Logarithm and space BMO

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Abstract. Functions with bounded mean oscillation create so called BMO space or John Nirenberg space. They have considerable applications in differential equations theory and harmonic analysis. Logarithm presents an important example of function which belongs to this space BMO but which is not bounded. The norm of this function can be obtained by simple means of infinitesimal calculus. The calculations of it can be viewed as a suitable contribution to the teaching/learning of functional analysis.

Keywords: Morrey Campanato spaces, John Nirenberg space, BMO, bounded mean oscillation, logarithm.

1. Introduction

Functional analysis is enrolled among special subjects which are taught only in some specialized courses of several universities of science or technology. During such lessons students are usually acquainted with the theoretical knowledge by the form of sequence of definitions and theorems, often without sufficient number of suitable supporting isolated models. We would like to offer one useful example which calculates BMO-norm of function logarithm and which can serve as a good training of limits.

Theory of function spaces is a special part of functional analysis which is developing significantly and which has extensive applications in other branches of mathematics, in particular in differential equations (see e.g. [5, 14, 15]). Many types of function spaces are very important for investigation of plenty of the various properties of many differential equations solutions. Also the so called Morrey Campanato spaces have proven to be a very useful tools in this way. They can be considered as a generalization of Lebesgue spaces $L^p(\Omega)$ of integrable functions (see [13, 18]). Space of special type of such functions is called BMO — functions with bounded mean oscillation. The bounded functions create the subset of BMO spaces but no any function with bounded means oscillation is bounded. Well known example of such function is logarithm. We would like to offer one useful calculation of BMO-norm of logarithm which can serve as a good training for calculations of limits.

2. Morrey Campanato spaces

Lets us recall the basic definitions and notations of Morrey Campanato spaces (see e.g. [16]).

We have a function $f : \Omega \to \mathbb{R}$, which is defined and locally integrable on area $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary.

For any measurable set $Q \subset \mathbb{R}^n$ we can denote mean value

$$f_Q = \frac{1}{|Q|} \int_Q f(x)\, dx.$$

The open ball with center $a \in \mathbb{R}^n$ and diameter $r > 0$ is usually denoted by

$$B(a;r) = \{x \in \mathbb{R}^n; |x-a| < r\}.$$

At first, for any $\lambda > 0$ and $p \geq 1$ we shall define norms or pseudo-norm

$$\|f\|_{L_M^{p,\lambda}(\Omega)} = \sup_{a\in\Omega, r>0} \left(\frac{1}{r^\lambda} \int_{B(a;r)\cap\Omega} |f|^p \; dx \right)^{\frac{1}{p}},$$

$$[f]_{L_C^{p,\lambda}(\Omega)} = \sup_{a\in\Omega, r>0} \left(\frac{1}{r^\lambda} \int_{B(a;r)\cap\Omega} \left|f - f_{B(a;r)}\right|^p \; dx \right)^{\frac{1}{p}} \quad \text{and}$$

$$\|f\|_{L_C^{p,\lambda}(\Omega)} = [f]_{L_C^{p,\lambda}(\Omega)} + \|f\|_{L^p(\Omega)}.$$

Function $f \in L^p(\Omega)$ belongs to Morrey space $L_M^{p,\lambda}(\Omega)$ if

$$\|f\|_{L_M^{p,\lambda}(\Omega)} < \infty$$

and to Campanato space $L_C^{p,\lambda}(\Omega)$ if

$$\|f\|_{L_C^{p,\lambda}(\Omega)} < \infty.$$

The definition can be formulated by means of the supremum over n-dimensional cubes Q (their sides are usually supposed to be parallel with axes) with the same result, too.

It holds

$$L_M^{p,\lambda}(\Omega) \subset L_C^{p,\lambda}(\Omega)$$

and for $1 \leq p \leq q < \infty$ and λ, $\nu > 0$, where $\frac{\lambda-n}{p} \leq \frac{\nu-n}{q}$, we have following embedding relations

$$L_M^{q,\nu}(\Omega) \subset L_M^{p,\lambda}(\Omega) \quad \text{and} \quad L_C^{q,\nu}(\Omega) \subset L_C^{p,\lambda}(\Omega)$$

for suitable Ω.

3. John Nirenberg spaces

A special type of these spaces is represented by functions with bounded mean oscillation. Such space called and denoted by BMO was first introduced by F. John and L. Nirenberg in 1961 (see [12]).

Let us present a possible definition of space $BMO(\Omega)$ where Ω is a cube (with sides parallel with axes) in $\mathbb{R}^n$. The function (class of equivalent functions, more precisely) f is said to in $BMO(\Omega)$ if the norm

$$
\begin{aligned}
\|f\|_{BMO(\Omega)} &= \sup_{a\in\Omega, r>0} \frac{1}{r^n} \int_{B(a;r)\cap\Omega} \left|f - f_{B(a;r)\cap\Omega}\right| \, dx \\
&= \sup_{Q\subset\Omega} \frac{1}{|Q|} \int_{Q\cap\Omega} |f - f_Q| \, dx
\end{aligned}
\tag{1}
$$

is finite, the second supremum is taken over all cubes $Q \subset \Omega$ (with sides parallel with axes).

This space BMO is very interesting and it is equal to space $L_C^{1,n}$. Its use is remarkable in harmonic analysis, especially for investigation of Calderon Zygmund type operators and interpolation (see [1, 2, 3, 17]). For instance, if we have a linear operator T continuous from L^2 to L^2 and continuous from L^∞ to BMO then it is also continuous from L^p to L^p for any $2 < p < \infty$.

The following diagram expresses the relationship for given $p \geq 1$ where $C^{0,\alpha}$ denotes a space of Holder functions.

$$
\begin{array}{ccccccc}
L^p & \approx & L_M^{p,0} & \approx & L_C^{p,0} & \approx & L^p \\
 & & \cup & & \cup & & \\
 & & L_M^{p,\lambda} & \approx & L_C^{p,\lambda} & & \\
 & & \cup & & \cup & & \\
L^\infty & \approx & L_M^{p,n} & \underset{\neq}{\subseteq} & L_C^{p,n} & \approx & BMO \\
 & & \cup & & \cup & & \\
 & & \{\ 0\ \} & & L_C^{p,\lambda} & \approx & C^{0,\frac{\lambda-n}{p}} \\
 & & & & \cup & & \\
 & & & & L_C^{p,n+p} & \approx & C^{0,1} \\
 & & & & \cup & & \\
 & & & & \{\text{const.}\} & &
\end{array}
$$

It was proved by C. Fefferman in 1971 that the space BMO is a dual of the Hardy space H^1 (see [8]).

There are plenty of generalizations of this space or alternative definitions (see [4, 6, 7, 9, 10, 11]).

We can simply see that the space $L^\infty(\Omega)$ is a subspace of the space $BMO(\Omega)$. But not every function from $BMO(\Omega)$ belongs to the space $L^\infty(\Omega)$.

Logarithm on $\Omega = \langle 1; \infty)$ is well known example of such function. Many proofs that logarithm belongs to the space $BMO(\Omega)$ are presented in many publications. We would like to offer a straight calculation of BMO norm. We are going to use only the results of simple calculus, in particular limits. The calculations used can also serve as a useful contribution to the teaching process of functional analysis and as an example of an interesting application of calculus.

Notice that $\log|x|$ is an element of the space $BMO(\mathbb{R})$ but function

$$g(x) = \begin{cases} \log x, & \text{on } (0;\infty) \\ 0, & \text{on } (-\infty;0\rangle \end{cases}$$

is not in $BMO(\mathbb{R})$.

4. Calculations

We consider $\Omega = \langle 1;\infty)$ and a positive function $f : \Omega \longrightarrow \mathbb{R}$. We denote by F the antiderivative to f on $(1;\infty)$.

For arbitrary interval $Q = \langle a;b\rangle$, $1 \le a < b$, we have by the mean value theorem that there is some point $c \in (a;b)$ such that

$$f(c) = \frac{F(b)-F(a)}{b-a} = \frac{1}{b-a}\int_a^b f\,dx = \frac{1}{|Q|}\int_Q f\,dx = f_Q.$$

Lemma 4.1. *For f increasing we have*

$$\frac{1}{|Q|}\int_Q |f-f_Q|\,dx = \frac{1}{b-a}\left(F(a)+F(b)-2F(c)+f(c)(2c-a-b)\right).$$

Proof. The result can be reached by simple calculations

$$\begin{aligned}
\frac{1}{|Q|}\int_Q |f-f_Q|\,dx &= \frac{1}{b-a}\int_a^b |f(x)-f(c)|\,dx \\
&= \frac{1}{b-a}\left(\int_a^c (f(c)-f(x))\,dx + \int_c^b (f(x)-f(c))\,dx\right) \\
&= \frac{1}{b-a}\left([xf(c)-F(x)]_a^c + [F(x)-xf(c)]_c^b\right) \\
&= \frac{1}{b-a}\left(F(a)+F(b)-2F(c)+f(c)\,(2c-a-b)\right).
\end{aligned}$$

Now we apply this lemma to the case of natural logarithm $f(x) = \log x$ and interval $Q = \langle a;b\rangle$, $1 \le a < b$. This function is increasing on $\Omega = \langle 1;\infty)$ and its antiderivative is $F(x) = x\log x - x = x(\log x - 1)$.

We have mean value

$$f_Q = \frac{1}{|Q|}\int_Q f\,dx = \frac{1}{b-a}\int_a^b \log x\,dx = \frac{1}{b-a}\left[x\log x - x\right]_a^b = \frac{b\log b - a\log a}{b-a} - 1.$$

Inverse function of logarithm is exponential function, hence the point $c \in (a;b)$ for which $f(c) = f_Q$ can be expressed as

$$c = \mathrm{e}^{f_Q} = \frac{1}{\mathrm{e}} \cdot \mathrm{e}^{\frac{b\log b - a\log a}{b-a}}\,, \qquad f(c) = \log c = \frac{b\log b - a\log a}{b-a} - 1.$$

Then the ratio from definition of the space BMO is according to the lemma 4.1

$$\frac{1}{|Q|}\int_Q |f - f_Q|\,dx = 2\left(\frac{\mathrm{e}^{\frac{b\log b - a\log a}{b-a}}}{\mathrm{e}(b-a)} - ab\frac{\log b - \log a}{(b-a)^2}\right), \tag{2}$$

because

$$\begin{aligned}
&\frac{1}{b-a}\left(F(a) + F(b) - 2F(c) + f(c)\,(2c - a - b)\right)\\
&= \frac{1}{b-a}\left(a\log a - a + b\log b - b - 2c\log c + 2c + \log c\,(2c - a - b)\right)\\
&= \frac{1}{b-a}\left(a\log a + b\log b + 2c - (a+b)\,(\log c + 1)\right)\\
&= \frac{1}{b-a}\left(\frac{(b-a)(a\log a + b\log b) - (a+b)(b\log b - a\log a)}{b-a} + 2c\right)\\
&= \frac{1}{b-a}\left(2c - 2ab\frac{\log b - \log a}{b-a}\right) = 2\left(\frac{\mathrm{e}^{\frac{b\log b - a\log a}{b-a}}}{\mathrm{e}(b-a)} - ab\frac{\log b - \log a}{(b-a)^2}\right).
\end{aligned}$$

Now we define

$$\phi(a,b) = \frac{\mathrm{e}^{\frac{b\log b - a\log a}{b-a}}}{b-a} - ab\mathrm{e}\frac{\log b - \log a}{(b-a)^2} \tag{3}$$

and for arbitrary $a \geq 1$ we define on $\langle a;\infty)$

$$\phi_1(b) = \phi(a,b). \tag{4}$$

We would like to prove that this function (for any a) is increasing,

$$\lim_{b\to a^+}\phi_1(b) = 0 \quad \text{and} \quad \lim_{b\to\infty}\phi_1(b) = 1. \tag{5}$$

Lemma 4.2. *It holds*

$$\phi_1(b) = \phi_2\left(\frac{b}{a}\right)\,, \quad \textit{where} \quad \phi_2(x) = \frac{\mathrm{e}^{\frac{x\log x}{x-1}} - \mathrm{e}\frac{x\log x}{x-1}}{x-1}.$$

Proof. At first we try to adjust the exponent of power

$$\begin{aligned}\frac{b\log b - a\log a}{b-a} &= \frac{b\log b - b\log a + b\log a - a\log a}{b-a}\\ &= b\frac{\log b - \log a}{b-a} + \log a = \frac{\frac{b}{a}\log\frac{b}{a}}{\frac{b}{a}-1} + \log a.\end{aligned}$$

Then, we can calculate

$$\begin{aligned}\phi_1(b) &= \frac{\mathrm{e}^{\frac{b\log b - a\log a}{b-a}} - ab\mathrm{e}^{\frac{\log b-\log a}{b-a}}}{b-a} = \frac{\mathrm{e}^{\frac{\frac{b}{a}\log\frac{b}{a}}{\frac{b}{a}-1}}\cdot a - ab\mathrm{e}^{\frac{\log\frac{b}{a}}{b-a}}}{b-a}\\ &= \frac{\mathrm{e}^{\frac{\frac{b}{a}\log\frac{b}{a}}{\frac{b}{a}-1}} - \mathrm{e}\frac{b}{a}^{\frac{\log\frac{b}{a}}{\frac{b}{a}-1}}}{\frac{b}{a}-1} = \phi_2\left(\frac{b}{a}\right).\end{aligned}$$

This relation between functions ϕ_1 and ϕ_2 is a consequence of linear transformation $x = \frac{b}{a}$. It enables us to investigate the properties of function ϕ_2 instead of function ϕ_1.

Lemma 4.3. *It holds* $\lim_{x\to 1^+}\phi_2(x) = 0$.

Proof. At first we will realize that

$$\lim_{x\to 1^+}\frac{x\log x}{x-1} = 1.$$

Using l'Hospital rule we calculate

$$\lim_{x\to 1^+}\frac{x-\log x - 1}{(x-1)^2} = \lim_{x\to 1^+}\frac{1-\frac{1}{x}}{2(x-1)} = \lim_{x\to 1^+}\frac{1}{2x} = \frac{1}{2}.$$

Then, we use again l'Hospital rule for calculation

$$\begin{aligned}&\lim_{x\to 1^+}\frac{\mathrm{e}^{\frac{x\log x}{x-1}} - \mathrm{e}\frac{x\log x}{x-1}}{x-1}\\ &= \lim_{x\to 1^+}\frac{\left(\mathrm{e}^{\frac{x\log x}{x-1}} - \mathrm{e}\right)\cdot\left(\frac{x\log x}{x-1}\right)'}{1} = \lim_{x\to 1^+}\left(\mathrm{e}^{\frac{x\log x}{x-1}} - \mathrm{e}\right)\cdot\frac{x-\log x - 1}{(x-1)^2}\\ &= \underbrace{\lim_{x\to 1^+}\left(\mathrm{e}^{\frac{x\log x}{x-1}} - \mathrm{e}\right)}_{=0}\cdot\underbrace{\lim_{x\to 1^+}\frac{x-\log x-1}{(x-1)^2}}_{=\frac{1}{2}} = 0\cdot\frac{1}{2} = 0.\end{aligned}$$

Lemma 4.4. *It holds*

$$\lim_{x\to\infty}\phi_2(x) = 1.$$

Proof. Again, at first we realize that

$$\lim_{x\to\infty} \frac{\log x}{x-1} = 0$$

and using relation

$$\frac{x\log x}{x-1} = \log x + \frac{\log x}{x-1}$$

we calculate

$$\begin{aligned}\lim_{x\to\infty} \frac{\mathrm{e}^{\frac{x\log x}{x-1}} - \mathrm{e}\frac{x\log x}{x-1}}{x-1} &= \lim_{x\to\infty} \frac{x\mathrm{e}^{\frac{\log x}{x-1}} - \mathrm{e}x\frac{\log x}{x-1}}{x-1} \\ = \lim_{x\to\infty}\frac{x}{x-1}\cdot\left(\lim_{x\to\infty} \mathrm{e}^{\frac{\log x}{x-1}} - \mathrm{e}\lim_{x\to\infty}\frac{\log x}{x-1}\right) &= 1\cdot(\mathrm{e}^0 - \mathrm{e}\cdot 0) = 1.\end{aligned}$$

Lemma 4.5. *Function ϕ_2 is increasing on interval $\langle 1;\infty)$.*

Proof. For one of the possible proofs of monotony of this function we can use derivative

$$\begin{aligned}\phi_2'(x) &= \left(\frac{\mathrm{e}^{\frac{x\log x}{x-1}} - \mathrm{e}\frac{x\log x}{x-1}}{x-1}\right)' \\ &= \frac{\left(\mathrm{e}^{\frac{x\log x}{x-1}} - \mathrm{e}\right)\cdot\frac{x-1-\log x}{(x-1)^2}\cdot(x-1) - \left(\mathrm{e}^{\frac{x\log x}{x-1}} - \mathrm{e}\frac{x\log x}{x-1}\right)}{(x-1)^2} \\ &= \frac{\mathrm{e}(x\log x + \log x + 1 - x) - \mathrm{e}^{\frac{x\log x}{x-1}}\cdot\log x}{(x-1)^3} = \frac{g_1(x)}{(x-1)^3}\ ,\end{aligned}$$

where $g_1(x) = \mathrm{e}(x\log x + \log x + 1 - x) - \mathrm{e}^{\frac{x\log x}{x-1}}\cdot\log x$.

In order to prove that function ϕ_2' is positive on $\langle 1;\infty)$ and so function ϕ_2 is increasing it is necessary to show that function g_1 is positive on this interval.

Function g_1 can be split into two parts

$$g_1(x) = \underbrace{\left(\mathrm{e}x + \mathrm{e} - 2\mathrm{e}^{\frac{x\log x}{x-1}}\right)}_{=g_3(x)}\cdot\frac{\log x}{2} + \frac{\mathrm{e}}{2}\cdot\underbrace{(x\log x + \log x + 2 - 2x)}_{=g_2(x)\geq 0}.$$

Function g_2 is positive on $\langle 1;\infty)$ because $g_2(1) = g_2'(1) = 0$ and $g_2''(x) = \frac{x-1}{x^2} \geq 0$ on $\langle 1;\infty)$.

We shall prove that function g_3 is positive, which is equivalent with inequality

$$g_4(x) = \frac{\mathrm{e}^{\frac{x\log x}{x-1}}}{x+1} \leq \frac{\mathrm{e}}{2}.$$

This can be shown similarly. Derivative of g_4 is negative and $\lim_{x\to 1^+} g_4(x) = \frac{e}{2}$. Let us calculate derivative

$$g_4'(x) = \frac{e^{\frac{x\log x}{x-1}} \cdot \frac{x-1-\log x}{(x-1)^2} \cdot (x+1) - e^{\frac{x\log x}{x-1}}}{(x+1)^2}$$
$$= -\frac{e^{\frac{x\log x}{x-1}}}{(x^2-1)^4} \cdot \underbrace{(x\log x + \log x + 2 - 2x)}_{=g_2(x)\geq 0} \leq 0.$$

Properties (5) of function ϕ_1 are proved by lemmas 4.3, 4.4 and 4.5.

5. BMO norm of logarithm

Now we can calculate BMO-norm of logarithm by definition (1) above using supremum over all $Q = \langle a; b\rangle$, $1 \leq a < b$,

$$\|\log\|_{BMO(1;\infty)} = \sup \int_Q |f - f_Q|\, dx = 2 \sup_{1\leq a<b} \left(\frac{e^{\frac{b\log b - a\log a}{b-a}}}{e(b-a)} - ab\frac{\log b - \log a}{(b-a)^2} \right)$$
$$= \frac{2}{e} \sup_{1\leq a<b} \phi(a,b) = \frac{2}{e} \lim_{b\to\infty} \phi(a,b) = \frac{2}{e} \lim_{b\to\infty} \phi_1(b) = \frac{2}{e}\ ,$$

where we have used relation (2), definitions (3), (4) and properties (5).

We can conclude that function logarithm has bounded mean oscilation (i.e. it is an element of $BMO(1;\infty)$) and its norm is equal to $\|\log\|_{BMO(1;\infty)} = \frac{2}{e}$.

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