Reciprocal sums of triple products of general second order recursion

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Abstract. By applying the method of generating function, the purpose of this paper is to give several summation of reciprocals related to triple product of general second order recurrence $\{W_{rn}\}$ for arbitrary positive integer r. As applications, some identities involving Fibonacci, Lucas numbers are obtained.
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1. Introduction

In the notation of Horadam [4], write

$$W_n = W_n(a, b; P, Q)$$

so that

$$(1) \qquad W_n = PW_{n-1} - QW_{n-2}, \quad (W_0 = a,\ W_1 = b,\ n \geq 2),$$

where a, b, P and Q are integers, with $PQ \neq 0$. In the sequel we shall suppose that $\Delta = P^2 - 4Q > 0$. Then it is easily to obtain the Binet formula [4]:

$$(2) \qquad W_n = \frac{A\alpha^n - B\beta^n}{\alpha - \beta},$$

where $\alpha = \frac{P+\sqrt{P^2-4Q}}{2}$, $\beta = \frac{P-\sqrt{P^2-4Q}}{2}$, $A = b - \beta a$ and $B = b - \alpha a$. In particular, if $U_n = W_n(0, 1; P, Q), V_n = W_n(2, P; P, Q)$, then $\alpha^n - \beta^n = (\alpha - \beta)U_n, \alpha^n + \beta^n = V_n$.

In [1], Andre-Jeannin obtained the following series identities:

$$(3) \qquad \sum_{n=1}^{m} \frac{Q^n}{W_n W_{n+k}} = U_m \sum_{n=1}^{k} \frac{Q^n}{W_n W_{n+m}},$$

$$(4) \qquad \sum_{n=1}^{\infty} \frac{Q^n}{W_n W_{n+k}} = \frac{1}{ABU_k} \left(\sum_{n=1}^{k} \frac{W_{n+1}}{W_n} - k\alpha \right),$$

where $P > 0$ and k and m are nonnegative integers. (3) and (4) are given by Good [3] with the case $Q = -1$. Brousseau [2] proved (4) for $W_n = F_n$. The Fibonacci sequence is a classical sequence generated from the question of the rabbits propagation. From the viewpoint of the rabbits propagationthe classical Fibonacci sequence is generalized in many forms and some general forms are obtained. If integers $F_0 = 0, F_1 = 1$, and $W_n(0, 1; 1, -1) = F_n = F_{n-1} + F_{n-2}, n \geq 2$ is called generalized Fibonacci number.

In [7], Mansour used the generating function techniques to get several summation of reciprocals related to generalized Fibonacci numbers.

Regarding taking l-th powers of terms in the sums, Xi [9] generalized the results of (3) and (4). For example, he derived the following infinitive reciprocal sums:

$$\begin{aligned}\sum_{n=k}^{\infty} \frac{Q^n}{W_n^l W_{n+m}^l} \sum_{i=0}^{l-1} &\Big[(W_{k+1} - W_k\beta)Q^{n-k}\alpha^m - (W_{k+1} - W_k\alpha) \\ &\times \beta^{2(n-k)+m}\Big]^{l-1-i} \Big[(W_{k+1} - W_k\beta)Q^{n-k}\beta^m \\ &-(W_{k+1} - W_k\alpha)\beta^{2(n-k)+m}\Big]^i \\ &= \frac{(P^2 - 4Q)^{l/2}Q^k}{(\alpha^m - \beta^m)(W_{k+1} - W_k\beta)} \sum_{i=0}^{m-1} \frac{\beta^{li}}{W_{k+i}^l}. \end{aligned} \tag{5}$$

In [6], Kilic and Irmak obtained the generalize the result of (5) regarding reciprocal sums of l-th powers of the terms with indices.

In [8], Xi obtained the summation of reciprocals related to of triple product of generalized Fibonacci sequences. By applying the method of generating function, the purpose of this paper is to give several summation of reciprocals related to triple product of general second order recurrence $\{W_{rn}\}$ for arbitrary positive integer r. As applications, some identities involving Fibonacci, Lucas numbers are obtained.

Throughout the paper, we assume that $r,\ k,\ l,\ m$ and t are five positive integers with $t \geq k$.

2. Main result

In this section we consider both finite and infinite reciprocal sums of square of triple product of r-consecutive terms of sequence $\{W_n\}$. First of all, we give the following general second order recurrence $\{W_{rn}\}$ of [6] which are to be used later.

Let W_n be the nth term of sequence $\{W_n\}$ (see [5]) . Then for $n, r > 0$,

$$W_{rn} = Y_r W_{r(n-1)} - Z_r W_{r(n-2)} \tag{6}$$

where $Y_r = \alpha^r + \beta^r$ and $Z_r = (\alpha\beta)^r$.

Theorem 2.1. *Let $P > 0$. Then*

$$\sum_{n=k}^{t} \frac{Q^{rn}}{W_{rn}W_{r(n+1)}W_{r(n+2)}} \left[\mu_{rk}\alpha^{r(n+1-k)} + \nu_{rk}\beta^{r(n+1-k)}\right] = \frac{Q^{rk}}{\mu_{rk}} \left[\frac{1}{W_{rk}} - \frac{\beta}{W_{r(k+1)}} - \frac{\beta^{r(t+1-k)}}{W_{r(t+1)}} + \frac{\beta^{r(t+2-k)}}{W_{r(t+2)}}\right], \tag{7}$$

where $\mu_{rk} = W_{r(k+1)} - W_{rk}\beta^r$, $\nu_{rk} = W_{r(k+1)} - W_{rk}\alpha^r$.

Proof. Let $f(x) = \sum_{n=k}^{\infty} W_{rn}x^n$, $\mu_{rk} = W_{r(k+1)} - W_{rk}\beta^r$, $\nu_{rk} = W_{r(k+1)} - W_{rk}\alpha^r$. From (6) we have

$$f(x) - W_{rk}x^k - W_{r(k+1)}x^{k+1} = Y_r x[f(x) - W_{rk}x^k] - Z_r x^2 f(x).$$

Hence, the following generating function is obtained:

$$f(x) = x^k \frac{W_{rk} + x[W_{r(k+1)} - Y_r W_{rk}]}{1 - Y_r x + Z_r x^2}.$$

Since $1 - Y_r x + Z_r x^2 = (1 - \alpha^r x)(1 - \beta^r x)$, we can decompose $f(x)$ into partial fractions:

$$f(x) = \frac{x^k}{\alpha^r - \beta^r}\left(\frac{\mu_{rk}}{1 - \alpha^r x} - \frac{\nu_{rk}}{1 - \beta^r x}\right).$$

Comparing the coefficients of x^n in both sides of above equation, we obtain that

$$W_{rn} = \frac{\mu_{rk}}{\alpha^r - \beta^r}\alpha^{r(n-k)} - \frac{\nu_{rk}}{\alpha^r - \beta^r}\beta^{r(n-k)}. \tag{8}$$

Let

$$T_{rn} = \frac{\beta^{r(n-k)}}{W_{rn}} = \frac{\beta^{r(n-k)}}{\frac{\mu_{rk}}{\alpha^r - \beta^r}\alpha^{r(n-k)} - \frac{\nu_{rk}}{\alpha^r - \beta^r}\beta^{r(n-k)}}. \tag{9}$$

Then

$$\begin{aligned} T_{rn} - T_{r(n+1)} &= \frac{\beta^{r(n-k)}}{W_{rn}} - \frac{\beta^{r(n+1-k)}}{W_{r(n+1)}} \\ &= \frac{\beta^{r(n-k)}[W_{r(n+1)} - \beta^r W_{rn}]}{W_{rn}W_{r(n+1)}} \\ &= \frac{\mu_{rk}Q^{r(n-k)}}{W_{rn}W_{r(n+1)}} \end{aligned}$$

and so

$$T_{r(n+1)} - T_{r(n+2)} = \frac{\mu_{rk}Q^{r(n+1-k)}}{W_{r(n+1)}W_{r(n+2)}}.$$

Hence, we have

$$\begin{aligned} &T_{rn} - T_{r(n+1)} - T_{r(n+1)} + T_{r(n+2)} \\ &= \frac{\beta^{r(n-k)}}{W_{rn}} - \frac{\beta^{r(n+1-k)}}{W_{r(n+1)}} - \frac{\beta^{r(n+1-k)}}{W_{r(n+1)}} + \frac{\beta^{r(n+2-k)}}{W_{r(n+2)}} \\ &= \frac{\mu_{rk} Q^{r(n-k)}}{W_{rn} W_{r(n+1)}} - \frac{\mu_{rk} Q^{r(n+1-k)}}{W_{r(n+1)} W_{r(n+2)}} \\ &= \frac{\mu_{rk} Q^{r(n-k)}}{W_{rn} W_{r(n+1)} W_{r(n+2)}} \left[\mu_{rk} \alpha^{r(n+1-k)} + \nu_{rk} \beta^{r(n+1-k)} \right]. \end{aligned}$$

Thus,

$$\begin{aligned} &\sum_{n=k}^{t} \frac{Q^{rn}}{W_{rn} W_{r(n+1)} W_{r(n+2)}} \left[\mu_{rk} \alpha^{r(n+1-k)} + \nu_{rk} \beta^{r(n+1-k)} \right] \\ &\quad = \frac{Q^{rk}}{\mu_{rk}} \left[\frac{1}{W_{rk}} - \frac{\beta}{W_{r(k+1)}} - \frac{\beta^{r(t+1-k)}}{W_{r(t+1)}} + \frac{\beta^{r(t+2-k)}}{W_{r(t+2)}} \right]. \end{aligned}$$

The proof of the theorem is completed. □

Theorem 2.2. *Let $P > 0$. Then*

$$\begin{aligned} &\sum_{n=k}^{\infty} \frac{Q^{rn}}{W_{rn} W_{r(n+1)} W_{r(n+2)}} \left[\mu_{rk} \alpha^{r(n+1-k)} + \nu_{rk} \beta^{r(n+1-k)} \right] \\ &\quad = \frac{Q^{rk}}{\mu_{rk}} \left[\frac{1}{W_{rk}} - \frac{\beta}{W_{r(k+1)}} \right], \end{aligned} \tag{10}$$

where $\mu_{rk} = W_{r(k+1)} - W_{rk}\beta^r$, $\nu_{rk} = W_{r(k+1)} - W_{rk}\alpha^r$.

Proof. By (7), we have

$$\begin{aligned} &\sum_{n=k}^{\infty} \frac{Q^{rn}}{W_{rn} W_{r(n+1)} W_{r(n+2)}} \left[\mu_{rk} \alpha^{r(n+1-k)} + \nu_{rk} \beta^{r(n+1-k)} \right] \\ &\quad = \frac{Q^{rk}}{\mu_{rk}} \lim_{t\to\infty} \left[\frac{1}{W_{rk}} - \frac{\beta}{W_{r(k+1)}} - \frac{\beta^{r(t+1-k)}}{W_{r(t+1)}} + \frac{\beta^{r(t+2-k)}}{W_{r(t+2)}} \right], \end{aligned}$$

where the limiting process has been justified by

$$\left| \frac{\alpha}{\beta} \right| = \left| \frac{P + \sqrt{P^2 - 4Q}}{P - \sqrt{P^2 - 4Q}} \right| > 1, \quad \text{for} \quad P > 0,$$

and

$$\begin{aligned} \lim_{t\to\infty} T_{rt} &= \lim_{t\to\infty} \frac{\beta^{r(t-k)}}{W_{rt}} \\ &= \lim_{t\to\infty} \frac{1}{\frac{W_{r(k+1)} - W_{rk}\beta^r}{\alpha^r - \beta^r} \left(\frac{\alpha}{\beta}\right)^{r(t-k)} - \frac{W_{r(k+1)} - W_{rk}\alpha^r}{\alpha^r - \beta^r}} = 0. \end{aligned} \tag{11}$$

The proof of the theorem is completed. □

Theorem 2.3. *Let $P > 0$. Then*

$$\sum_{n=k}^{t} \frac{Q^{rn}}{W_{rn}W_{r(n+m)}W_{r(n+m+l)}} \Big\{\mu_{rk}\alpha^{r(n+m-k)}\Big[\alpha^{r(m+l)} + \beta^{r(m+l)} - 2\alpha^{(rl)}\beta^{(rm)}\Big]$$
$$+\nu_{rk}\beta^{r(n+m-k)}\Big[\alpha^{r(m+l)} + \beta^{r(m+l)} - 2\alpha^{(rm)}\beta^{(rl)}\Big]\Big\}$$
$$= \frac{(\alpha^r - \beta^r)^2 Q^{rk}}{\mu_{rk}} \left\{\sum_{i=0}^{m-1}\left[\frac{\beta^{ri}}{W_{r(k+i)}} - \frac{\beta^{r(t+i+1-k)}}{W_{r(t+i+1)}}\right]\right.$$
$$\left. - \sum_{i=0}^{l-1}\left[\frac{\beta^{r(m+i)}}{W_{r(k+m+i)}} - \frac{\beta^{r(t+m+i+1-k)}}{W_{r(t+m+i+1)}}\right]\right\}, \tag{12}$$

where $\mu_{rk} = W_{r(k+1)} - W_{rk}\beta^r$, $\nu_{rk} = W_{r(k+1)} - W_{rk}\alpha^r$.

Proof. From (9) and (8), we have

$$T_{rn} - T_{r(n+m)} = \frac{(\alpha^{rm} - \beta^{rm})\mu_{rk}Q^{r(n-k)}}{(\alpha^r - \beta^r)W_{rn}W_{r(n+m)}},$$

$$T_{r(n+m)} - T_{r(n+m+l)} = \frac{(\alpha^{rl} - \beta^{rl})\mu_{rk}Q^{r(n+m-k)}}{(\alpha^r - \beta^r)W_{r(n+m)}W_{r(n+m+l)}}.$$

Hence,

$$T_{rn} - T_{r(n+m)} - T_{r(n+m)} + T_{r(n+m+l)}$$
$$= \frac{(\alpha^{rm} - \beta^{rm})\mu_{rk}Q^{r(n-k)}}{(\alpha^r - \beta^r)W_{rn}W_{r(n+m)}} - \frac{(\alpha^{rl} - \beta^{rl})\mu_{rk}Q^{r(n+m-k)}}{(\alpha^r - \beta^r)W_{r(n+m)}W_{r(n+m+l)}}$$
$$= \frac{\mu_{rk}Q^{r(n-k)}}{(\alpha^r - \beta^r)^2 W_{rn}W_{r(n+m)}W_{r(n+m+l)}} \Big\{\mu_{rk}\alpha^{r(n+m-k)}\Big[\alpha^{r(m+l)} + \beta^{r(m+l)}$$
$$-2\alpha^{(rl)}\beta^{(rm)}\Big] + \nu_{rk}\beta^{r(n+m-k)}\Big[\alpha^{r(m+l)} + \beta^{r(m+l)} - 2\alpha^{(rm)}\beta^{(rl)}\Big]\Big\}.$$

Thus,

$$\sum_{n=k}^{t} \frac{Q^{rn}}{W_{rn}W_{r(n+m)}W_{r(n+m+l)}} \Big\{\mu_{rk}\alpha^{r(n+m-k)}\Big[\alpha^{r(m+l)} + \beta^{r(m+l)} - 2\alpha^{(rl)}\beta^{(rm)}\Big]$$
$$+\nu_{rk}\beta^{r(n+m-k)}\Big[\alpha^{r(m+l)} + \beta^{r(m+l)} - 2\alpha^{(rm)}\beta^{(rl)}\Big]\Big\}$$
$$= \frac{(\alpha^r - \beta^r)^2 Q^{rk}}{\mu_{rk}} \left\{\sum_{i=0}^{m-1}\left[\frac{\beta^{ri}}{W_{r(k+i)}} - \frac{\beta^{r(t+i+1-k)}}{W_{r(t+i+1)}}\right]\right.$$
$$\left. - \sum_{i=0}^{l-1}\left[\frac{\beta^{r(m+i)}}{W_{r(k+m+i)}} - \frac{\beta^{r(t+m+i+1-k)}}{W_{r(t+m+i+1)}}\right]\right\}.$$

The proof of the theorem is completed. □

Theorem 2.4. *Let $P > 0$. Then*

$$\sum_{n=k}^{\infty} \frac{Q^{rn}}{W_{rn}W_{r(n+m)}W_{r(n+m+l)}} \Big\{\mu_{rk}\alpha^{r(n+m-k)}\left[\alpha^{r(m+l)}+\beta^{r(m+l)}-2\alpha^{rl}\beta^{rm}\right]$$
$$+\nu_{rk}\beta^{r(n+m-k)}\left[\alpha^{r(m+l)}+\beta^{r(m+l)}-2\alpha^{rm}\beta^{rl}\right]\Big\}$$
$$= \frac{(\alpha^r-\beta^r)^2Q^{rk}}{\mu_{rk}}\left[\sum_{i=0}^{m-1}\frac{\beta^{ri}}{W_{r(k+i)}}-\sum_{i=0}^{l-1}\frac{\beta^{r(m+i)}}{W_{r(k+m+i)}}\right], \tag{13}$$

where $\mu_{rk} = W_{r(k+1)} - W_{rk}\beta^r$, $\nu_{rk} = W_{r(k+1)} - W_{rk}\alpha^r$.

Proof. By (12) and (11), we have

$$\sum_{n=k}^{\infty} \frac{Q^{rn}}{W_{rn}W_{r(n+m)}W_{r(n+m+l)}} \Big\{\mu_{rk}\alpha^{r(n+m-k)}\left[\alpha^{r(m+l)}+\beta^{r(m+l)}-2\alpha^{rl}\beta^{rm}\right]$$
$$+\nu_{rk}\beta^{r(n+m-k)}\left[\alpha^{r(m+l)}+\beta^{r(m+l)}-2\alpha^{rm}\beta^{rl}\right]\Big\}$$
$$= \frac{(\alpha^r-\beta^r)^2Q^{rk}}{\mu_{rk}}\lim_{t\to\infty}\Bigg\{\sum_{i=0}^{m-1}\left[\frac{\beta^{ri}}{W_{r(k+i)}}-\frac{\beta^{r(t+i+1-k)}}{W_{r(t+i+1)}}\right]$$
$$-\sum_{i=0}^{l-1}\left[\frac{\beta^{r(m+i)}}{W_{r(k+m+i)}}-\frac{\beta^{r(t+m+i+1-k)}}{W_{r(t+m+i+1)}}\right]\Bigg\}.$$

The theorem is proved. □

Corollary 2.5. *Let $P > 0$. Then*

$$\sum_{n=1}^{\infty} \frac{Q^{rn}}{W_{rn}W_{r(n+m)}W_{r(n+m+l)}} \Big\{\mu_{r}\alpha^{r(n+m-1)}\left[\alpha^{r(m+l)}+\beta^{r(m+l)}-2\alpha^{rl}\beta^{rm}\right]$$
$$+\nu_{r}\beta^{r(n+m-1)}\left[\alpha^{r(m+l)}+\beta^{r(m+l)}-2\alpha^{rm}\beta^{rl}\right]\Big\}$$
$$= \frac{(\alpha^r-\beta^r)^2Q^{r}}{\mu_{r}}\left[\sum_{i=0}^{m-1}\frac{\beta^{ri}}{W_{r(1+i)}}-\sum_{i=0}^{l-1}\frac{\beta^{r(m+i)}}{W_{r(1+m+i)}}\right],$$

where $\mu_r = W_{2r} - W_r\beta^r$, $\nu_r = W_{2r} - W_r\alpha^r$.

Proof. Take $k = 1$ in the identity (13), respectively. □

3. Some applications

In this section we can obtain some interesting identities involving Fibonacci, Lucas numbers by taking special values for a, b, P and Q.

3.1 Fibonacci numbers

We note that, $W_n(0,1;1,-1)=F_n$, the Fibonacci number. Then according to above theorems, corollaries and the Binet formula (2) we obtain

$$\sum_{n=1}^{\infty}\frac{(-1)^{rn}}{F_{rn}F_{r(n+m)}F_{r(n+m+l)}}\Big\{\alpha^{r(n+m)}\Big[\alpha^{r(m+l)}+\beta^{r(m+l)}-2\alpha^{rl}\beta^{rm}\Big] \\ +\beta^{r(n+m)}\Big[\alpha^{r(m+l)}+\beta^{r(m+l)}-2\alpha^{rm}\beta^{rl}\Big]\Big\} \\ =\frac{5(-1)^r}{\alpha^r}\left[\sum_{i=0}^{m-1}\frac{\beta^{ri}}{F_{r(1+i)}}-\sum_{i=0}^{l-1}\frac{\beta^{r(m+i)}}{F_{r(1+m+i)}}\right].$$

In particular,

$$\sum_{n=1}^{\infty}\frac{1}{F_{2n}F_{2n+4}F_{2n+8}}\left(\alpha^{2n+4}+\beta^{2n+4}\right)=\frac{151-48\sqrt{5}}{1512}.$$

3.2 Lucas numbers

We note that, $W_n(2,1;1,-1)=L_n$, the Lucas number. Then according to above theorems, corollaries and the Binet formula (2) we obtain

$$\sum_{n=1}^{\infty}\frac{(-1)^{rn}}{L_{rn}L_{r(n+m)}L_{r(n+m+l)}}\Big\{\alpha^{r(n+m)}\Big[\alpha^{r(m+l)}+\beta^{r(m+l)}-2\alpha^{rl}\beta^{rm}\Big] \\ -\beta^{r(n+m)}\Big[\alpha^{r(m+l)}+\beta^{r(m+l)}-2\alpha^{rm}\beta^{rl}\Big]\Big\} \\ =\frac{(-1)^r}{\alpha^r}\left[\sum_{i=0}^{m-1}\frac{\beta^{ri}}{L_{r(1+i)}}-\sum_{i=0}^{l-1}\frac{\beta^{r(m+i)}}{L_{r(1+m+i)}}\right].$$

In particular,

$$\sum_{n=1}^{\infty}\frac{1}{L_{2n}L_{2n+2}L_{2n+6}}\Big[(15+\sqrt{5})\alpha^{2n+4}-(15-\sqrt{5})\beta^{2n+4}\Big]=\frac{383\sqrt{5}}{5922}.$$

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