On periodicities in cluster algebras

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Abstract. The aim of this paper is to study periodicities in cluster algebras. Firstly, we give a proof of Restriction and Extension Theorem for cluster algebras without coefficients. Then, we show that the periodicity of a labeled seed with coefficients from an arbitrary semifield depends only on the corresponding extended exchange matrix, which implies the independence of periodicity relative to coefficients.
Keywords: cluster algebra, periodicity, labeled seed, coefficient.

1. Introduction

Cluster algebras were invented by Fomin and Zelevinsky in a series of papers. They are defined as commutative $\mathbb{ZP}$-algebras generated by cluster variables in which $\mathbb{P}$ is a semifield. Many relations between cluster algebras and other branches of mathematics have been discovered, such as periodicities of T-systems and Y-systems, representation theory of quivers, combinatorics, Poisson geometry and higher Teichmüler spaces.

Cluster algebras of finite type were classified by Fomin and Zelevinsky in [5], which is the same as the Cartan-Killing classification of semisimple Lie algebras and finite root systems. Periodicities in cluster algebras were firstly studied by Fomin and Zelevinsky in [6]. They proved Zamolodchikovs periodicity conjecture on Y-systems from indecomposable Cartan matrices of finite type. The periodicity conjecture for a pair of Dynkin diagrams was proved by Keller using additive categorification of cluster algebras in [15]. Another related concept closed with cluster algebras and periodicities is T-systems. Periodicity conjecture and half-periodicity conjecture for T-systems and Y-systems associated with the quantum affine algebras of Dynkin type at any level were proved by In-

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oue, Iyama, Keller, Kunima, and Nakanishi [11, 12]. The theory of periodicities in cluster algebras studied above has also important applications in dilogarithm identities. Indeed for any period of a seed, there exists a dilogarithm identity.

Nakanishi studied periodicities in cluster algebras in general cases rather than finite type in [16]. Nakanishi also proved an interesting theorem (Restriction and Extension Theorem) for cluster algebras with coefficients from subtraction-free semifields. Even though it is hard to determine all periods of labeled seeds, it is possible to find a large number of periods by Extension Theorem. He also formulated dilogarithm identities associated to periodicities of labeled seeds. These results are extended to more general cases, in [17] for generalized cluster algebras and in [18] for quantum generalized cluster algebras.

In this paper, we prove Restriction and Extension Theorem for cluster algebras without coefficients by using positive property in cluster algebras and sign-coherence of g-vectors and c-vectors, see Proposition 3.2. Following from Proposition 3.2, Restriction and Extension Theorem holds for cluster algebras with coefficients from an arbitrary semifield, and it also implies periods of labeled seeds only depend on the extended exchange matrix and thus are irrelevant to coefficients in the labeled seeds.

The organization of this paper is as follows. In Section 2, we give the notions and basic properties for cluster algebras including c-vectors, g-vectors and F-polynomials. In Section 3, we introduce periodicities of labeled seeds and exchange matrices following from [16]. Firstly, we give a proof of Restriction and Extension Theorem for cluster algebras without coefficients. Then, we show that the periodicity of a labeled seed with coefficients from an arbitrary semifield depends only on the corresponding extended exchange matrix. This consequence implies the independence of periodicity relative to coefficients.

2. Preliminaries

In this section, we recall basic concepts and important properties of cluster algebras. Let $\mathbb{P}$ be a semifield, that is an abelian multiplicative group equipped with a binary operation $\oplus$ which is associative, commutative and distributive with respect to the multiplication in $\mathbb{P}$. Note that $\mathbb{P}$ is a torsion-free multiplicative group and thus $\mathbb{ZP}$ is a domain. Let $\mathbb{QP}$ denote the quotient field of $\mathbb{ZP}$. And take a field $\mathcal{F}$ isomorphic to the field of rational functions in n independent variables with coefficients in $\mathbb{QP}$. In the following, for two positive integers $a < b$ we use $[a, b]$ to denote the subset $\{a, a+1, \ldots, b\}$ of natural numbers for simplicity.

A *labeled seed* is a triplet $(\mathbf{x}, \mathbf{y}, B)$ where $\mathbf{x}$ is an n-tuple of free generators of $\mathcal{F}$, $\mathbf{y}$ is an n-tuple of elements of $\mathbb{P}$, and B is an $n \times n$ skew-symmetrizable integer matrix. Note that B is said to be skew-symmetrizable if there exists an positive definite diagonal matrix D such that DB is skew-symmetric. Moreover we call $(\mathbf{y}, B)$ an Y-seed. For $k \in [1, n]$, define another triplet $(\mathbf{x}', \mathbf{y}', B') = \mu_k(\mathbf{x}, \mathbf{y}, B)$ which is called the *mutation* of $(\mathbf{x}, \mathbf{y}, B)$ at k and obtained by the following rules:

(1) $\mathbf{x}' = (x'_1, \ldots, x'_n)$ is given by $x'_k x_k = \frac{y_k \prod x_i^{[b_{ik}]_+} + \prod x_i^{[-b_{ik}]_+}}{y_k \oplus 1}$ and $x'_i = x_i$ for $i \neq k$;

(2) $\mathbf{y}' = (y'_1, \ldots, y'_n)$ is given by $y'_i = y_k^{-1}$ for $i = k$; and otherwise $y'_i = y_i y_k^{[b_{ki}]_+}(y_k \oplus 1)^{-b_{ki}}$;

(3) $b'_{ij} = \begin{cases} -b_{ij}, & \text{if } i = k \text{ or } j = k; \\ b_{ij} + \mathrm{sgn}(b_{ik})\max[b_{ik}b_{kj}]_+, & \text{otherwise.} \end{cases}$.

where $[x]_+ = \max\{x, 0\}$. Note that $(\mathbf{x}', \mathbf{y}', B')$ is also a labeled seed and μ_k is an involution for any $k \in [1, n]$. In a labeled seed $(\mathbf{x}, \mathbf{y}, B)$, $\mathbf{x}$ is called a *labeled cluster*, the elements in $\mathbf{x}$ are called *cluster variables*, $\mathbf{y}$ is the coefficient tuple and B is called an *exchange matrix*.

Remark 2.1. There is a bijection between skew-symmetric integer matrices and finite quivers without loops or 2-cycles. Such quivers are called *cluster quivers*. This bijection is kept under mutations of matrices and quiver mutations, see [15] for details.

Definition 2.2 ([7])**.** *Let* T_n *be a* n*-regular tree and valencies emitting from each vertex are labelled by* $1, 2, \ldots, n$*. A cluster pattern is a* n*-regular tree* T_n *such that for each vertex* $t \in \mathrm{T}_n$*, there is a labeled seed* $\Sigma_t = (\mathbf{x}_t, \mathbf{y}_t, B_t)$ *and for each edge labelled by* k*, two labeled seeds in the endpoints are obtained from each other by seed mutation at* k*. And* $\Sigma_t = (\mathbf{x}_t, \mathbf{y}_t, B_t)$ *are written as follows:*

$$\mathbf{x}_t = (x_{1,t}, x_{2,t}, \ldots, x_{n,t}), \mathbf{y}_t = (y_{1,t}, y_{2,t}, \ldots, y_{n,t}), B_t = (b_{ij}^t).$$

Note that a cluster pattern is uniquely determined by one labeled seed, thus for a labeled seed $(\mathbf{x}, \mathbf{y}, B)$, we may associate with a cluster pattern $T_n(\mathbf{x}, \mathbf{y}, B)$.

The cluster algebra $\mathcal{A} = \mathcal{A}(\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$ associated to the initial seed $(\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$ is a $\mathbb{ZP}$-subalgebra of $\mathcal{F}$ generated by cluster variables appeared in $T_n(\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$. For simplicity, we always denote the initial seed $(\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$ of the cluster algebra with principle coefficients by

$$\mathbf{x}_t = (x_1, x_2, \ldots, x_n), \;\; \mathbf{y}_t = (y_1, y_2, \ldots, y_n), \;\; B_t = (b_{ij}).$$

One of most important properties in cluster algebras is the Laurent phenomenon, that is, any cluster variable can be expressed as a Laurent polynomial in terms of cluster variables in the initial labeled seed with coefficients in $\mathbb{ZP}$. It was conjectured that these Laurent polynomials always have positive coefficients in $\mathbb{NP}$, which has been proved in the skew-symmetrizable case in [8].

Let I be a finite set. The tropical semifield $\mathbb{P} = \mathrm{trop}(y)$ of $y = (y_i)_{i \in I}$ is a multiplicative group generated freely by the elements $y_i (i \in I)$. And the binary operation $\oplus$ is defined as follows:

$$\prod_{i \in I} y_i^{a_i} \oplus \prod_{i \in I} y_i^{b_i} = \prod_{i \in I} y_i^{\min(a_i, b_i)}.$$

The cluster algebra $\mathcal{A} = \mathcal{A}(\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$ is said to have *principal coefficients*, if the semifield is given by $\mathbb{P} = \mathrm{trop}(y_{1,t_0}, y_{2,t_0}, \dots, y_{n,t_0})$. If $\mathbb{P} = \mathbf{1}$, the cluster algebra is said to have *no coefficients*. In the rest of this section, we concentrate on cluster algebras with principal coefficients.

Let $\mathcal{A} = \mathcal{A}(\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$ be a cluster algebra with principle coefficients. In this case, for any $t \in T_n$ and $i \in [1, n]$, the coefficient $y_{i,t}$ is actually a Laurent monomial in $y_j (j \in [1, n])$, i.e. $y_{i,t} = \Pi_{j=1}^n y_j^{c_{ji}^t}$. The vector $\mathbf{c}_{i,t} = (c_{1i}^t, c_{2i}^t, \dots, c_{ni}^t)^T$ is called the *c-vector* corresponding to $y_{i,t}$, and $C_t = (\mathbf{c}_{1,t}, \mathbf{c}_{2,t}, \dots, \mathbf{c}_{n,t})$ is called *c-matrix* at t. For a vector $\mathbf{v} = (v_1, v_2, \dots, v_n)^T$, we write $\mathbf{v} \geqslant 0$ to denote $v_i \geqslant 0$ for any $i \in [1, n]$ and $\mathbf{v} \leqslant 0$ to denote $v_i \leqslant 0$ for any $i \in [1, n]$. For row vectors, we have similar statements.

Proposition 2.3 ([7, 8])**.** *The c-vectors have the following properties:*

(1) *(recurrence relations) for any edge $t \overset{k}{—} t'$ in T_n, the c-vectors satisfy that*

$$c_{ij}^{t'} = \begin{cases} -c_{ik}^t, & \text{if } j = k; \\ c_{ij}^t + \mathrm{sgn}(c_{ik}^t)[c_{ik}^t b_{jk}^t]_+, & \text{otherwise.} \end{cases}$$

(2) *(sign-coherence) for any $t \in T_n, i \in [1, n]$, the c-vector $\mathbf{c}_{i,t}$ satisfies either $\mathbf{c}_{i,t} \geqslant 0$ or $\mathbf{c}_{i,t} \leqslant 0$.*

It is natural to consider extended matrices $\begin{bmatrix} B_t \\ C_t \end{bmatrix}$, and they are called extended exchange matrices at t. Note that $\begin{bmatrix} B_t \\ C_t \end{bmatrix}$ is also obtained from $\begin{bmatrix} B_0 \\ I \end{bmatrix}$ by the mutation rule of matrices.

Remark 2.4. If $B = B_0$ is skew-symmetric, its frmaed quiver is obtained from its cluster quiver by adding n (frozen) vertices $\{n+1, n+2, \dots, 2n\}$ and arrows from $n+i$ to i for each $i \in [1, n]$, which is the corresponding quiver of $\begin{bmatrix} B \\ I \end{bmatrix}$. The corresponding quiver of $\begin{bmatrix} B_t \\ C_t \end{bmatrix}$ is also obtained from the framed quiver of B by quiver mutations.

For a cluster algebra with principle coefficients, the Laurent phenomenon and positive property are stated as follows.

Proposition 2.5 ([7])**.** *Let $\mathcal{A} = \mathcal{A}(\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$ be the cluster algebra with principle coefficients. Each cluster variable $x_{i,t}$ is contained in $\mathbb{Z}_{\geqslant 0}[x_1^{\pm 1}, x_2^{\pm 1}, \dots, x_n^{\pm 1}, y_1, y_2, \dots, y_n]$.*

Let $F_{i,t} = x_{i,t}(1, 1, \dots, 1, y_1, y_2, \dots, y_n) \in Z_{\geqslant 0}[y_1, y_2, \dots, y_n]$ be polynomials with nonnegative coefficients and they are called F-polynomials.

Proposition 2.6 ([7, 8])**.** *The F-polynomials satisfy the following properties:*

(1) *(recurrence relations) for any edge $t \overset{k}{-} t'$ in T_n, the c-vectors satisfy that*

$$F_{i,t'} = \begin{cases} F_{i,t}^t, & \text{if } i \neq k; \\ (F_{k,t})^{-1}(\prod_{j=1}^n y_j^{c_{jk}^t}(F_{j,t})^{[b_{jk}^t]_+} + \prod_{j=1}^n y_j^{-c_{jk}^t}(F_{j,t})^{[-b_{jk}^t]_+}), & \text{otherwise.} \end{cases}$$

(2) *each F-polynomial $F_{i,t}$ has a constant term 1.*

(3) *let $\mathcal{A}_{\mathbb{P}_1} = \mathcal{A}_{\mathbb{P}_1}(\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$ be the cluster algebra over an arbitrary semifield $\mathbb{P}_1$, and $x_{i,t,\mathbb{P}_1}$ be the corresponding cluster variable in the i-th position of the labeled cluster at t. Then, we have*

$$x_{i,t,\mathbb{P}_1} = \frac{x_{i,t}|_{\mathcal{F}_1}(x_1, x_2, \ldots, x_n; y_1, \ldots, y_n)}{F_{i,t}|_{\mathbb{P}_1}(y_1, y_2, \ldots, y_n)}.$$

In addition, any cluster variable $x_{i,t}$ is homogeneous with respect to a given $\mathbb{Z}^n$-grading in $\mathbb{Z}_{\geqslant 0}[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_n^{\pm 1}, y_1, y_2, \ldots, y_n]$, which is given by

$$deg(x_i) = \mathbf{e}_i, \quad deg(y_i) = -\mathbf{b}_i,$$

where $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are the standard basis vectors in $\mathbb{Z}_n$, and $\mathbf{b}_i$ is the i-th column vector of B. Let $\mathbf{g}_{i,t}$ denote the degree (column) vector of $x_{i,t}$. They are called *g-vectors* and the matrix $G_t = (\mathbf{g}_{1,t}, \mathbf{g}_{2,t}, \ldots, \mathbf{g}_{n,t})$ is called the *g-matrix* at t.

Proposition 2.7 ([7, 8, 19])**.** *The g-vectors and g-matrices satisfy the following properties:*

(1) *(recurrence relations) for any edge $t \overset{k}{-} t'$ in T_n, the g-vectors satisfy that*

$$g_{ij}^{t'} = \begin{cases} g_{ij}^t, & \text{if } j \neq k; \\ -g_{ik}^t + \Sigma_{s=1}^n g_{is}^t[-b_{sk}^t]_+ - \Sigma_{s=1}^n b_{is}^t[-c_{sk}^t]_+, & \text{otherwise.} \end{cases}$$

(2) *(sign-coherence) any row vector $\mathbf{v}$ of the g-matrix G_t satisfies either $\mathbf{v} \geqslant 0$ or $\mathbf{v} \leqslant 0$.*

(3) *for any $t \in T_n$, we have*

$$G_t^T D(C_t) D^{-1} = I,$$

where D is the skew-symmetriclizer of B.

(4) *let $\mathcal{A} = \mathcal{A}(\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$ be the cluster algebra with coefficients from an arbitrary semifield $\mathbb{P}$. For any $t \in T_n, i \in [1, n]$, the cluster variable can be expressed by*

$$x_{i,t} = (\prod_{j=1}^n x_j^{g_{j,t}}) \frac{F_{i,t}|_{\mathcal{F}}(\widehat{y_1}, \widehat{y_2}, \ldots, \widehat{y_n})}{F_{i,t}|_{\mathbb{P}}(y_1, y_2, \ldots, y_n)},$$

where $\widehat{y_j} = y_j \prod_i x_i^{b_{ij}}$ $(1 \leqslant j \leqslant n)$.

3. Periodicities in cluster algebras

In this section, we recall basic concepts on periodicities, and we prove main results in this section. Let I be a subset of $[1,n]$, a sequence $\mathbf{i}=(i_1,i_2,\ldots,i_s)$ is called an I-sequence if $i_p\in I$ for any $p\in[1,s]$. Moreover, $\mathbf{i}$ is called an essential I-sequence if $i_p\neq i_{p+1}$ for any $p\in[1,s-1]$. Let $(\mathbf{x},\mathbf{y},B)$ be a labeled seed and $\sigma\in\mathfrak{S}_n$, where $\mathfrak{S}_n$ is the symmetric group of degree n. Then we define $(\mathbf{x},\mathbf{y},B)^\sigma:=(\mathbf{x}^\sigma,\mathbf{y}^\sigma,B^\sigma)$, where $\mathbf{x}^\sigma,\mathbf{y}^\sigma,B^\sigma$ are given by

$$\mathbf{x}^\sigma=(x_{\sigma(1)},x_{\sigma(2)},\ldots,x_{\sigma(n)}),\quad \mathbf{y}^\sigma=(y_{\sigma(1)},y_{\sigma(2)},\ldots,y_{\sigma(n)}),\quad B^\sigma=(b_{\sigma(i)\sigma(j)}).$$

Moreover if $B=(b_{ij})_{m\times n}$ is an $m\times n$ matrix with $m>n$ and $\sigma\in\mathfrak{S}_n$, we define $B^\sigma=(b_{\sigma(i)\sigma(j)})_{m\times n}$ where $\sigma(k)=k$ for any $k\in[n+1,m]$.

Definition 3.1. *Let $(\mathbf{x}_t,\mathbf{y}_t,B_t)$, $(\mathbf{x}_{t'},\mathbf{y}_{t'},B_{t'})$ be two labeled seeds of the cluster algebra $\mathcal{A}=\mathcal{A}(\mathbf{x},\mathbf{y},B)$, $\mathbf{i}=(i_1,i_2,\ldots,i_s)$ be an I-sequence, and $\sigma\in\mathfrak{S}_n$, where $\mathfrak{S}_n$ is the symmetric group of degree n.*

(i) If $B_{t'}=\mu_{\mathbf{i}}(B_t)$ and $B_{t'}=B_t^\sigma$, we call $\mathbf{i}$ a σ-period of the exchange matrix B_t; furthermore, if $\sigma=\mathrm{id}$, we simply call it a period of B'.

(ii) If $(\mathbf{y}_{t'},B_{t'})=\mu_{\mathbf{i}}(\mathbf{y}_t,B_t)$ and $(\mathbf{y}_{t'},B_{t'})=(\mathbf{y}_t^\sigma,B_t^\sigma)$, we call $\mathbf{i}$ a σ-period of the Y-seed $(\mathbf{y}_t,B_t)$; furthermore, if $\sigma=\mathrm{id}$, we simply call it a period of $(\mathbf{y}_t,B_t)$.

(iii) If $(\mathbf{x}_{t'},\mathbf{y}_{t'},B_{t'})=\mu_{\mathbf{i}}(\mathbf{x}_t,\mathbf{y}_t,B_t)$ and $(\mathbf{x}_{t'},\mathbf{y}_{t'},B_{t'})=(\mathbf{x}_t,\mathbf{y}_t,B_t)^\sigma$, we call $\mathbf{i}$ a σ-period of the labeled seed $(\mathbf{x}_t,\mathbf{y}_t,B_t)$; furthermore, if $\sigma=\mathrm{id}$, we simply call it a period of $(\mathbf{x}_t,\mathbf{y}_t,B_t)$.

The definition of σ-period is a little different from the one in [16]. Note that it is well-known that labeled seeds are determined by clusters, thus to show $\mathbf{i}$ is a σ-period of $(\mathbf{x}_t,\mathbf{y}_t,B_t)$, it is enough to show $(\mathbf{x}_{t'},\mathbf{y}_{t'},B_{t'})=\mu_{\mathbf{i}}(\mathbf{x}_t,\mathbf{y}_t,B_t)$ and $\mathbf{x}_{t'}=\mathbf{x}_t^\sigma$.

Let $\mathbf{i}=(i_1,i_2,\ldots,i_s)$ and $\mathbf{j}=(j_1,j_2,\ldots,j_p)$ be two arbitrary I-sequences. Suppose that $(\mathbf{x}_t,\mathbf{y}_t,B_t)=\mu_{\mathbf{j}}(\mathbf{x},\mathbf{y},B)$ and $\mathbf{i}$ is a period of $(\mathbf{x},\mathbf{y},B)$. Then we have

$$\mu_{\mathbf{j}}\mu_{\mathbf{i}}\mu_{\mathbf{j}^{-1}}(\mathbf{x}_t,\mathbf{y}_t,B_t)=\mu_{\mathbf{j}}\mu_{\mathbf{i}}\mu_{\mathbf{j}^{-1}}\mu_{\mathbf{j}}(\mathbf{x},\mathbf{y},B)=\mu_{\mathbf{j}}\mu_{\mathbf{i}}(\mathbf{x},\mathbf{y},B)=(\mathbf{x}_t,\mathbf{y}_t,B_t),$$

where $\mathbf{j}^{-1}=(j_p,\ldots,j_2,j_1)$. Thus, the sequence $\mathbf{j}^{-1}\mathbf{i}\mathbf{j}$ is a period of $(\mathbf{x}_t,\mathbf{y}_t,B_t)=\mu_{\mathbf{j}}(\mathbf{x},\mathbf{y},B)$, which induces a natural bijection between the sets of periods of two labeled seeds. Thus we may, without loss of generality, mainly consider periods of the initial labeled seeds.

The Extension Theorem was partially formulated by Keller in [14] and was generalized by Plamondon. Nakanishi proved Restriction/Extension Theorem of periodicities of labeled seeds for cluster algebras with coefficients from the universal semifield, see [16]. We now prove this theorem for cluster algebras without coefficients in Proposition 3.2.

Let $I,\tilde{I}$ be two index sets with $I\subset\tilde{I}$, and $B=(b_{ij})_{i,j\in I}$ is the principal submatrix of a skew-symmetrizable matrix $\tilde{B}=(\tilde{b}_{ij})_{i,j\in\tilde{I}}$ such that $B=\tilde{B}|_I$ under the restriction of the index set I. In this case, B is called the I-restriction of $\tilde{B}$ and $\tilde{B}$ is called the $\tilde{I}$-extension of B.

Proposition 3.2. *For $I \subset \tilde{I}$, let B be the I-restriction of the skew-symmetrizable matrix $\tilde{B}$, $\tilde{B}$ be the $\tilde{I}$-extension of B, and $\sigma \in \mathfrak{S}_n$.*

(i) (Restriction) Assume that an I-sequence $\mathbf{i} = (i_1, i_2, \ldots, i_s)$ is a σ-period of the labeled seed $(\tilde{\mathbf{x}}, \tilde{B})$ in $\mathcal{A}(\tilde{\mathbf{x}}, \tilde{B})$, then $\mathbf{i}$ is also an σ-period of the labeled seed $(\mathbf{x}, B)$ in $\mathcal{A}(\mathbf{x}, B)$.

(ii) (Extension) Assume that an I-sequence $\mathbf{i} = (i_1, i_2, \ldots, i_s)$ is a σ-period of the labeled seed $(\mathbf{x}, B)$ in $\mathcal{A}(\mathbf{x}, B)$, then $\mathbf{i}$ is also an σ-period of the labeled seed $(\tilde{\mathbf{x}}, \tilde{B})$ in $\mathcal{A}(\tilde{\mathbf{x}}, \tilde{B})$.

Proof. (i) For any I-sequence $\mathbf{j}$, the labeled seed $\mu_{\mathbf{j}}(\mathbf{x}, B)$ can be obtained from $\mu_{\mathbf{j}}(\tilde{\mathbf{x}}, \tilde{B})$ by specializing all cluster variables $x_k (k \in \tilde{I} \backslash I)$ to 1. Then the result follows.

(ii) It is enough to prove this result for $I = \{1, 2, \ldots, n\}$ and $\tilde{I} = \{1, 2, \ldots, n+1\}$. Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, $\tilde{\mathbf{x}} = (x_1, x_2, \ldots, x_n, x_{n+1})$. Since $\mathbf{i}$ is an I-sequence and a σ-period of $(\mathbf{x}, B)$, we have $\mu_{\mathbf{i}}(\mathbf{x}, B) = (\mathbf{x}^{\sigma}, B^{\sigma})$. Suppose that $(\tilde{\mathbf{x}}', \tilde{B}') = \mu_{\mathbf{i}}(\mathbf{x}, B)$, where $\tilde{\mathbf{x}}' = (x'_1, \ldots, x'_n, x'_{n+1})$. Note that $x'_{n+1} = x_{n+1}$ follows from the mutation rules and the fact that $\mathbf{i}$ is an I-sequence. Since every cluster variable is a Laurent polynomial in terms of the initial seed with nonnegative coefficients, for each $i \in [1, n]$, suppose that

$$x'_i = f_i(x_1, x_2, \ldots, x_{n+1})$$

be the corresponding Laurent polynomial in terms of $x_1, x_2, \ldots, x_{n+1}$ with positive coefficients. Then, we have

$$x_{\sigma(i)} = f_i(x_1, x_2, \ldots, x_n, 1)$$

for any $i \in [1, n]$. Since $x_1, x_2, \ldots, x_n$ are algebraically independent, $f_i(x_1, x_2, \ldots, x_{n+1})$ must be a Laurent monomial in the form $x_{\sigma(i)} x_{n+1}^{a_i}$ for some $a_i \in \mathbb{Z}$, that is,

$$x'_i = x_{\sigma(i)} x_{n+1}^{a_i}$$

for each $i \in [1, n]$ and some $a_i \in \mathbb{Z}$.

Using terms of F-polynomials and g-vectors, we have another expression of x'_i. By Proposition 2.7(4), for each $i \in [1, n]$, we have

$$x_{\sigma(i)} x_{n+1}^{a_i} = x'_i = F_{i,\mathbf{i}}^{\tilde{B}}|_{\mathcal{F}}(\prod_i x_i^{\tilde{b}_{i1}}, \ldots, \prod_i x_i^{\tilde{b}_{in}}, \prod_i x_i^{\tilde{b}_{i(n+1)}}) x_1^{g_{1i}} \ldots x_n^{g_{ni}} x_{n+1}^{g_{n+1,i}}.$$

Note that $F_{i,\mathbf{i}}^{\tilde{B}} \in \mathbb{Z}_{\geqslant 0}[y_1, \ldots, y_n, y_{n+1}]$ is a polynomial with nonnegative integer coefficients, and $F_{i,\mathbf{i}}^{\tilde{B}}$ has a constant term 1. Then we must have

$$x_{\sigma(i)} x_{n+1}^{a_i} = x_1^{g_{1i}} \ldots x_n^{g_{ni}} x_{n+1}^{g_{n+1,i}}.$$

Then $g_{\sigma(i)i} = 1, g_{n+1,i} = a_i, g_{qi} = 0$ for $q \in [1, n+1] \backslash \{\sigma(i), n+1\}$. The corresponding G-matrix has the form

$$G = \begin{bmatrix} P & 0 \\ \alpha & 1 \end{bmatrix}, \qquad (1)$$

where P is the permutation matrix corresponding to σ (note that $P^T = P^{-1}$), and $\alpha = (a_1, a_2, \ldots, a_n)$.

By the property of sign-coherence of g-vectors, the vector $\alpha \geq 0$. By Proposition 2.7(3), the tranpose of DCD^{-1} is inverse to G, that is

$$G^T DCD^{-1} = I_{n+1}, \tag{2}$$

where D is the positive definite diagonal integer matrix such that $D\tilde{B}$ is skew-symmetric.

Assume that

$$DCD^{-1} = \begin{bmatrix} A & \beta \\ \gamma & w \end{bmatrix}, \tag{3}$$

we have

$$G^T DCD^{-1} = \begin{bmatrix} P^T A + \alpha^T \gamma & P^T \beta + \alpha^T w \\ \gamma & w \end{bmatrix}. \tag{4}$$

Replacing (4) into (2), since $P^T = P^{-1}$, we obtain that $A = P, \ \ \beta = -P\alpha^T, \ \ \gamma = 0, \ \ w = 1$. Thus,

$$DCD^{-1} = \begin{bmatrix} P & -P\alpha^T \\ 0 & 1 \end{bmatrix}. \tag{5}$$

By the property of sign-coherence of c-vectors, $-P\alpha^T \geqslant 0$ which implies $a_i \leqslant 0$ for each $i \in [1, n]$. Then we have $a_i = 0$ for each $i \in [1, n]$, and $x'_i = x_{\sigma(i)}$ follows. Thus $\mathbf{i}$ is also an σ-period of the labeled seed $(\tilde{\mathbf{x}}, \tilde{B})$ in $\mathcal{A}(\tilde{\mathbf{x}}, \tilde{B})$ □

Even though Restriction Theorem holds for exchange matrices, in general, Extension Theorem does not hold. For exchange matrices, we observe the following lemma.

Lemma 3.3. *Let $I = \{1, 2, \ldots, n\}$ and $\tilde{I} = \{1, 2, \ldots, m\}$ with $m \geq n$, and B be the I-restriction of the skew-symmetrizable matrix $\tilde{B} = \begin{bmatrix} B & H \\ C & M \end{bmatrix}$, and $\mathbf{i}$ be an I-sequence. Suppose that $\mu_{\mathbf{i}}(\tilde{B}) = \begin{bmatrix} B' & H' \\ C' & M' \end{bmatrix}$, then $\mu_{\mathbf{i}} \begin{bmatrix} B \\ C \end{bmatrix} = \begin{bmatrix} B' \\ C' \end{bmatrix}$.*

Proof. Note that $\mathbf{i}$ is an I-sequence. The statement follows easily from the definition of mutation of matrices.

□

As an application of Proposition 3.2, we prove the following theorem.

Theorem 3.4. *An I-sequence $\mathbf{i}$ is a σ-period of the initial labeled seed $(\mathbf{x}, B)$ if and only if it is a σ-period of the $\begin{bmatrix} B \\ I_n \end{bmatrix}$. In particular, if B is skew-symmetric, then $\mathbf{i}$ is a σ-period of the initial labeled seed $(\mathbf{x}, B)$ if and only if $\mathbf{i}$ is a σ-period of the corresponding framed quiver.*

Proof. Let $\tilde{\mathbf{x}} = (x_1, x_2, \dots, x_{2n})$, and $\tilde{B} = \begin{bmatrix} B & -I_n \\ I_n & 0 \end{bmatrix}$. If $\mu_{\mathbf{i}}(\mathbf{x}, B) = (\mathbf{x}^\sigma, B^\sigma)$, by Extension Theorem, $\mu_{\mathbf{i}}(\tilde{\mathbf{x}}, \tilde{B}) = (\tilde{\mathbf{x}}^\sigma, \tilde{B}^\sigma)$. Then the result follows from Lemma 3.3.

Conversely, suppose $\mu_{\mathbf{i}} \begin{bmatrix} B \\ I_n \end{bmatrix} = \begin{bmatrix} B \\ I_n \end{bmatrix}^\sigma$, we consider the cluster algebra with principal coefficients and the initial labeled seed (X, Y, B). It is easy to see $\mu_{\mathbf{i}}(Y) = Y^\sigma$. Since labeled clusters are determined by their coefficients, i.e. C-matrices, thus we have $\mu_{\mathbf{i}}(X, Y, B) = (X^\sigma, Y^\sigma, B^\sigma)$. Then the conclusion follows from Proposition 2.6(3). □

Note that by [19] exchange matrices are determined by c-matrices, Corollary 3.4 essentially says a σ-period of the initial labeled seed is exactly the σ-period of the corresponding c-matrices. Actually Theorem 3.4 holds for cluster algebras with coefficients from arbitrary semifields.

Theorem 3.5. *Let $\mathcal{A} = \mathcal{A}(\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$ be a cluster algebra over an arbitrary semifield $\mathbb{P}$, with the initial seed $(\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$ given by*

$$\mathbf{x}_{t_0} = (x_1, x_2, \dots, x_n),\ \mathbf{y}_{t_0} = (y_1, y_2, \dots, y_n),\ B_{t_0} = (b_{ij})_{n\times n}.$$

Let $I = \{1, 2, \dots, n\}$ and $\mathbf{i}$ be an I-sequence and $(\mathbf{x}, \mathbf{y}, B)$ be a labeled seed of $\mathcal{A}$. Then $\mathbf{i}$ is a σ-period of $(\mathbf{x}, \mathbf{y}, B)$ if and only if $\mathbf{i}$ is a σ-period of $\begin{bmatrix} B \\ I_n \end{bmatrix}$.

Proof. Without loss of generality, assume that $(\mathbf{x}, \mathbf{y}, B) = (\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$.

Suppose that $\mu_{\mathbf{i}}(\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0}) = (\mathbf{x}_{t_0}^\sigma, \mathbf{y}_{t_0}^\sigma, B_{t_0}^\sigma)$. Since $\mu_{\mathbf{i}}(\mathbf{x}_{t_0}, B_{t_0})$ is obtained from $\mu_{\mathbf{i}}(\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$ by specializing all $y_i (1 \leqslant i \leqslant n)$ to 1. Thus, we know that $\mathbf{i}$ is a σ-period of $(\mathbf{x}_{t_0}, B_{t_0})$ in $\mathcal{A}(\mathbf{x}_{t_0}, B_{t_0})$, which is also a σ-period of $\begin{bmatrix} B_{t_0} \\ I_n \end{bmatrix}$ by Theorem 3.4.

Conversely, assume that $\mathbf{i}$ is a σ-period of $\begin{bmatrix} B_{t_0} \\ I_n \end{bmatrix}$. Let $\mathcal{A}(X_{t_0}, Y_{t_0}, B_{t_0})$ be the corresponding cluster algebra of $\mathcal{A}$ with principal coefficients and with the initial labeled seed $(X_{t_0}, Y_{t_0}, B_{t_0})$. We have $\mathbf{i}$ is a σ-period of $(X_{t_0}, Y_{t_0}, B_{t_0})$ in $\mathcal{A}(X_{t_0}, Y_{t_0}, B_{t_0})$, since labeled seeds are determined by c-matrices by [[1], Theorem 2.5]. In particular, $\mathbf{i}$ is a σ-period of X_{t_0}. By Proposition 2.6, $\mathbf{i}$ is also a σ-period of $\mathbf{x}_{t_0}$ and thus is a σ-period of $(\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$. □

Corollary 3.6. *Periods of labeled seeds are independent of coefficients.*

Remark 3.7. Corollary 3.6 is also proved by Peigen Cao and the first author in a different way, see e.g. [2] for details. In this paper, we emphasise that it can also be shown by Extension Theorem.

Example 3.8. Let $\mathcal{A}$ be a cluster algebra with the initial labeled seed $(\mathbf{x}, B)$, where $\mathbf{x} = (x_1, x_2)$ and $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is the exchange matrix corresponding to the cluster quiver of type A_2. Notice that $\mathcal{A}$ is of finite type, all labeled seeds of $\mathcal{A}$ are shown in the following graph. There are ten various labeled seeds and five unlabeled seeds.

Let $I = [1, 2]$, then the two I-sequences $\mathbf{i} = (1, 2, 1, 2, 1)$ and $\mathbf{j} = (2, 1, 2, 1, 2)$ are (12)-periods of the labeled seed $(\mathbf{x}, B)$, and the two I-sequences

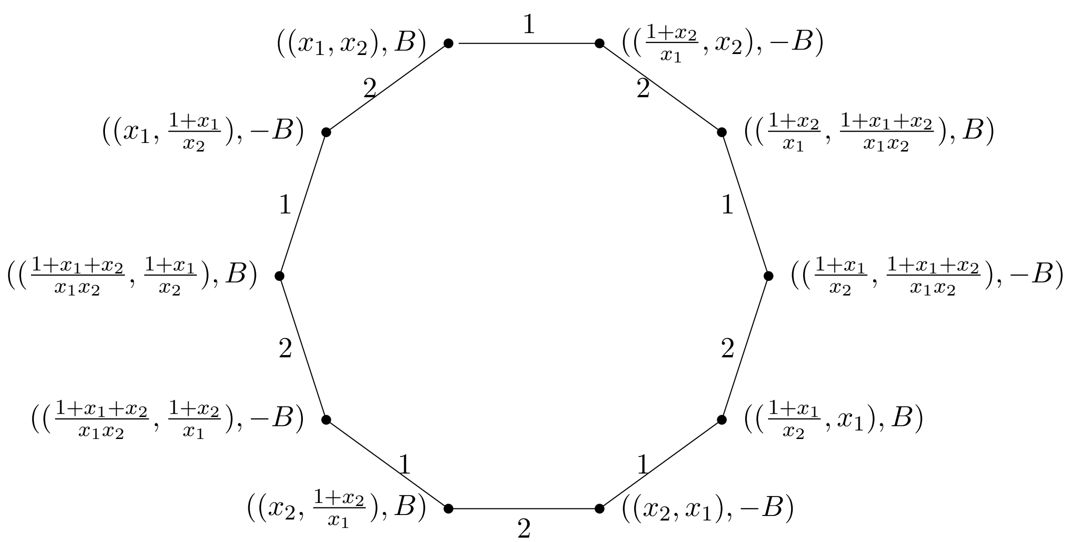
$$\mathbf{p} = (1, 2, 1, 2, 1, 2, 1, 2, 1, 2) \text{ and } \mathbf{q} = (2, 1, 2, 1, 2, 1, 2, 1, 2, 1)$$

are periods of $(\mathbf{x}, B)$.

Since $\mu_{\mathbf{p}}\mu_{\mathbf{q}} = id = \mu_{\mathbf{q}}\mu_{\mathbf{p}}$, any period of $(\mathbf{x}, B)$ is copy of $\mathbf{p}$ or $\mathbf{q}$. The two I-sequences $\mathbf{a} = (1, 2)$ and $\mathbf{b} = (2, 1)$ are two periods of the exchange matrix B, and any period of B is copy of $\mathbf{a}$ or $\mathbf{b}$.

On the other hand, by the Extension Theorem, for any cluster quiver which has a simple edge connecting two vertices i and j, the actions of two mutation sequences $\mu_i\mu_j\mu_i\mu_j\mu_i$ and $\mu_j\mu_i\mu_j\mu_i\mu_j$ on the labeled seed $(\mathbf{x}, B)$ are both equivalent to the action of the permutation (ij) .

In general, for any skew-symmetrizable matrix B, let $B[i, j]$ be the principal submatrix obtained from B by removing its rows or columns indexed by $[1, n] \setminus I$, where $I = \{i, j\}$. Then there exists an essential I-sequence $\mathbf{i}$ such that $\mathbf{i}$ is a period of $(\mathbf{x}, B)$ if and only if $B[i, j]$ is of finite type, i.e. $|b_{ij}b_{ji}| \leq 3$.



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