Existence and exponential stability of second-order neutral stochastic functional differential equations with infinite delay and Poisson jumps

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Abstract. In this work, we study the existence and uniqueness of mild solutions to second-order neutral stochastic functional differential equations (NSFDEs) with infinite delay and Poisson jumps under global and local Carathéodory conditions by means of the successive approximation. The p-th moment exponential stability of mild solution to second-order NSFDEs with infinite delay and poisson jumps is also studied. Further, example is given to illustrate the proposed theory.
Keywords: Existence, stability, stochastic differential equations, infinite delay, Poisson jumps.

1. Introduction

In recent days, The study of stochastic differential equations (SDEs) has attracted the researchers because of its applicability to diverse fields [1, 30, 8, 11, 17, 21].

Stochastic systems depend on the present state and a period of past state as well, this system is said to be stochastic functional differential equation (SFDE). In many areas of science, there has been an increasing interest in the investigation of SFDEs incorporating memory or aftereffect i.e., there is the effect of infinite delay on state equations. The importance of SFDEs with infinite delay can be found in [4, 29, 14] and references there in. The development of the theory of functional differential equations with infinite delay depends on a choice of a phase space. In fact, various phase spaces have been considered and each different phase space required a separate development of the theory [12]. The common phase space $\mathcal{B}$ is proposed by Hale and Kato in [9]. Kolmanovskii, in [14], introduced the NSFDE and its applications to chemical engineering and

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aero elasticity. Further, it has been drawn the attention of many researchers [3, 6, 10, 17, 18, 22, 23, 31] and references there in. In [5], the existence of the solutions of first order NSFDEs with delay and Poisson jumps are studied. It may be noted here that the mentioned works are confined to first-order systems.

To model the problems like mechanical vibrations, charge on a capacitor, condenser subjected to white noise excitation, second-order SDEs are more appropriate. With the advent of applications, second order SDEs, FSDEs, NSFDEs have attracted the focus of many researchers since last decade. The second-order damped functional stochastic evolution equations are studied by McKibben [19] and one can refer [16] for further works on this topic. Moreover, McKibben [20] established the existence and uniqueness of mild solutions for a class of second-order neutral stochastic evolution equations with finite delay. Balasubramaniam et al. [2] gave the sufficient conditions for the approximate controllability of the second-order neutral stochastic evolution equations with infinite delay. In [26], the authors established the asymptotic stability of second-order neutral stochastic differential equations using fixed point theorem.

The existence and uniqueness of solution of SFDEs with Poisson jumps are established in [15, 24]. Very recently, existence, uniqueness and stability of mild solutions for second-order neutral stochastic evolution equations with infinite delay and Poisson jumps are discussed in [25] and references there in. Further, Sakthivel et al. [27] presented the exponential stability of nonlinear second-order stochastic evolution equations with Poisson jumps by using a fixed point argument, Jiang et al.[13] discussed stability analysis for second-order stochastic neutral partial functional systems subject to infinite delays and impulses by the new integral inequality together with the stochastic analysis technique.

In this paper, inspired by the aforementioned works [13, 25, 23, 27], second-order NSFDEs with infinite delay and Poisson jumps is considered. To the best of our knowledge, there are no results on the existence of mild solutions of second-order NSFDEs with infinite delay and Poisson jumps under Carath$\acute{e}$odory conditions in available literature. Motivated by the above attention, using Carath$\acute{e}$odory conditions we aim to establish the existence and uniqueness of mild solutions to second-order NSFDEs with infinite delay and Poisson jumps in which the initial value belongs to the space $\mathcal{B}((-\infty, 0], H)$ (for more details refer section 2). Besides, the exponential stability in p-th moment of the considered NSFDEs with infinite delay and Poisson jumps is studied to obtain the required sufficient conditions.

The paper is structured as follows. In section 2, some preliminaries are presented. Existence and uniqueness of mild solutions are discussed in section 3. In section 4, the p-th moment exponential stability of mild solutions are presented. An example is provided in last section to illustrate the theory.

2. Preliminaries

In this section, some basic concepts that are useful for the development of our results are presented. For more details, the reader may refer to Da Prato and Zabczyk [6], Frattorini [7], Hale and Kato [9] and the references therein.

Let H and K be two real separable Hilbert spaces. $L(K, H)$ stands for the set of all bounded linear operators form K into H. We will use the notation $|.|$ and $\langle . \, , . \rangle$ to denote the norm and inner product for H and $L(K, H)$ respectively. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a filtered complete probability space with an increasing right continuous family $\{\mathcal{F}_t\}_{t\geq 0}$ of complete sub $\sigma-$ algebras of $\mathcal{F}$.

Definition 2.1 ([9]). *$\mathcal{B}$ is a linear space of family of $\mathcal{F}_0$- measurable functions from $(-\infty, 0]$ into H endowed with a norm $\|.\|_{\mathcal{B}}$ which satisfies the following axioms:*

$(A1)$ *If $x : (-\infty, T] \to H$ is continuous on $[0, T]$ and $x_0 \in \mathcal{B}$, then, for every $t \in [0, T]$, the following conditions hold.*

(1) $x_t \in \mathcal{B}$

(2) $|x(t)| \leq L\|x_t\|_{\mathcal{B}}$

(3) $\|x_t\|_{\mathcal{B}} \leq M(t) \sup_{0\leq s\leq t} |x(s)| + N(t)\|x_0\|_{\mathcal{B}}$

where $L > 0$ is a constant, $M, N : [0, \infty) \to [1, \infty)$ are continuous, $N(t)$ is locally bounded and L, M, N are independent of x(.).

$(A2)$ *The space $\mathcal{B}$ is complete.*

Remark 2.1. For convenience, the condition (3) in (A1) can be replaced by the following condition

$$\|x_t\|_{\mathcal{B}} \leq \sup_{0\leq s\leq t} |x(s)| + N\|x_0\|_{\mathcal{B}},$$

where $N = \sup_{0\leq s\leq t} |N(s)|$.

Let $Q \in L(K, H)$ be an operator defined by $Qe_n = \lambda_n e_n$ with finite trace $Tr(Q) = \sum_{n=1}^{\infty} \lambda_n < \infty$ where $\lambda_n \geq 0$ (n = 1, 2, ...) are some nonnegative real numbers and $\{e_n\}$ (n = 1, 2, ...) is a complete orthonormal system $\{e_n\}$ in K. Then the above K valued stochastic process $w(t)$ is called a Q-wiener process. Let $\beta_n(t)$ $(n = 1, 2, ...)$ be a sequence of real valued one dimensional Brownian motion. Set

$$\langle w(t), e\rangle = \sum_{n=1}^{\infty} \sqrt{\lambda_n} \langle e_n, e\rangle \beta_n(t), \quad e \in K.$$

Let $\sigma \in L(K, H)$ and define

$$\|\sigma\|^2_{L_2^0} = Tr(\sigma Q\sigma^*) = \sum_{n=1}^{\infty} |\sqrt{\lambda_n}\sigma e_n|^2.$$

If $|\sigma|^2_{L_2^0} < \infty$, then $\|\sigma\|$ is called a Q-Hilbert-Schmidt operator and $L_2^0(K,H)$ denotes the space of all Q-Hilbert-Schmidt operators $\sigma : K \to H$. Let $p = p(t), t \geq 0$ be a stationary $\mathcal{F}_t-$ Poisson process with characteristic measure λ. $N(dt, du)$ denotes the Poisson counting measure associated with λ, i.e., $N(t,Z) = \sum_{t_1<s<t_2} I_Z(p(s))$ with a measurable set $Z \in \mathcal{B}(K-\{0\})$, which denotes the Borel $\sigma-$field of $K-\{0\}$. $\tilde{N}(dt,du) = N(dt,du) - dt\lambda(du)$ represents the compensated Poisson measure that is independent of $w(t)$.

The one parameter family $\{C(t) : t \in \mathbb{R}\} \subset L(H,H)$ satisfying

(i) $C(0) = I$,

(ii) $C(t)x$ is continuous in t on $\mathbb{R}$, for all $x \in H$,

(iii) $C(t+s) + C(t-s) = 2C(t)C(s)$, for all $t, s \in \mathbb{R}$

is called a strongly continuous cosine family.

The corresponding strongly continuous sine family $\{S(t) : t \in \mathbb{R}\} \subset L(H,H)$ is defined by $S(t)x = \int_0^t C(s)xds, t \in \mathbb{R}, x \in H$.

The generator $A : H \to H$ of $\{C(t) : t \in \mathbb{R}\}$ is given by $Ax = \frac{d^2}{dt^2}C(t)x|_{t=0}$ for all $x \in D(A) = \{x \in H : C(.)x \in C^2(\mathbb{R}; H)\}$. It is well known that the infinitesimal generator A is closed, densely defined operator H. Such cosine and the corresponding sine families and their generators satisfy the following properties.

Lemma 2.1 ([7])**.** *Suppose that A is the infinitesimal generator of a cosine family of operators $\{C(t) : t \in \mathbb{R}\}$. Then the following properties hold:*

(i) there exists $G^* \geq 1$ and $a \geq 0$ such that $\|C(t)\| \leq G^*e^{at}$ and therefore $\|S(t)\| \leq G^*e^{at}$

(ii) $A\int_s^u S(r)xdr = [C(u) - C(r)]x$ for all $0 \leq s \leq u < \infty$

(iii) there exists $G_1^* \geq 1$ such that $\|S(s) - S(u)\| \leq G_1^* \int_u^s e^{a|\theta|}d\theta$ for all $0 \leq u \leq s < \infty$.

Definition 2.2. *Denote by $\mathcal{M}^2((-\infty,T],H)$ the space of all H-valued continuous $\mathcal{F}_t$- adapted process $x = \{x(t)\}_{-\infty<t\leq T}$ such that*

(i) $x_0 = \phi \in \mathcal{B}$ and $x(t)$ is càdlàg on $[0,T]$;

(ii) define the norm $\|.\|_{\mathcal{M}}$ in $\mathcal{M}^2((-\infty,T],H)$ by

$$\|x\|^2_{\mathcal{M}} = E\|x_0\|^2_{\mathcal{B}} + E\int_0^T |x(t)|^2dt < \infty. \tag{1}$$

Then, $\mathcal{M}^2((-\infty,T],H)$ with the norm (1) is a Banach space. In the sequel, in case without confusion, we just use $\|.\|$ for the norm.

Lemma 2.2 ([6]). *For arbitrary $L_2^0(K,H)$- valued predictable process g such that*

$$\sup_{s\in[0,T]} E\left\|\int_0^s g(r)dw(r)\right\|^{2m} \leq (m(m-1))^m \left(\int_0^t (E\|g(s)\|_{L_2^0}^{2m})^{1/m} ds\right)^m, t \in [0,\infty).$$

Lemma 2.3 ([15]). *Let the space $M_\lambda^\nu([0,T]\times Z\times\Omega,H)$ $(\nu\geq 2)$ be the set of all random process $\rho(t,u)$ with values in H, predictable with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ such that*

$$E\left(\int_0^t\int_Z \|\rho(s,u)\|^\nu \lambda(du)ds\right) < +\infty.$$

Suppose $\rho\in M_\lambda^2([0,t]\times Z\times\Omega,H)\cap M_\lambda^4([0,T]\times Z\times\Omega,H)$, then for any $t\in[0,T]$,

$$E\left[\sup_{s\in[0,t]}\left\|\int_0^s\int_Z S(s-s_1)\rho(s_1,u)\tilde{N}(ds_1,du)\right\|^2\right]$$
$$\leq C\left\{E\left(\int_0^t\int_Z \|\rho(s,u)\|^2\lambda(du)ds\right) + E\left(\int_0^t\int_Z \|\rho(s,u)\|^4\lambda(du)ds\right)^{\frac{1}{2}}\right\},$$

for some number $C>0$, dependent on $T>0$.

In this paper, we consider the second-order neutral stochastic functional differential equation with infinite delay and Poisson jumps of the form, for $t\in[0,T]$,

$$\begin{aligned} d[x'(t)-f_1(t,x_t)] &= [Ax(t)+f_2(t,x_t)]dt + f_3(t,x_t)dw(t) \\ &\quad + \int_Z f_4(t,x_t,u)\tilde{N}(dt,du) \\ x_0 &= \phi\in\mathcal{B}, \\ x'(0) &= \xi \end{aligned} \tag{2}$$

where $A: D(A)\subset H\to H$ is the infinitesimal generator of a strongly continuous cosine family $C(t)$ on H; $f_i\in[0,T]\times\mathcal{B}\to H (i=1,2)$, $f_3:[0,T]\times\mathcal{B}\to L_2^0(K,H)$ and $f_4:[0,T]\times\mathcal{B}\times(Z-\{0\})\to H$ are some suitable measurable mappings. The history $x_t:(-\infty,0]\to H, x_t(\theta)=x(t+\theta)$, for $t\geq 0$, belongs to the phase space $\mathcal{B}$. The initial data $\phi=\{\phi(t): -\infty<t\leq 0\}$ is an $\mathcal{F}_0$-measurable, $\mathcal{B}-$ valued stochastic process with finite second moment and ξ ia an $\mathcal{F}_0-$ measurable H-valued random variable independent of the wiener process with finite second moment.

Definition 2.3. *A stochastic process $x:(-\infty,T]\to H$ is called a mild solution of* (2) *if*

(*i*) *$x(t)$ is $\mathcal{F}_t$- adapted and $\{x_t: t\in[0,T]\}$ is $\mathcal{B}-$valued;*

(*ii*) *$\int_0^T |x(s)|^2 ds<\infty$, P-a.s.;*

(iii) for each $t \in [0,T]$, $x(t)$ satisfies the following integral equation

$$
\begin{aligned}
x(t) = C(t)\phi(0) + S(t)[\xi - f_1(0,\phi)] + \int_0^t C(t-s)f_1(s,x_s)ds \\
+ \int_0^t S(t-s)f_2(s,x_s)ds + \int_0^t S(t-s)f_3(s,x_s)dw(s) \\
+ \int_0^t S(t-s)\int_Z f_4(s,x_s,u)\tilde{N}(ds,du)
\end{aligned} \tag{3}
$$

where $x_0(.) = \phi \in \mathcal{B}$.

In order to obtain existence and uniqueness of mild solutions to (2), we need the following assumptions:

(H1) the cosine family of operators $\{C(t) : t \in [0,T]\}$ on H and the corresponding sine family $\{S(t) : t \in [0,T]\}$ satisfy the conditions $\|C(t)\|^2 \le K_1$ and $\|S(t)\|^2 \le K_1, t \ge 0$ for a positive constant K_1;

(H2) f_i $(i = 1,2,3,4)$ satisfy the following conditions

(2a) there exists a function $\Gamma(t,v) : [0,T] \times [0,\infty) \to [0,\infty)$ such that

$$
\begin{aligned}
E\int_0^t |f_i(s,\psi)|^2 ds + E\int_0^t |f_3(s,\psi)|^2 ds + E\int_0^t \int_Z |f_4(s,\psi,u)|^2 \lambda(du)ds \\
+ E\left(\int_0^t \int_Z |f_4(s,\psi,u)|^4 \lambda(du)ds\right)^{\frac{1}{2}} \\
\le \int_0^t \Gamma(s, E\|\psi\|_{\mathcal{B}}^2)ds
\end{aligned}
$$

for all $\psi \in \mathcal{B}$ and $t \in [0,T]$ $\quad (i = 1,2)$,

(2b) $\Gamma(t,v)$ is locally integrable in t for each fixed $v \in [0,\infty)$ and is continuous concave, and monotone nondecreasing in v for each fixed $t \in [0,T]$,

(2c) for any constant $M > 0$, the deterministic ordinary differential equation

$$\frac{dv}{dt} = M\Gamma(t,v), \quad 0 \le t \le T$$

has a global solution for any initial value v_0;

(H3) (3a) there exists a function $\Upsilon(t,v) : [0,T] \times [0,\infty) \to [0,\infty)$ such that

$$\begin{aligned} &E\int_0^t |f_i(s,\psi) - f_i(s,\varphi)|^2 ds + E\int_0^t |f_3(s,\psi) - f_3(s,\varphi)|^2 ds \\ &\qquad + E\int_0^t \int_Z |f_4(s,\psi,u) - f_4(s,\varphi,u)|^2 \lambda(du) ds \\ &\qquad + E\left(\int_0^t \int_Z |f_4(s,\psi,u) - f_4(s,\varphi,u)|^4 \lambda(du) ds\right)^{\frac{1}{2}} \\ &\leq \int_0^t \Upsilon(s, E\|\psi - \varphi\|_{\mathcal{B}}^2) ds \end{aligned}$$

for all $\psi, \varphi \in \mathcal{B}$ and $t \in [0,T]$ $\quad (i = 1,2)$

(3b) $\Upsilon(t,v)$ is locally integrable in t for each fixed $v \in [0,\infty)$ and is continuous, nondecreasing and concave in v for each fixed $t \geq 0$. Moreover, $\Upsilon(t,0) = 0$ and if a nonnegative continuous function $Y(t), 0 \leq t \leq T$ satisfies

$$Y(t) \leq D\int_0^t \Upsilon(s, Y(s)) ds, \quad 0 \leq t \leq T,$$

where $D > 0$ is a positive constant, then $Y(t) \equiv 0$ for all $0 \leq t \leq T$;

(H4) (the local condition)

(4a) for any integer $N > 0$, there exists a function $\Upsilon_N(t,v) : [0,T] \times [0,\infty) \to [0,\infty)$ such that

$$\begin{aligned} &E\int_0^t |f_i(s,\psi) - f_i(s,\varphi)|^2 ds + E\int_0^t |f_3(s,\psi) - f_3(s,\varphi)|^2 ds \\ &\qquad + E\int_0^t \int_Z |f_4(s,\psi,u) - f_4(s,\varphi,u)|^2 \lambda(du) ds \\ &\qquad + E\left(\int_0^t \int_Z |f_4(s,\psi,u) - f_4(s,\varphi,u)|^4 \lambda(du) ds\right)^{\frac{1}{2}} \\ &\leq \int_0^t \Upsilon_N(s, E\|\psi - \varphi\|_{\mathcal{B}}^2) ds \end{aligned}$$

for all $\psi, \varphi \in \mathcal{B}$ with $\|\psi\|_{\mathcal{B}}, \|\varphi\|_{\mathcal{B}} \leq N$ and $t \in [0,T]$ $\quad (i = 1,2)$,

(4b) $\Upsilon_N(t,v)$ is locally integrable in t for each fixed $v \in [0,\infty)$ and is continuous, nondecreasing, and concave in v for each fixed $t \geq 0$. Moreover, $\Upsilon_N(t,0) = 0$ and if a nonnegative continuous function $Y(t), 0 \leq t \leq T$ satisfies

$$Y(t) \leq D\int_0^t \Upsilon_N(s, Y(s)) ds, \quad 0 \leq t \leq T,$$

where $D > 0$ is a positive constant, then $Y(t) \equiv 0$ for all $0 \leq t \leq T$.

3. Existence and uniqueness

In this section, we prove the existence and uniqueness of mild solution of (2).

Let $x^0(t) = C(t)\phi(0) + S(t)[\xi - f_1(0,\phi)], \quad t \in [0,T]$.

For each $n \geq 1$, the sequence of successive approximation is defined as follows

$$
\begin{aligned}
x^n(t) &= C(t)\phi(0) + S(t)[\xi - f_1(0,\phi)] + \int_0^t C(t-s) f_1(s, x_s^{n-1}) ds \\
&\quad + \int_0^t S(t-s) f_2(s, x_s^{n-1}) ds + \int_0^t S(t-s) f_3(s, x_s^{n-1}) dw(s) \\
&\quad + \int_0^t S(t-s) \int_Z f_4(s, x_s^{n-1}, u) \tilde{N}(ds, du), \quad t \in [0,T], \\
x^n(t) &= \phi(t), \quad -\infty < t \leq 0.
\end{aligned} \tag{4}
$$

Theorem 3.1. *If the assumptions (H1) - (H3) hold. Then, there exists a unique mild solution of* (2) *in space* $\mathcal{M}^2((-\infty,T],H)$.

Proof. The proof of this theorem is divided into the following three steps.

Step 1. Boundedness of $\{x^n(t); n \geq 0\}$ in the space $\mathcal{M}^2((-\infty,T],H)$.

$$
i.e., \quad E(\sup_{0\leq s\leq t} |x^n(s)|^2) \leq u_t \leq u_T < \infty.
$$

It is obvious that $x^0(t) \in \mathcal{M}^2((-\infty,T],H)$ and now we prove that $x^n(t) \in \mathcal{M}^2((-\infty,T],H)$. From (4), using the Hölder inequality and the Doobs martingale inequality, we have

$$
\begin{aligned}
E\left(\sup_{0\leq s\leq t} |x^n(s)|^2\right) &\leq 6K_1 E|\phi(0)|^2 + 12K_1 E|\xi|^2 + 12K_1 E|f_1(0,\phi)|^2 \\
&\quad + 6K_1 T E \int_0^t |f_1(s, x_s^{n-1})|^2 ds + 6K_1 T E \int_0^t |f_2(s, x_s^{n-1})|^2 ds \\
&\quad + 6K_1 E \int_0^t |f_3(s, x_s^{n-1})|^2 ds \\
&\quad + 6CE \int_0^t \int_Z |f_4(s, x_s^{n-1}, u)|^2 \lambda(du) ds \\
&\quad + 6CE \left(\int_0^t \int_Z |f_4(s, x_s^{n-1}, u)|^4 \lambda(du) ds\right)^{\frac{1}{2}} \\
&\leq K_2 + K_3 \int_0^t \Gamma(s, E\|x_s^{n-1}\|_{\mathcal{B}}^2) ds
\end{aligned}
$$

where $K_2 = 6K_1 E\|\phi\|_{\mathcal{B}}^2 + 12K_1 E|\xi|^2 + 12K_1 \Gamma(0, E\|\phi\|_{\mathcal{B}}^2)$, $K_3 = 6(K_1(2T+1) + 2C)$. By Remark 2.1, we have

$$
E(\sup_{0\leq s\leq t} |x^n(s)|^2) \leq K_2 + K_3 \int_0^t \Gamma(s, 2E(N^2\|\phi\|_{\mathcal{B}}^2 + \sup_{0\leq r\leq s} |x^{n-1}(r)|^2)) ds \tag{5}
$$

For any $p \geq 1$, from (5), we get

$$\begin{aligned}
&\max_{1\leq n\leq p} E\left(\sup_{0\leq s\leq t}|x^n(s)|^2\right)\\
&\leq K_2 + K_3\int_0^t \Gamma\left(s, 2E\left(N^2\|\phi\|^2_{\mathcal{B}} + |x^0(s)|^2 + \max_{1\leq n\leq p}\left(\sup_{0\leq r\leq s}|x^n(r)|\right)^2\right)\right)ds\\
&\leq K_2+K_3\int_0^t \Gamma\left(s, \left(\frac{2}{3}K_2 + 2N^2E\|\phi\|^2_{\mathcal{B}}+2E\left(\max_{1\leq n\leq p}\left(\sup_{0\leq r\leq s}|x^n(r)|\right)^2\right)\right)\right)ds
\end{aligned}$$

or

$$\begin{aligned}
\max_{1\leq n\leq p} E&\left(\frac{2}{3}K_2 + 2N^2E\|\phi\|^2_{\mathcal{B}} + 2\sup_{0\leq s\leq t}|x^n(s)|^2\right)\\
&\leq \frac{5}{3}K_2 + 2N^2E\|\phi\|^2_{\mathcal{B}} + K_3\int_0^t\Gamma\bigg(s, \bigg(\frac{2}{3}K_2 + 2N^2E\|\phi\|^2_{\mathcal{B}}\\
&\quad + 2E\bigg(\max_{1\leq n\leq p}\left(\sup_{0\leq r\leq s}|x^n(r)|\right)^2\bigg)\bigg)\bigg)ds.
\end{aligned}$$

Using assumption (2c), it can be observed that u_t satisfies

$$u_t = \frac{5}{3}K_2 + 2N^2E\|\phi\|^2_{\mathcal{B}} + K_3\int_0^t \Gamma(s, u_s)ds.$$

Since $E\|\phi\|^2_{\mathcal{B}} < \infty$, we deduce that

$$\max_{1\leq n\leq p} E(\sup_{0\leq s\leq t}|x^n(s)|^2) \leq u_t \leq u_T < \infty. \tag{6}$$

Since p is arbitrary, we have

$$E|x^n(t)|^2 \leq u_T \quad \text{for all} \;\; 0\leq t\leq T, n\geq 1. \tag{7}$$

From Definition (2.4) and the above result, we obtain

$$\begin{aligned}
\|x^n\|^2 &= E\|x^n_0\|^2_{\mathcal{B}} + E\int_0^T |x^n(t)|^2dt\\
&\leq E\|\phi\|^2_{\mathcal{B}} + Tu_T < \infty,
\end{aligned}$$

which shows that the sequence $\{x^n(t), n\geq 0\}$ is bounded in $\mathcal{M}^2((-\infty, T], H)$.

Step 2. The sequence $\{x^n(t), n\geq 1\}$ is Cauchy. For $m, n \geq 0$ and $t\in[0,T]$, from (4), we get

$$E(\sup_{0\leq s\leq t}|x^{n+1}(s) - x^{m+1}(s)|^2)$$

$$
\begin{aligned}
&\leq 4K_1TE\int_0^t |f_1(s,x_s^n)-f_1(s,x_s^m)|^2 ds\\
&\quad + 4K_1TE\int_0^t |f_2(s,x_s^n)-f_2(s,x_s^m)|^2 ds\\
&\quad + 4K_1E\int_0^t |f_3(s,x_s^n)-f_3(s,x_s^m)|^2 ds\\
&\quad + 4CE\int_0^t\int_Z |f_4(s,x_s^n)-f_4(s,x_s^m,u)|^2\lambda(du)ds\\
&\quad + 4CE\left(\int_0^t\int_Z |f_4(s,x_s^n)-f_4(s,x_s^m,u)|^4\lambda(du)ds\right)^{\frac{1}{2}}\\
&\leq K_4\int_0^t \Upsilon(s,E\|x_s^n-x_s^m\|_{\mathcal{B}})ds, \quad \text{where} \quad K_4 = 4(K_1(2T+1)+2C)\\
&\leq K_4\int_0^t \Upsilon(s,E(\sup_{0\leq r\leq s}|x^n(r)-x^m(r)|^2))ds.
\end{aligned}
$$

Let $Y(t) = \lim_{n,m\to\infty} E(\sup_{0\leq s\leq t}|x^n(s)-x^m(s)|^2)$. From (6), (2b) and the Fatou's lemma, we obtain

$$Y(t) \leq K_4\int_0^t \Upsilon(s,Y(s))ds$$

by (3b), we get $Y(t)=0$, hence $\{x^n(t), n\geq 1\}$ is a Cauchy sequence in $\mathcal{M}^2$. As $n\to\infty$, using Borel-Cantelli lemma, $x^n(t)\to x(t)$ uniformly for $0\leq t\leq T$ and hence (4) tends to the solution $x(t)$ of (2), for all $-\infty < t\leq T$.

Step 3. Uniqueness of solutions of (2)**.** Suppose $x_1(t)$ and $x_2(t)$ are two solutions of (2). From step 1, we can see that $x_1(t), x_2(t)\in\mathcal{M}^2((-\infty,T],H)$. From step 2, it can be shown that

$$E(\sup_{0\leq s\leq t}|x_1(s)-x_2(s)|^2) \leq K_4\int_0^t \Upsilon(s,E(\sup_{0\leq r\leq s}|x_1(r)-x_2(r)|))ds.$$

or

$$Y(t) \leq K_4\int_0^t \Upsilon(s,Y(s))ds.$$

By assumption (3b), we obtain $Y(t)\equiv 0$, which implies that $x_1(t)=x_2(t)$ a.s. for any $t_0\leq t\leq T$. Therefore, for all $-\infty<t\leq T$, $x_1(t)=x_2(t)$ a.s. This completes the proof of the uniqueness of a solution. □

Next, we present the existence and uniqueness of mild solution of (2) with the local Carathéodory conditions.

Theorem 3.2. *Let* $(H1),(H2)$ *and* $(H4)$ *hold. Then there exists a unique mild solution of* (2) *in space* $\mathcal{M}^2((-\infty,T],H)$.

Proof. Let N be a positive integer and $T_0 \in (0,T)$. Let the sequence of the functions $f_i^N(t,v)(i=1,2,3,4)$ for $(t,v) \in [0,T_0] \times \mathcal{B}$ be defined as follows

$$f_i^N(t,v) = \begin{cases} f_i(t,v), & \text{if } \|v\|_{\mathcal{B}} \le N, \\ f_i\left(t, \frac{Nv}{\|v\|_{\mathcal{B}}}\right), & \text{if } \|v\|_{\mathcal{B}} > N, \end{cases} (i=1,2,3)$$

$$f_4^N(t,v,u) = \begin{cases} f_4(t,v,u), & \text{if } \|v\|_{\mathcal{B}} \le N, \\ f_4\left(t, \frac{Nv}{\|v\|_{\mathcal{B}}}, u\right), & \text{if } \|v\|_{\mathcal{B}} > N. \end{cases}$$

Then the function $f_i^N(t,v)$ $(i=1,2,3,4)$ satisfy the assumptions (H2) and (H4). As a consequence of Theorem 3.1, there exists a unique mild solution $x^N(t)$ and $x^{N+1}(t)$ to the following integral equations respectively;

$$\begin{aligned} x^N(t) &= C(t)\phi(0) + S(t)[\xi - f_1^N(0,\phi)] + \int_0^t C(t-s) f_1^N(s,x_s^N)ds \\ &\quad + \int_0^t S(t-s) f_2^N(s,x_s^N)ds + \int_0^t S(t-s) f_3^N(s,x_s^N)dw(s) \\ &\quad + \int_0^t S(t-s) \int_Z f_4^N(s,x_s^N,u)\tilde{N}(ds,du) \end{aligned} \tag{8}$$

$$\begin{aligned} x^{N+1}(t) &= C(t)\phi(0) + S(t)[\xi - f_1^{N+1}(0,\phi)] + \int_0^t C(t-s) f_1^{N+1}(s,x_s^{N+1})ds \\ &\quad + \int_0^t S(t-s) f_2^{N+1}(s,x_s^{N+1})ds + \int_0^t S(t-s) f_3^{N+1}(s,x_s^{N+1})dw(s) \\ &\quad + \int_0^t S(t-s) \int_Z f_4^{N+1}(s,x_s^{N+1},u)\tilde{N}(ds,du). \end{aligned} \tag{9}$$

Define the stopping times

$$\begin{aligned} \delta_N &= T_0 \wedge \inf\{t \in [0,T] : \|x_t^N\|_{\mathcal{B}} \ge N\} \\ \delta_{N+1} &= T_0 \wedge \inf\{t \in [0,T] : \|x_t^{N+1}\|_{\mathcal{B}} \ge N\} \\ \tau_N &= \delta_N \wedge \delta_{N+1}. \end{aligned}$$

From (8) and (9), we have

$$\begin{aligned} &E(\sup_{0 \le s \le t\wedge\tau_N} |x^{N+1}(s) - x^N(s)|^2) \\ &\quad \le 5K_1 E|f_1^N(0,\phi) - f_1^{N+1}(0,\phi)|^2 \\ &\qquad + 5K_1 T E \int_0^{t\wedge\tau_N} |f_1^{N+1}(s,x_s^{N+1}) - f_1^N(s,x_s^N)|^2 ds \\ &\qquad + 5K_1 T E \int_0^{t\wedge\tau_N} |f_2^{N+1}(s,x_s^{N+1}) - f_2^N(s,x_s^N)|^2 ds \end{aligned}$$

$$
\begin{aligned}
&+ 5K_1 E \int_0^{t\wedge\tau_N} |f_3^{N+1}(s, x_s^{N+1}) - f_3^N(s, x_s^N)|^2 ds \\
&+ 5CE \int_0^{t\wedge\tau_N} \int_Z |f_4^{N+1}(s, x_s^{N+1}, u) - f_4^N(s, x_s^N, u)|^2 \lambda(du) ds \\
&+ 5CE \left(\int_0^{t\wedge\tau_N} \int_Z |f_4^{N+1}(s, x_s^{N+1}, u) - f_4^N(s, x_s^N, u)|^4 \lambda(du) ds \right)^{\frac{1}{2}}.
\end{aligned}
$$

Clearly $f_1^N(0,\phi) = f_1^{N+1}(0,\phi)$ and since for $0 \le s \le \tau_N$

$$
\begin{aligned}
f_i^{N+1}(s, x_s^N) &= f_i^N(s, x_s^N), \quad i = 1,2,3, \\
f_4^{N+1}(s, x_s^N, u) &= f_4^N(s, x_s^N, u).
\end{aligned}
$$

Note that $\|x_s^{N+1} - x_s^N\|_{\mathcal{B}}^2 \le \sup_{0\le v\le s} |x^{N+1}(v) - x^N(v)|^2$ and hence we obtain that

$$
\begin{aligned}
&E(\sup_{0\le s\le t\wedge\tau_N} |x^{N+1}(s) - x^N(s)|^2) \\
&\qquad \le 5K_1 T E \int_0^{t\wedge\tau_N} |f_1^{N+1}(s, x_s^{N+1}) - f_1^{N+1}(s, x_s^N)|^2 ds \\
&\qquad + 5K_1 T E \int_0^{t\wedge\tau_N} |f_2^{N+1}(s, x_s^{N+1}) - f_2^{N+1}(s, x_s^N)|^2 ds \\
&\qquad + 5K_1 E \int_0^{t\wedge\tau_N} |f_3^{N+1}(s, x_s^{N+1}) - f_3^{N+1}(s, x_s^N)|^2 ds \\
&\qquad + 5CE \int_0^{t\wedge\tau_N} \int_Z |f_4^{N+1}(s, x_s^{N+1}, u) - f_4^{N+1}(s, x_s^N, u)|^2 \lambda(du) ds \\
&\qquad + 5CE \left(\int_0^{t\wedge\tau_N} \int_Z |f_4^{N+1}(s, x_s^{N+1}, u) - f_4^{N+1}(s, x_s^N, u)|^4 \lambda(du) ds \right)^{\frac{1}{2}}.
\end{aligned}
$$

Therefore, for all $0 \le t \le T_0$, employing assumption (H4), it follows that

$$
\begin{aligned}
&E(\sup_{0\le s\le t} |x^{N+1}(s\wedge\tau_N) - x^N(s\wedge\tau_N)|^2) \\
&\qquad \le 5(K_1(2T+1) + C) \int_0^t \Upsilon_{N+1}\left(s\wedge\tau_N, E\|x_{s\wedge\tau_N}^{N+1} - x_{s\wedge\tau_N}^N\|_{\mathcal{B}}^2\right) ds \\
&\qquad \le 5(K_1(2T+1) + C) \int_0^t \Upsilon_{N+1}\Bigg(s\wedge\tau_N, E\bigg(\sup_{0\le v\le s} |x^{N+1}(v\wedge\tau_N) \\
&\qquad\qquad - x^N(v\wedge\tau_N)|^2\bigg)\Bigg) ds.
\end{aligned}
$$

By (4b), we get

$$
E(\sup_{0\le s\le t} |x^{N+1}(s\wedge\tau_N) - x^N(s\wedge\tau_N)|^2) = 0.
$$

Thus, for a.e., $\omega \in \Omega$, $x^{N+1}(t) = x^N(t)$ for $0 \leq t \leq T_0 \wedge \tau_N$. Note that for each $\omega \in \Omega$, there exists an $N_0(\omega) > 0$ such that $0 \leq T_0 \leq \tau_{N_0}$. Define $x(t)$ by

$$x(t) = x^{N_0}(t), \qquad \text{for} \quad t \in [0, T_0].$$

Since $x(t \wedge \tau_N) = x^N(t \wedge \tau_N)$, it holds that

$$\begin{aligned} x(t \wedge \tau_N) &= C(t)\phi(0) + S(t)[\xi - f_1^N(0, \phi)] + \int_0^{t\wedge\tau_N} C(t-s) f_1^N(s, x_s^N) ds \\ &\quad + \int_0^{t\wedge\tau_N} S(t-s) f_2^N(s, x_s^N) ds + \int_0^{t\wedge\tau_N} S(t-s) f_3^N(s, x_s^N) dw(s) \\ &\quad + \int_0^{t\wedge\tau_N} S(t-s) \int_Z f_4^N(s, x_s^N, u) \tilde{N}(ds, du) \\ &= C(t)\phi(0) + S(t)[\xi - f_1(0, \phi)] + \int_0^{t\wedge\tau_N} C(t-s) f_1(s, x_s) ds \\ &\quad + \int_0^{t\wedge\tau_N} S(t-s) f_2(s, x_s) ds + \int_0^{t\wedge\tau_N} S(t-s) f_3(s, x_s) dw(s) \\ &\quad + \int_0^{t\wedge\tau_N} S(t-s) \int_Z f_4(s, x_s, u) \tilde{N}(ds, du). \end{aligned}$$

Taking $N \to \infty$, for all $t \in [0, T]$, we have

$$\begin{aligned} x(t) &= C(t)\phi(0) + S(t)[\xi - f_1(0, \phi)] + \int_0^t C(t-s) f_1(s, x_s) \\ &\quad + \int_0^t S(t-s) f_2(s, x_s) ds + \int_0^t S(t-s) f_3(s, x_s) dw(s). \\ &\quad + \int_0^t S(t-s) \int_Z f_4(s, x_s, u) \tilde{N}(ds, du). \end{aligned}$$

□

4. Exponential stability

Definition 4.1. *The solution of integral equation* (3) *is said to be exponentially stable in p ($p \geq 2$) moment, if there exists a pair of positive constants $\mu > 0$ and $M_1 > 0$ such that*

$$E\|x(t)\|^p \leq M_1 e^{-\mu t}, \qquad t \geq 0, \quad p \geq 2.$$

In this section, we need the following assumptions to establish the exponential stability of (2).

(A1) The cosine family of operators $\{C(t) : t \leq 0\}$ on H and the corresponding sine family $\{S(t) : t \leq 0\}$ satisfy the conditions $\|C(t)\| \leq Me^{-\beta t}$ and $\|S(t)\| \leq Me^{-\alpha t}$, $t \geq 0$ for some constants $M \geq 1, \alpha > 0$ and $\beta > 0$.

(A2) There exist constants $\lambda_i > 0$ $(i = 1, 2, 3, 4)$ and a function $z : (-\infty, \infty) \to [0, \infty)$ with $\int_{-\infty}^{0} z(t)dt = 1$ and $\int_{-\infty}^{0} z(t)e^{-\rho t}dt < \infty$ $(\rho > 0)$ such that

$$
\begin{aligned}
\|f_i(t,x) - f_i(t,y)\| &\le \lambda_i \int_{-\infty}^{0} z(\theta)\|x(t+\theta) - y(t+\theta)\|d\theta, \\
& f_i(t,0) = 0, i = 1,2, \\
\|f_3(t,x) - f_3(t,y)\|_{L_2^0} &\le \lambda_3 \int_{-\infty}^{0} z(\theta)\|x(t+\theta) - y(t+\theta)\|d\theta, \\
& f_3(t,0) = 0, \\
\int_Z \|f_4(t,x,u) - f_4(t,y,u)\|\lambda(du) &\le \lambda_4 \int_{-\infty}^{0} z(\theta)\|x(t+\theta) - y(t+\theta)\|d\theta, \\
& f_4(t,0,u) = 0, \\
& x, y \in \mathcal{B}, t \ge 0.
\end{aligned} \tag{10}
$$

(A3) $6^{p-1}M^p \left[\beta^{-p}\lambda_1^p + \alpha^{-p}\lambda_2^p + \alpha^{-p/2}\lambda_3^p \left(\frac{2(p-1)}{p-2}\right)^{1-p/2} \left(\frac{p(p-1)}{4}\right)^{p/2} + \alpha^{-p}\lambda_4^p\right] < 1, \quad (p \ge 2)$.

Lemma 4.1. *Let $L_1, L_2 \in (0, \rho]$ and assume that there exist some positive constants $C_i > 0$ (i=1,2,3,4) and a function $\hat{y} : (-\infty, \infty) \to [0, \infty)$ such that*

$$
\hat{y}(t) \le \begin{cases} C_1 e^{-L_1 t} + C_2 e^{-L_2 t} + C_3 \int_0^t e^{-L_1(t-s)} \int_{-\infty}^{0} z(\theta)\hat{y}(s+\theta)d\theta ds \\ +C_4 \int_0^t e^{-L_2(t-s)} \int_{-\infty}^{0} z(\theta)\hat{y}(s+\theta)d\theta ds, t \ge 0, \\ C_1 e^{-L_1 t} + C_2 e^{-L_2 t}, t \in (-\infty, 0], \end{cases} \tag{11}
$$

holds. If $\frac{C_3}{L_1} + \frac{C_4}{L_2} < 1$, then,

$$
\hat{y}(t) \le M_2 e^{-\mu t}, t \in (-\infty, \infty),
$$

where $\mu \in (0, L_1 \wedge L_2)$ is a positive root of the algebra equation:

$$
\left(\frac{C_3}{L_1 - \mu} + \frac{C_4}{L_2 - \mu}\right) \int_{-\infty}^{0} z(\theta)e^{-\mu\theta}d\theta = 1
$$

and

$$
M_2 = \max\left\{\frac{C_1(L_1 - \mu)}{C_3 \int_{-\infty}^{0} z(\theta)e^{-\mu\theta}d\theta}, \frac{C_2(L_2 - \mu)}{C_4 \int_{-\infty}^{0} z(\theta)e^{-\mu\theta}d\theta}, C_1 + C_2\right\} > 0.
$$

Proof. Let $F(\lambda) = \left(\frac{C_3}{L_1-\lambda} + \frac{C_4}{L_2-\lambda}\right) \int_{-\infty}^{0} z(\theta)e^{-\mu\theta}d\theta - 1$, then it is obvious that there exists a positive constant $\mu \in (0, L_1 \wedge L_2)$ such that $F(\mu) = 0$.
For any $\epsilon > 0$ and Let

$$N_\epsilon = \max\left\{\frac{(C_1+\epsilon)(L_1-\mu)}{C_3\int_{-\infty}^{0} z(\theta)e^{-\mu\theta}d\theta}, \frac{(C_2+\epsilon)(L_2-\mu)}{C_4\int_{-\infty}^{0} z(\theta)e^{-\mu\theta}d\theta}, C_1+C_2\right\} > 0. \tag{12}$$

Now , in order to show this Lemma, we only claim that (11) implies

$$\hat{y}(t) \le N_\epsilon e^{-\mu t}, \quad t \in (-\infty, \infty). \tag{13}$$

Obviously, for any $t \in (-\infty, 0], (13)$ holds. Now we will prove (13) by the contradiction method. Assume that there exists a $t_1 > 0$ such that

$$\hat{y} < N_\epsilon e^{-\mu t}, \quad t \in (-\infty, t_1), \quad \hat{y}(t_1) = N_\epsilon e^{-\mu t_1}. \tag{14}$$

However, from (11)

$$\begin{aligned}
\hat{y}(t_1) \le{}& C_1e^{-L_1t_1} + C_2e^{-L_2t_1} + C_3N_\epsilon\int_0^{t_1} e^{-L_1(t_1-s)}\int_{-\infty}^{0} z(\theta)e^{-\mu(s+\theta)}d\theta ds\\
&+ C_4N_\epsilon\int_0^{t_1} e^{-L_2(t_1-s)}\int_{-\infty}^{0} z(\theta)e^{-\mu(s+\theta)}d\theta ds\\
\le{}& \left(C_1 - \frac{N_\epsilon C_3}{L_1-\mu}\int_{-\infty}^{0} z(\theta)e^{-\mu\theta}d\theta\right)e^{-L_1t_1}\\
&+ \left(C_2 - \frac{N_\epsilon C_4}{L_2-\mu}\int_{-\infty}^{0} z(\theta)e^{-\mu\theta}d\theta\right)e^{-L_2t_1}\\
&+ \left(\frac{C_3}{L_1-\mu}\int_{-\infty}^{0} z(\theta)e^{-\mu\theta}d\theta + \frac{C_4}{L_2-\mu}\int_{-\infty}^{0} z(\theta)e^{-\mu\theta}d\theta\right)N_\epsilon e^{-\mu t_1}
\end{aligned} \tag{15}$$

Note that $\mu \in (0, L_1 \wedge L_2)$, From (12), we obtain

$$\frac{C_3}{L_1-\mu}\int_{-\infty}^{0} z(\theta)e^{-\mu\theta}d\theta + \frac{C_4}{L_2-\mu}\int_{-\infty}^{0} z(\theta)e^{-\mu\theta}d\theta = 1$$

and

$$\begin{aligned}
&C_1 - \frac{N_\epsilon C_3}{L_1-\mu}\int_{-\infty}^{0} z(\theta)e^{-\mu\theta}d\theta\\
&\le C_1 - \frac{C_3}{L_1-\mu}\int_{-\infty}^{0} z(\theta)e^{-\mu\theta}d\theta\frac{(C_1+\epsilon)(L_1-\mu)}{C_3\int_{-\infty}^{0} z(\theta)e^{-\mu\theta}d\theta} < 0\\
&C_2 - \frac{N_\epsilon C_4}{L_2-\mu}\int_{-\infty}^{0} z(\theta)e^{-\mu\theta}d\theta\\
&\le C_2 - \frac{C_4}{L_2-\mu}\int_{-\infty}^{0} z(\theta)e^{-\mu\theta}d\theta\frac{(C_2+\epsilon)(L_2-\mu)}{C_4\int_{-\infty}^{0} z(\theta)e^{-\mu\theta}d\theta} < 0.
\end{aligned}$$

Thus, (15) yields

$$\hat{y}(t_1) < N_\epsilon e^{-\mu t}, \tag{16}$$

which contradicts (14), that is, (13) holds. As $\epsilon > 0$ is arbitrarily small, in view of (13), it follows $\hat{y}(t) \le M_2 e^{-\mu t}, t \ge 0$. $\square$

Theorem 4.1. *Assume that (A1)-(A3) hold and $\alpha, \beta \in (0, \rho]$, then the mild solution of (2) is exponentially stable in p-th moment for $p \ge 2$.*

Proof. From (3), we have

$$\begin{aligned} E\|x(t)\|^p &\le 6^{p-1} M^p E\|\phi\|^p e^{-\beta t} + 6^{p-1} M^p E\|\xi - f_1(0,\phi)\|^p e^{-\alpha t} \\ &\quad + 6^{p-1} E\left\| \int_0^t C(t-s) f_1(s, x_s) ds \right\|^p \\ &\quad + 6^{p-1} E\left\| \int_0^t S(t-s) f_2(s, x_s) ds \right\|^p \\ &\quad + 6^{p-1} E\left\| \int_0^t S(t-s) f_3(s, x_s) dw(s) \right\|^p \\ &\quad + 6^{p-1} E\left\| \int_0^t S(t-s) \int_Z f_4(s, x_s, u) \tilde{N}(ds, du) \right\|^p \\ &= 6^{p-1} \sum_{i=1}^{6} \Phi_i. \end{aligned} \tag{17}$$

From (A1), (A2) and the Hölder inequality, we have

$$\begin{aligned} \Phi_3 &\le M^p \left(\int_0^t \left(e^{-\beta(t-s)(1-\frac{1}{p})} \right)^{\frac{p}{p-1}} ds \right)^{p-1} \int_0^t e^{-\beta(t-s)} E\|f_1(s, x_s)\|^p ds \\ &= M^p \left(\int_0^t e^{-\beta(t-s)} ds \right)^{p-1} \int_0^t e^{-\beta(t-s)} E\|f_1(s, x_s)\|^p ds \\ &\le M^p \lambda_1^p \beta^{1-p} \int_0^t e^{-\beta(t-s)} E\left(\int_{-\infty}^0 z(\theta) \|x(s+\theta)\| d\theta \right)^p ds. \end{aligned}$$

Similarly, we have

$$\Phi_4 \le M^p \lambda_2^p \alpha^{1-p} \int_0^t e^{-\alpha(t-s)} E\left(\int_{-\infty}^0 z(\theta) \|x(s+\theta)\| d\theta \right)^p ds.$$

By using the conditions (A1), (A2) and Lemma 2.2, it follows that

$$\begin{aligned} \Phi_5 &\le M^p \left(\frac{p(p-1)}{4} \right)^{p/2} \left(\int_0^t \left(e^{-\alpha p(t-s)} E\|f_3(s, x_s)\|^p_{L_2^0} \right)^{2/p} ds \right)^{p/2} \\ &= M^p \left(\frac{p(p-1)}{4} \right)^{p/2} \left(\int_0^t e^{-2\alpha(t-s)} \left(E\|f_3(s, x_s)\|^p_{L_2^0} \right)^{2/p} ds \right)^{p/2} \end{aligned}$$

$$
\begin{aligned}
&\leq M^p\left(\frac{p(p-1)}{4}\right)^{p/2}\left(\int_0^t\left(e^{-2\alpha(t-s)(1-\frac{1}{p})}\right)^{p/(p-2)}ds\right)^{(p/2)-1}\\
&\qquad\times\int_0^t\left(e^{-2\alpha(t-s)(1/p)}\left(E\|f_3(s,x_s)\|^p_{L_2^0}\right)^{2/p}\right)^{p/2}ds\\
&\leq M^p\left(\frac{p(p-1)}{4}\right)^{p/2}\left(\int_0^t e^{-2\alpha(t-s)((p-1)/(p-2))}ds\right)^{(p/2)-1}\\
&\qquad\times\int_0^t e^{-\alpha(t-s)}E\|f_3(s,x_s)\|^p_{L_2^0}ds\\
&\leq M^p\lambda_3^p\left(\frac{p(p-1)}{4}\right)^{p/2}\left(\frac{2\alpha(p-1)}{p-2}\right)^{1-(p/2)}\\
&\qquad\times\int_0^t e^{-\alpha(t-s)}E\left(\int_{-\infty}^0 z(\theta)\|x(s+\theta)\|d\theta\right)^p ds.
\end{aligned}
$$

From (A1)and (A2), we obtain

$$
\begin{aligned}
\Phi_6&\leq M^p\left(\int_0^t e^{-\alpha(t-s)}ds\right)^{p-1}\int_0^t e^{-\alpha(t-s)}\int_Z E\|f_4(s,x_s,u)\|^p\lambda(du)ds\\
&\leq M^p\alpha^{1-p}\lambda_4^p\int_0^t e^{-\alpha(t-s)}E\left(\int_{-\infty}^0 z(\theta)\|x(s+\theta)\|d\theta\right)^p ds.
\end{aligned}
$$

These together with (17) yields

$$
\begin{aligned}
E\|x(t)\|^p&\leq 6^{p-1}M^pE\|\phi\|^pe^{-\beta t}+6^{p-1}M^pE\|\xi-f_1(0,\phi)\|^pe^{-\alpha t}\\
&\quad+6^{p-1}M^p\beta^{1-p}\lambda_1^p\int_0^t e^{-\beta(t-s)}\int_{-\infty}^0 z(\theta)E\|x(s+\theta)\|^pd\theta ds\\
&\quad+6^{p-1}M^p\alpha^{1-p}\lambda_2^p\int_0^t e^{-\alpha(t-s)}\int_{-\infty}^0 z(\theta)E\|x(s+\theta)\|^pd\theta ds\\
&\quad+6^{p-1}M^p\lambda_3^p\left(\frac{p(p-1)}{4}\right)^{p/2}\left(\frac{2\alpha(p-1)}{p-2}\right)^{1-(p/2)}\\
&\qquad\times\int_0^t e^{-\alpha(t-s)}\int_{-\infty}^0 z(\theta)E\|x(s+\theta)\|^pd\theta ds\\
&\quad+6^{p-1}M^p\alpha^{1-p}\lambda_4^p\int_0^t e^{-\alpha(t-s)}\int_{-\infty}^0 z(\theta)E\|x(s+\theta)\|^pd\theta ds.
\end{aligned}
$$

It can be easily verified that there exists two positive numbers $M'>0$ and $M''>0$ such that $E\|x(t)\|^2\leq M'e^{-\beta t}+M''e^{-\alpha t}$, for any $t\in(-\infty,0]$. Let $\tilde{C}_1=6^{p-1}M^pE\|\phi\|^p,\tilde{C}_2=6^{p-1}M^pE\|\xi-f_1(0,\phi)\|^p,\tilde{C}_3=6^{p-1}M^p\beta^{1-p}\lambda_1^p,\tilde{C}_4=6^{p-1}M^p\alpha^{1-p}\lambda_2^p+6^{p-1}M^p\lambda_3^p\left(\frac{p(p-1)}{4}\right)^{p/2}\left(\frac{2\alpha(p-1)}{p-2}\right)^{1-(p/2)}+6^{p-1}M^p\alpha^{1-p}\lambda_4^p$, if $\frac{\tilde{C}_3}{\beta}+\frac{\tilde{C}_4}{\alpha}<1$, i.e., (A3) holds, then by using Lemma 4.1, we can obtain

$$
E\|x(t)\|^2\leq M_1e^{-\mu t},\quad t\in[0,\infty)\ \ (\mu\in(0,L_1\wedge L_2)),
$$

where

$$M_1 = \max\Big\{6^{p-1}M^p(E\|\phi\|^p + E\|\xi - f_1(0,\phi)\|^p), 6^{p-1}M^p\beta^{-p}\lambda_1^p,$$
$$6^{p-1}M^p\left(\alpha^{-p}\lambda_2^p + \alpha^{-p/2}\lambda_3^p\left(\frac{p(p-1)}{2}\right)^{p/2}\left(\frac{2\alpha(p-1)}{p-2}\right)^{1-(p/2)}\right.$$
$$\left.+\,\alpha^{-p}\lambda_4^p\right)\Big\} > 0.$$

□

5. Example

In this section, we present an example to illustrate the results obtained in previous sections. Let $H = L^2[0,\pi]$ with the norm $\|.\|$. And let $e_n = \sqrt{\frac{2}{\pi}}\sin(ny)$ $(n = 1,2,...)$ denote the complete orthonormal basis in H.

Let $w(t) := \sum_{n=1}^{\infty}\sqrt{\lambda_n}\beta_n(t)e_n$, $(\lambda_n > 0)$, where $\{\beta_n(t)\}$ are one dimensional standard Brownian motion mutually independent on a usual complete probability space. $\tilde{N}(ds,du)$ is a compensated Poisson random measure on $[1,\infty)$ with parameter $\lambda(du)dt$. The Wiener process $w(t)$ is independent of $\tilde{N}(du)dt$.

Define the operator $A : H \to H$ by $(Ax)(y) = \frac{\partial^2}{\partial y^2}x(y)$ with the domain $D(A) = \{x \in H : x(0) = x(\pi)\}$. Then

$$Ax = \sum_{n=1}^{\infty} -n^2\langle x, e_n\rangle e_n, x \in D(A),$$

where $\{e_n : n \in N\}$ is the orthonormal set of eigen vectors of A corresponding to the eigen values $-n^2$ for $n \in N$. The operator $C(t)$ defined by

$$C(t)x = \sum_{n=1}^{\infty}\cos(nt)\langle x, e_n\rangle e_n, \quad t \in \mathbb{R},$$

from a cosine function on H, with associated sine function

$$S(t)x = \sum_{n=1}^{\infty}\frac{\sin(nt)}{n}\langle x, e_n\rangle e_n, \quad t \in \mathbb{R},$$

for all $x \in H$, with $\|C(t)\| \le e^{-\pi^2 t}$ and $\|S(t)\| \le e^{-\pi^2 t}, t \ge 0$.

Now, we consider the following second-order neutral stochastic partial differential equations with infinite delay and Poisson jumps

$$\begin{aligned}\partial&\left[\frac{\partial x(t,y)}{\partial t} - p_1 x(t+\theta,y)\right]\\ &= \left[\frac{\partial^2 x(t,y)}{\partial y^2} + p_2 x(t+\theta,y)\right]\partial t + p_3 x(t+\theta,y)dw(t)\end{aligned} \tag{18}$$

$$+\int_Z p_4 x(t+\theta,y)u\tilde{N}(dt,du),$$
$$t\in[0,T], y\in[0,\pi], \theta\in(-\infty,0),$$

subject to the conditions

$$\begin{aligned} x(t,y) &= \phi(t,y), \quad -\infty < t \le 0, 0 < y < \pi, \\ x(t,0) &= x(t,\pi) = 0, \quad 0 \le t \le T, \\ \frac{\partial x(0,y)}{\partial t} &= \xi(y), \quad 0 < y < \pi, \end{aligned}$$

where $\xi \in L^2[0,\pi], \phi \in \mathcal{B}$ and $\pi > 0$, $(i = 1,2,3,4)$. Define

$$\begin{aligned} f_1(t,x_t) &= p_1 x(t+\theta,y), \\ f_2(t,x_t) &= p_2 x(t+\theta,y), \\ f_3(t,x_t) &= p_3 x(t+\theta,y), \\ f_4(t,x_t,u) &= p_4 x(t+\theta,y)u, \end{aligned}$$

where $\theta \in (-\infty,0]$.

The system (18) can be rewritten in the form of (2). Assume that f_i $(i = 1,2,3,4)$ satisfy the conditions of Theorem 3.1 and Theorem 3.2. Then, the system (18) has a unique mild solution.

It is easy to see all the conditions are satisfied with $M = 1, \alpha = \beta = \pi$. By virtue of Theorem 4.1, the mild solution of (18) is p-th moment exponentially stable provided that, $p \ge 2$,

$$6^{p-1}\left[\pi^{-p}p_1^p + \pi^{-p}p_2^p + \pi^{-p}p_4^p + \pi^{-p/2}p_3^p\left(\frac{2(p-1)}{p-2}\right)^{1-(p/2)}\left(\frac{p(p-1)}{4}\right)^{p/2}\right] < 1.$$

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