angular momentum integral. Hènon-Heiles introduced a two-dimensional axil-symmetric, potential admit a third isolating integral which is analytically and sufficiently complicated to give the trajectories which are far from trivial, this integral existed only for some values of the constant of energy, and in escape energy, the problem is not integrable.

The HH and GHH (Generalized Hènon-Heiles) has been studied in great numbers of scientific works, to get a complete picture of the motion. The Hamiltonian Toda system [30] is possessed another integral for the motion of three particles moving on a ring and, Ford et al. [17] gave the the same result nu-

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merically. The Hamiltonian can be reduced to the 2-D system by a canonical transformation [25], and the expansion of the system up to the third order terms coincided with HH system as shown in G. Contopoulos and C. Polymilis [9]. They expanded the exponential terms up to 10th order approximation. The periodic orbits of HH was studied by many authors: El-Sabaa and Sherief [11] classified and discussed nine classes of the main periodic orbits and its stability while Davies et al. [10] applied the monodromy method, for calculating the periodic trajectories, the monodromy method was computationally very efficient where the periodic orbits including a number of simple bifurcations. Ozaki and Kurosaki [27] found the periodic orbits and the bifurcation of it by calculating Poincaré surface-section and the residues of their orbits. Carrasco and Vidal [7] used small parameter in order to apply the average method of second order to proved the existence of different families of periodic solutions for some cases of parameters and characterized the stability of the family of periodic orbits. For the study of fractal HH, Barrio et al. [4] presented the appearance of different kinds of fractal structures in the paradigmatic HH Hamiltonian, and as the KAM (Kolmogrov, Arnold, Mozer) regime continues, the regular bounded region presents a fractal structure. For non integrability of HH, Holmes [24] used the Melnikov's method to prove that the all neighboring systems of HH type are non-integrable and estimate the width of the primary stochastic layer which is the cause of non-integrability.

The HH system was study too in quantum mechanics: Bixon and Jortner [5] considered the dynamics of wave packets of bound states in the nonlinear HH system, and explored the correspondence between classical and quantum dynamics. Akhiezer et al. [2] studied the chaos of charged particles in a crystal regarding HH potential. Bastida et al. [32] investigated the possibility of using oblique coordinates to determine the energy levels and wave functions of HH coupled oscillator systems.

The aim of this article is to study GHH the Painlevé property [20] for the problem is applied to show the identification of specific integrable cases. The topology of the level set is given and, the phase portrait of the separated function of the integrable cases are determined. The surface-section was introduced by Poincaré to get the regular motion in all integrable cases for different values of the constant of energy.

2. Painlevé analysis

The GHH Hamiltonian system is

$$H = \frac{1}{2}(p_x^2 + p_y^2 + Ax^2 + By^2 + 2Dx^2y - \frac{2}{3}Cy^3). \tag{1}$$

Where the HH Hamiltonian can be written when $A = B = C = D = 1$. The GHH given a nonlinear dynamical system governed by a system of the equations:

$$\begin{aligned} &\ddot{x} + Ax + 2Dxy = 0, \\ &\ddot{y} + By - Dx^2 + Cy^2 = 0. \end{aligned} \tag{2}$$

By using Painlevé analysis, its required that a Laurent series of the solution, considering time as a complex variable, have only simple poles at all movable singularities, where these points are those that depend on the initial condition of the system. If the singular points independent to the constants of integration, then they are fixed singular points, and among these points, the branch and the essential singularities are said to be a critical points. When system satisfy Painlevé property (P-property), then they can be integrable and the solution expressed in Laurent expansion in neighborhood of the movable singular point. Many authors [21, 23, 6]have applied the Painlevé analysis to variety of the Hamiltonian dynamical system, and identified a considerable number of new integrable. Ablowitz et al. (1980) [1] announced an algorithm (ARS algorithm) to determine whether a non-linear system is integrable or not, this algorithm consists of the following steps:
The first step is to put

$$x \quad = a\tau^{\alpha}, y \quad = b\tau^{\beta}, \tau = t - t_0, \tag{3}$$

where α and β are integers and less than zero. So the system (2) becomes:

$$\begin{aligned} &a\alpha(\alpha - 1)\tau^{\alpha-2} = -2Dab\tau^{\alpha+\beta}, \\ &b\beta(\beta - 1)\tau^{\beta-2} = -Da^2(\tau^{2\alpha}) + Cb^2\tau^{2\beta}, \end{aligned} \tag{4}$$

and hance we can determine the leading order: we have two cases:

- *Case (i)* $\alpha = -2, a = \pm\frac{3}{D}\sqrt{2 + \frac{1}{\lambda}}, \beta = -2, b = \frac{6}{D}$.
- *Case (ii)* $\alpha = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 48\lambda}, a = arbitrary, \beta = -2, b = \frac{6}{C}$.

where $\lambda = D/C$.

The second step is determine the resonances, that is the power at which arbitrary constant of the solution can enter into the Laurent series expansion, so we have

$$\begin{aligned} x &= \pm\frac{3}{D}\sqrt{2 + 1/\lambda}t^{-2} + pt^{r-2}, \\ y &= -\frac{3}{D}t^{-2} + qt^{r-2}, \end{aligned} \tag{5}$$

where p and q are arbitrary parameters. Setting up the linear equations for p and q from the dominant balance terms in equation (2), and then the resonances are:

- for case (i)

$$r=-1,6,\frac{5}{2}\pm\frac{1}{2}\sqrt{1-24(1+\lambda)}, \tag{6}$$

- for case (ii)

$$r=-1,0,6,\pm\sqrt{1-\frac{48}{\lambda}}. \tag{7}$$

The third step is to verified that sufficient number of arbitrary constants exist without the introduction of the movable critical points

According to [29], the arbitrary constants exist where the root $r=-1$ is represent the arbitrary of pole position, while the root $r=0$ is associated to the arbitrary A. As a Painlevé criterion, the values of r must be integer. So, the problem is a Painlevé type and the integrable cases are:

1. $\lambda=-1, \frac{A}{B}=1$
2. $\lambda=-\frac{1}{6}$ any A, B
3. $\lambda=\frac{1}{16}, \frac{A}{B}=\frac{1}{16}$

3. Topological analysis

In the first case, we recall that the Hamilton-Jacobi equation corresponding to the system (2) separates to u, v defined as [33]

$$x=\frac{1}{2}(u+v),\ y=\frac{1}{2}(u-v), \tag{8}$$

which the first case $\lambda=-1$, then p_x, p_y can be determined through the coordinates u, v as follows:

$$\begin{aligned} p_x &= \frac{1}{\sqrt{6}}(\sqrt{F_1(u)+F_2(v)}),\\ p_y &= \frac{1}{\sqrt{6}}(\sqrt{F_1(u)-F_2(v)}). \end{aligned} \tag{9}$$

Where $F_1(u)$,$F_2(v)$ denoted by the polynomials:

$$F_1(u) = f-3h-\frac{3}{2}u^2-u^3, \tag{10}$$

$$F_2(v) = -f-3h-\frac{3}{2}v^2+v^3, \tag{11}$$

and f is the constant of separation. So the Hamilton-Jacobi equations leads to

$$\frac{du}{\sqrt{F_1(u)}}-\frac{dv}{\sqrt{F_2(v)}}=\frac{1}{\sqrt{6}}\beta, \tag{12}$$

$$\frac{du}{\sqrt{F_1(u)}}+\frac{dv}{\sqrt{F_2(v)}}=\sqrt{\frac{3}{2}}(t_0-t). \tag{13}$$

Therefore, the differential equations satisfied by u and v are

$$\sqrt{\frac{3}{2}}\int_{u_0}^{u}\frac{du}{\sqrt{F_1(u)}}=\int_{t_0}^{t}dt, \tag{14}$$

$$\sqrt{\frac{3}{2}}\int_{v_0}^{v}\frac{dv}{\sqrt{F_2(v)}}=\int_{t_0}^{t}dt. \tag{15}$$

In the second case where $\lambda=-\frac{1}{6}$, Wojciechowski [31]showed that the GHH can be separated shifted parabolic coordinates ξ,η such that

$$x^2=-4\xi\eta,\ y=\xi+\eta+(B-4A)/4, \tag{16}$$

and the Hamiltonian-Jacobi equation in this case

$$\begin{aligned}S=-ht+&\int[\frac{1}{2}(h-\frac{A(4A-B)^2}{32D})+\frac{k}{2}\xi-\frac{1}{2}D\xi^4+(\frac{3}{2}A-\frac{1}{4}B)\xi^3\\&-\frac{1}{2}(\frac{D}{2}(B(4A-B)+\frac{3}{2}(4A-B)^2))\xi^2]d\xi\\&+\int[\frac{1}{2}(h-\frac{A(4A-B)^2}{32D})-\frac{k}{2}\eta+\frac{1}{2}D\eta^4-(\frac{3}{2}A-\frac{1}{4}B)\eta^3\\&+\frac{1}{2}(\frac{D}{2}(B(4A-B)+\frac{3}{2}(4A-B)^2))\eta^2]d\eta,\end{aligned} \tag{17}$$

where h is the energy constant, k is the separation constant. And according to the equation of motion, we have the system of differential equations:

$$\frac{d\xi}{\sqrt{\xi P_1(\xi)}}-\frac{d\eta}{\sqrt{\eta P_2(\eta)}}=0, \tag{18}$$

$$\frac{\xi d\xi}{\sqrt{\xi P_1(\xi)}}+\frac{\eta d\eta}{\sqrt{\eta P_2(\eta)}}=-\frac{1}{2}dt. \tag{19}$$

these equations gives a separated functions

$$P(q)=16q^2-8(6A-B)q^3+(12A-B)(4A-B)q^2+(8h-A(4A-B)^2)q-k, \tag{20}$$

where q$\equiv(\xi,\eta)$

Fomenko [15] proposed a new approach in the qualitative theory of integrable Hamiltonian system, given the separation of the system, the determination of critical values of the energy momentum map boils down to the analysis of the discriminant surface of a polynomial. An Integrable Hamiltonian of a system has n-degrees of freedom always has a set of n-integrals of motion in n-involution consequently, the trajectory of the system is confined to n-dimensional manifold phase space. According to the Arnold Liouville theorem [?], for non-critical values of two constants of the problem, the regular level sets of a completely integrable Hamiltonian system consists of tori. Their number depends only upon the number and the location of the ovals of an associated Riemann surface.

We will study the topology of the first case ($\lambda = -1$) by using the Fomenko's classification theorem [16], this classification was studied in Jamil in the different cases in the rigid body problem [26], [18] . The second case ($\lambda = -\frac{1}{6}$) was studied by L.Gavrilov [19], which they are only separated of all integrable cases.

Let L_s be the real phase space topology of the system (9), where

$$L_s = [(x, y, p_x, p_y) \in R^4 : H = h, F = f] \subset R^4, \tag{21}$$

and let $\Sigma = \Sigma_1 \bigcup \Sigma_2$, where Σ_1 and Σ_2, are the discriminate locus of the polynomials $F_1(u)$ *and* $F_2(v)$, respectively:

$$\Sigma{=}\Sigma_1{\cup}\Sigma_2{=}[(h,f){\in}R^2/disc(F_1(u)){=}0]{\cup}[(h,f){\in}R^2/disc(F_2(v)){=}0]. \tag{22}$$

The set R^2/Σ consists of 16 connected component denoted by $D_i, i = 1, ..., 16$. Table 1 is presented the real roots of both $F_1(u)$, and $F_2(v)$ to find the ovals of ($\Gamma_1 : \omega_1 = \sqrt{F_1(u)}, \omega_2 = \sqrt{F_2(v)}$) to study the topological type of L_s (see Table 2).

To determine these ovals, it suffices to study the real roots of the polynomials $F_1(u)$, and $F_2(v)$ for different values of h and f. It is found that there exist exactly two "admissible" ovals whose projections on the u-plane and the v-plane which are denoted by $\triangle_1$ and $\triangle_2$, respectively as shown in Table 2.

To study the topological type of L_s, it is should be noted that the topological type of L_s depends only on passing the point $(h, f) \in R^2$ through Σ, for more details [12].

The topological type of L_s is either a torus as in $D_i, i = 1, 16$ or empty as in $D_i, i = 2, ..., 15$ (Liouville theorem [16]) as shown in Figure 1 and Table 2.

Table 1: The real roots of the polynomials $F_1(u)$ and $F_2(v)$ for $(h, f) \in R^2/\Sigma$

Domain	*Roots of* $F_1(u)$	*Roots of* $F_2(v)$
1	$u_1 < 0$	$v_1 < 0$
2	0	$v_1 < 0$
3	0	$v_1 < 0$
4	0	$v_1 < 0$
5	0	$v_1 < 0$
6	0	0
7	0	0
8	0	0
9	$u_1 < 0$	0
10	0	0
11	$u_1 < 0$	0
12	0	0
13	0	0
14	$u_1 < 0$	0
15	$u_1 < 0$	0
16	$u_1 < 0$	$v_1 > 0$

Table 2: Admissible ovals on diagram Σ.

Domain	$u - plane \triangle_1$	$v - plane \triangle_2$	*Topological type*
1	$[0, u_1]$	$[0, v_1]$	T
2	$\emptyset$	$[0, v_1]$	$\emptyset$
3	$\emptyset$	$[0, v_1]$	$\emptyset$
4	$\emptyset$	$[0, v_1]$	$\emptyset$
5	$\emptyset$	$[0, v_1]$	$\emptyset$
6	$\emptyset$	$\emptyset$	$\emptyset$
7	$\emptyset$	$\emptyset$	$\emptyset$
8	$\emptyset$	$\emptyset$	$\emptyset$
9	$[0, u_1]$	$\emptyset$	T
10	$\emptyset$	$\emptyset$	$\emptyset$
11	$\emptyset$	$\emptyset$	$\emptyset$
12	$\emptyset$	$\emptyset$	$\emptyset$
13	$\emptyset$	$\emptyset$	$\emptyset$
14	$[0, u_1]$	$\emptyset$	$\emptyset$
15	$[0, u_1]$	$\emptyset$	$\emptyset$
16	$[0, u_1]$	$[0, v_1]$	T

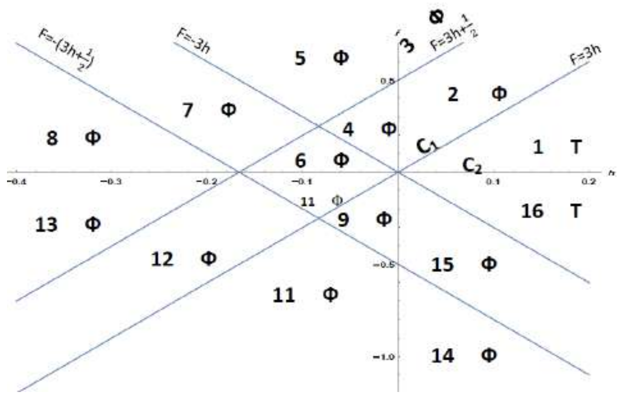

Figure 1: Diagram of bifurcation $\Sigma = \Sigma_1 \cup \Sigma_2$.

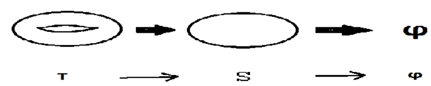

Figure 2: A torus T is contracted to the axial circle S and then vanishes.

A bifurcation of Liouville tori is due to the bifurcation of the polynomials $F_1(u)$ *and* $F_2(v)$ roots. In the present problem, there are two type of bifurcation as follows:

1. A torus contracts to circle and then vanished, as shown in Figure 2.
2. A symmetric bifurcation of one tori into one tori, as shown in Figure 3.

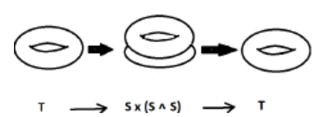

Figure 3: Bifurcation of Liouville tori, where $S \bigwedge S$ is union of two circles having exactly one common point.

Table 3 gives All generic bifurcations of Liouville tori of the system (9), Figures 4, and 5 show the bifurcations of the polynomials $F_1(u)$ and $F_2(v)$.

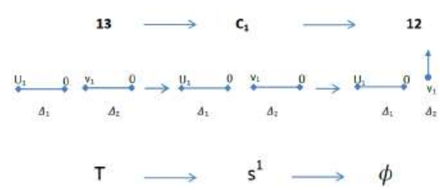

Figure 4: Correspondence between bifurcation of roots of polynomials $F_1(u)$ and $F_2(v)$ and bifurcation of invariant Liouville tori.

Domains	*The generic bifurcation*
$1 \to i, i = 2, ..., 15$	$T \to \emptyset$
$16 \to i, i = 2, ..., 15,$	$T \to \emptyset$
$1 \to 16$	$T \to T$

Table 3: Generic bifurcations of the level set L_s passing from domain i to domain j.

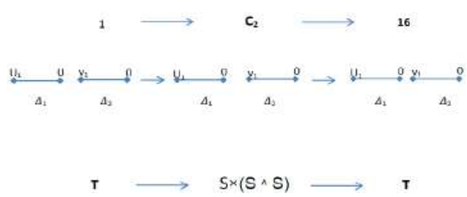

Figure 5: Correspondence between bifurcation of roots of polynomials $F_1(u)$ and $F_2(v)$ and bifurcation of invariant Liouville tori.

4. Periodic solutions

On the curve C_1 where $f = 3h$, we note that the tori T contracted to axial circle S and then vanish as shown in Figure 1, where the u parameter takes value in interval $[u_1, 0]$, and $v = \frac{-3}{2}$, then their exist a periodic solution on this curve as shown in the Table 4.

Domain	$u-plane \triangle_1$	$v-plane \triangle_2$	*Topological type*
C_1	$[u_1, 0]$	$[v_1, 0]$	S
C_2	$[u_1, 0]$	$[v_1, 0]$	$s \times (s \wedge s)$

Table 4: Topological type of L_s for $(h, f) \in B$.

By solving the second equation of (15), the function $F_1(u)$ is a polynomial of fourth degree and has three roots $u_1,\ u_2,\ andu_3$ such that

If $f > 3h$ or $f < 3h$, the real motion bounded by e_1, and e_2 such that $e_2 < u < e_1$.

Put

$$u = \frac{e_2 e_{31} - e_3 e_{21} sin^2(\phi)}{e_{31} - e_{21} sin^2(\phi)}, \tag{23}$$

where $e_{ij} = e_j - e_i$, and

$$du = \frac{2 e_{21} e_{32} sin(\phi) cos(\phi)}{e_{31}(1 - k^2 sin^2(\phi))^2} d\phi, \tag{24}$$

where

$$k^2 = \frac{e_{21}}{e_{31}}, \tag{25}$$

the equations (15), become:

$$\int_{t_0}^{t} dt = \sqrt{\frac{3}{2}}\sqrt{\frac{4}{e_{31}}}\int_{\phi_0}^{\phi}\sqrt{\frac{d\phi}{1-k^2 sin^2(\phi)}}. \tag{26}$$

Then, the solution is:

$$u = \frac{e_1e_{31} - e_3e_{21}sn^2(\sqrt{\frac{e_{31}}{6}}(t_0-t_1),k)}{e_{31} - e_{21}sn^2(\sqrt{\frac{e_{31}}{6}}(t_0-t_1),k)}. \tag{27}$$

Then, the period T of $F_1(u)$

$$T = g_1 sn^{-1}(-1,k) = g_1K(k). \tag{28}$$

where

$$g_1 = \frac{\sqrt{6}}{\sqrt{e_{31}}} \tag{29}$$

and $K(k)$is complete elliptic function of the first kind.

5. Phase portrait of the separated functions

The aim of this section is to find topological interpretation of the trajectory of the second case by using the phase portrait as manner it was studied in [13], [14]

The phase portrait of the separated functions is:

$$\begin{aligned} P_q = {} & 16q^4 - 8(6A-B)q^3 - (12A-B)(4A-B)q^2 \\ & + A(4A-B)^2q - 8hq - k - 4qp^2. \end{aligned} \tag{30}$$

We first study the singular points of P_q. These points can be found from the equations

$$\begin{aligned} &\frac{\partial P_q}{\partial p} = -8qp = 0, \\ &\frac{\partial P_q}{\partial q} = 64q^3 - 24(6A-B)q^2 - 2(12A-B)(4A-B)q - A(B-4A) \\ &\quad - 8h - 4p^2 = 0, \end{aligned} \tag{31}$$

and hence, when $p = 0$, the second equation of (31) became

$$64q^3 - 24(6A-B)q^2 - 2(12A-B)(4A-B)q - A(B-4A) - 8h = 0, \tag{32}$$

and when $q = 0$, we get the two equations

$$p^2 = \frac{1}{4}(A(B-4A)+8h), \tag{33}$$

then, from equation (31 and 32) we have the points $(\pm\sqrt{A(B-4A)+8h}, 0)$ and $(0, q_1)$, where q_1 the real root of the equation 31.

Now, we study the points in the following manner:

1. From equation (33) there are one singular point with p coordinate $(\pm\sqrt{A(B-4A)+8h},0)$ To get the type of the two points, put

$$q = y, \ \ p = \pm\sqrt{A(B-4A)+8h} + x, \tag{34}$$

in the function P_q, neglecting terms of degree greater than 2, then we have

$$P_q = -(12A-B)(4A-B)y^2 - 4(\pm\sqrt{A(B-4A)+8h})xy + A_0, \tag{35}$$

where A_0 contains the terms of zeros and first degree of x, y. These points are hyperbolic in two case:

1. $h > 0$ (Figure 6),
2. $h < 0$ (Figure 7),

where

$$\begin{bmatrix} \frac{\partial^2 P_q}{\partial x^2} & \frac{\partial^2 P_q}{\partial x \partial y} \\ \frac{\partial^2 P_q}{\partial x \partial y} & \frac{\partial^2 P_q}{\partial y^2} \end{bmatrix}_{x=y=0} < 0. \tag{36}$$

2. From equation (32) there are one singular point with q coordinate $(0, q_1)$. To get the type of the point, put

$$q = q_1 + y, \ \ p = x, \tag{37}$$

in the function P_q, neglecting terms of degree greater than 2, then we have

$$P_q = [96q_1^2 - 24(6A-B)q_1 - (12A-B)(4A-B)]y^2 - 4q_1x^2 + A_0, \tag{38}$$

where A_0 contains the terms of zeros and first degree of x, y. This point is hyperbolic point in two case:

1. $h > 0$ (Figure 8),
2. $h < 0$ (Figure 9),

where

$$\begin{bmatrix} \frac{\partial^2 P_q}{\partial x^2} & \frac{\partial^2 P_q}{\partial x \partial y} \\ \frac{\partial^2 P_q}{\partial x \partial y} & \frac{\partial^2 P_q}{\partial y^2} \end{bmatrix}_{x=y=0} < 0. \tag{39}$$

To study the phase portrait of the first case($\lambda = -1$) we have the separated function R:

$$R = p^2 + q^3 + \frac{3}{2}q^2 + 3h - f. \tag{40}$$

the singular points of this case are $(0,0)$ and $(0,-1)$. These points are elliptic and hyperbolic respectively as they shown in Figure 10.

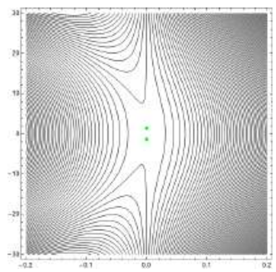

Figure 6: The two-hyperbolic point, where $h > 0$.

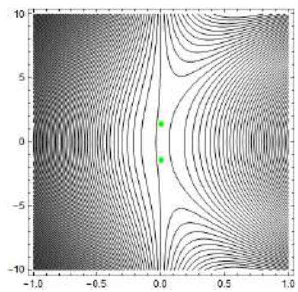

Figure 7: The two-hyperbolic point, where $h < 0$.

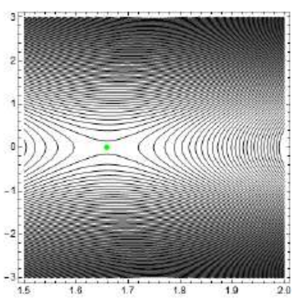

Figure 8: The hyperbolic point, where $h > 0$.

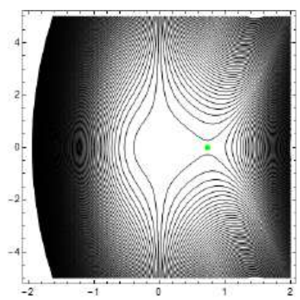

Figure 9: The hyperbolic point, where $h < 0$.

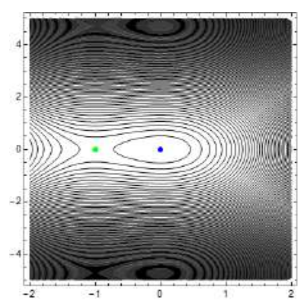

Figure 10: The one elliptic point, and the one hyperbolic point.

6. Poincaré surface section

To complete the picture of the present problem, we solve the GHH system numerically by using a surface section introducing by Poincaré [28]. The trajectory of the system may be treated in three dimension invariant $(x, \dot{x}, y, \dot{y})$, with h equals a constant. We find the Poincaré surface section $(x, \dot{x})$. If the regular lies on the smooth curves in the surface section then the system is integrable, and the regular orbits are called the invariant curves. If the system is non-integral then a part of the curves seem to destroy and chaotic orbits appear instead, filling a stochastic region.

Putting $\dot{x} = \dot{y} = 0$ in (1), then we have the equipotential lines

$$V(x, y) = h. \tag{41}$$

Putting $\dot{x} = 0$ in integral energy equation, we have

$$\dot{y} = \sqrt{2(h - V(0, y))}. \tag{42}$$

Now, we deduce the surface-section in all integrable cases to confirm that the orbits of the motion lie on invariant tori and the motion is ordered:

1. The first case $\lambda = -1, \frac{A}{B} = 1$, the equipotential lines at different values of h are shown in Figure 11, the invariant curves at $h = 0.08333$, $h = 0.1667$, and $h = 1$ are shown in Figures 12− 14.
2. The second case $\lambda = -\frac{1}{6}$ any A, B, the equipotential lines at different values of h are shown in Figure 15, the invariant curves at $h = 0.08333$, $h = 0.1667$, and $h = 1$ are shown in Figures 16− 18.
3. The third case $\lambda = \frac{1}{16}, \frac{A}{B} = \frac{1}{16}$, the equipotential lines at different values of h are shown in Figure 19, the invariant curves at $h = 0.08333$, $h = 0.1667$, and $h = 1$ are shown in Figures 20− 22.
4. Their are another case which GHH can be integrated, this case suggest by Chang et al. [8], this may exist for values of λ for which $\sqrt{1 - 48\lambda}$ = n/m with n and m two relatively prime integers. Then in this case the equipotential lines at different values of h are shown in Figure 23, the invariant curves at $h = 0.08333$, $h = 0.1667$, and $h = 1$ are shown in Figures 24− 26.

7. Conclusion

We have studied the complete description of the real phase topology of the problem and concluded that the topological type of the the level set is torus, tours or empty set as shown in Table 2. And, the periodic solution of the problem is determined. It is separated which allows to construct the topological translation of the trajectory by using the phase portrait. The singular points of the separated functions which are elliptic or hyperbolic points are determined. The elliptic points in the figures are stable in the Lyapunov sense, because a small disturbance will result in a closed trajectory that surrounds it and along which the state of the system remains close to these points, which the hyperbolic points are unstable because any small disturbance will result in a trajectory on which the state of the system deviates more and more from these points as t goes to infinity.

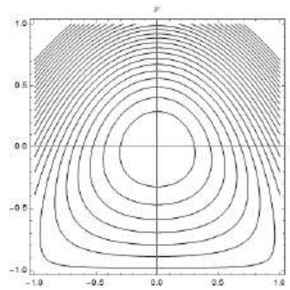

Figure 11: The equipotential line for $\lambda = -1$ with a different values of h.

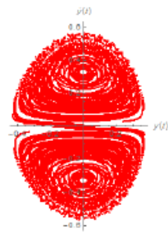

Figure 12: The invariant curves for $\lambda = -1$, and $h = 0.0833$.

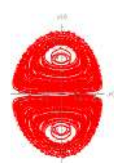

Figure 13: The invariant curves for $\lambda = -1$, and $h = 0.1667$.

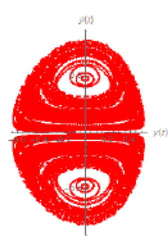

Figure 14: The invariant curves for $\lambda = -1$, and $h = 1$.

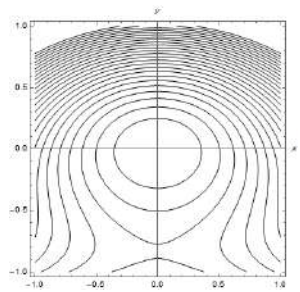

Figure 15: The equipotential line for $\lambda = -\frac{1}{6}$ with a different values of h.

Figure 16: The invariant curves for $\lambda = -\frac{1}{6}$, and $h = 0.0833$.

Figure 17: The invariant curves for $\lambda = -\frac{1}{6}$, and $h = 0.1667$.

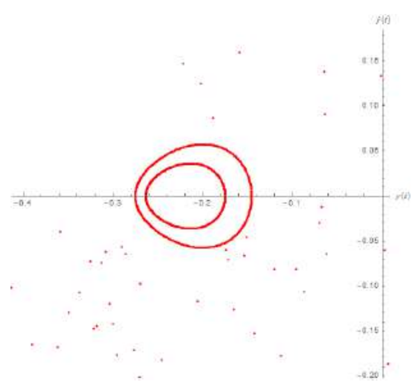

Figure 18: The invariant curves for $\lambda = -\frac{1}{6}$, and $h = 1$.

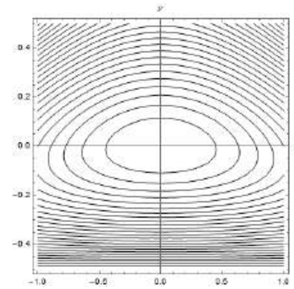

Figure 19: The equipotential line for$\lambda = -\frac{1}{16}$ with a different values of h.

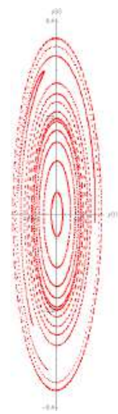

Figure 20: The invariant curves for$\lambda = -\frac{1}{16}$, and $h = 0.0833$.

Figure 21: The invariant curves for $\lambda = -\frac{1}{16}$, and $h = 0.1667$.

Figure 22: The invariant curves for $\lambda = -\frac{1}{16}$, and $h = 1$.

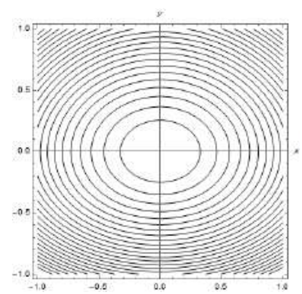

Figure 23: The equipotential line for $\lambda = \frac{1}{48}(1-(\frac{n}{m})^2)$with a different values of h.

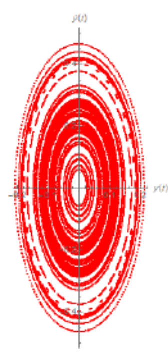

Figure 24: The invariant curves for $\lambda = \frac{1}{48}(1-\frac{4}{9})$, and $h = 0.0833$.

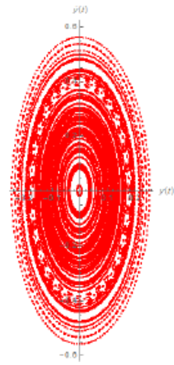

Figure 25: The invariant curves for $\lambda = \frac{1}{48}(1 - \frac{4}{9})$, and $h = 0.1667$.

Figure 26: The invariant curves for $\lambda = \frac{1}{48}(1 - \frac{4}{9})$, and $h = 1$.

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