Closed linear transformations of complex space-time endowed with Euclidean or Lorentz metric

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Abstract. Linear transformations (LTs) are essential for the development of Relativity Theory. Special Relativity is based on Lorentz Boost (LB). This cancels the transitive attribute in parallelism (which is equivalent to the 5^{th} Euclidean Postulate), when three observers are related (successive transformations), because LB is not closed LT. So, LB is combined with Euclidean spatial rotation, in order to obtain Lorentz transformation (which is closed LT) and the corresponding Lorentz group. In this paper, a new closed isometric LT in spaces (V^4) of dimension four ($n = 4$), with Euclidean or Lorentz metric (Minkowski Space), is presented (correlating frames with parallel spatial axes). This LT is represented by a matrix (Λ_B) containing real and imaginary numbers. Thus, V^4 is based on the field of complex numbers (C), by using real 0-(temporal) and complex 1, 2, 3- (spatial) coordinates.
Keywords: 5^{th} Euclidean postulate, Euclidean metric, Euclidean space, linear transformation, spacetime, Lorentz Boost, Lorentz Transformation, Minkowski space, special relativity, Thomas rotation.

Abbreviations and annotations

E^3 ▶ **three-dimensional Euclidean**	M^4 ▶ **Minkowski Space**
E^3 ▶ **four-dimensional Euclidean Space**	**RT** ▶ **Relativity Theory**
LB ▶ **Lorentz Boost**	**SR** ▶ **Special Relativity**
LT ▶ **Linear Transformation**	V^4 ▶ **Four-dimensional Space**

1. Introduction

Linear transformations (LTs) are essential for the development of Relativity Theory (RT), Quantum Mechanics (QMs) and generally in Modern Physics. In

Special Relativity (SR), Hermann Minkowski combined [1] (pp. 39-53) the one-dimensional time (T) with the *Three-dimensional Euclidean space* (E^3) endowed with Euclidean metric [2] (p. 14):

$$g_{E^3} = \mathrm{diag}(1, 1, 1) \tag{1.1}$$

and he produced a four-dimensional space M^4 (since known as *Minkowski space-time*), endowed with *Lorentz metric* [2] (p. 8):

$$g_L = \mathrm{diag}(-1, 1, 1, 1). \tag{1.2}$$

In this space, the position four-vector is written as

$$\vec{X} = x^0\vec{e}_0 + x^1\vec{e}_1 + x^2\vec{e}_2 + x^3\vec{e}_3 = \vec{e}_\mu x^\mu, \tag{1.3}$$

where

$$[\vec{e}_\mu] = [\vec{e}_0\, \vec{e}_1\, \vec{e}_2\, \vec{e}_3]; \quad X = \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \tag{1.4}$$

are the basis of M^4 and the coordinates of the position four-vector, respectively. The *Einstein's summation convention* [2] (p. 3) has been used in (1.3ii) and the following equations. Besides the *Lorentz length* $|\vec{X}|$ of the position four-vector is defined [2] (p. 17):

$$|\vec{X}|^2 = x_\mu g_{L\mu\nu} x^\nu = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2. \tag{1.5}$$

Correspondingly in E^3, we have

$$\vec{X} = x\vec{e}_1 + y\vec{e}_2 + z\vec{e}_3 = x^1\vec{e}_1 + x^2\vec{e}_2 + x^3\vec{e}_3 = \vec{e}_i x^i. \tag{1.6}$$

The common choice is the usage of real coordinates

$$x^1, x^2, x^3 \in R, \tag{1.7}$$

in order to be easily perceived by human senses.

We shall see that this field is not enough, in case that we wish to produce closed isometric LTs in Four-dimensional Spaces (V^4). So, we prefer complex coordinates

$$x^1, x^2, x^3 \in C. \tag{1.8}$$

For simplicity reasons, wherever we write i (the imaginary unit), we mean $\pm i$:

$$\mathrm{i} \to \pm\mathrm{i} \;\; ; \;\; \text{-i} \to \mp\mathrm{i}. \tag{1.9}$$

Besides, V^4 endowed with *Euclidean metric* [2] (p. 8):

$$g_{E^4} = \text{diag}(1,\ 1,\ 1,\ 1), \tag{1.10}$$

is called *Euclidean Four-dimensional Space* (E^4). In this space, the position four-vector is written as

$$\vec{X} = X^0\vec{E}_0 + x^1\vec{e}_1 + x^2\vec{e}_2 + x^3\vec{e}_3 = \vec{e}_\mu x^\mu, \tag{1.11}$$

where

$$[\vec{e}_\mu] = [\vec{E}_0\ \vec{e}_1\ \vec{e}_2\ \vec{e}_3] \quad ; \quad X = \begin{bmatrix} X^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \tag{1.12}$$

are the basis of E^4 and the coordinates of the position four-vector, respectively. Moreover, the *Euclidean length* of the position four-vector is

$$|\vec{X}|^2 = x_\mu g_{E^4\mu\nu} x^\nu = (X^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2. \tag{1.13}$$

2. Connection between spaces endowed with Euclidean and Lorentz metric and their transformations

From (1.2) and (1.10), we respectively have

$$\vec{e}_0 \cdot \vec{e}_0 = -1 \quad ; \quad \vec{E}_0 \cdot \vec{E}_0 = 1, \tag{2.1}$$

where dot "$\cdot$" is *Euclidean inner product* [2] (p. 7).
Thus, we understand that

$$\vec{E}_0 = \mathrm{i}\vec{e}_0. \tag{2.2}$$

After replacing the above to (1.11), we have

$$\vec{X} = iX^0\vec{e}_0 + x^1\vec{e}_1 + x^2\vec{e}_2 + x^3\vec{e}_3 = \vec{e}_\mu x^\mu. \tag{2.3}$$

Comparing this to (1.3), we obtain

$$X^0 = \frac{1}{\mathrm{i}} x^0. \tag{2.4}$$

The above procedure shows that the difference between *Euclidean* and *Lorentz metric* is caused by the different 0-four-vector of the used basis: $\vec{E}_0$ and $\vec{e}_0$, respectively. So, E^4 and M^4 are related via (2.2) and (2.4).

The corresponding LTs are also easily related. For instance, the *active interpretation* of LT [2] (p. 6) is

$$X' = \Lambda_{(\beta)} X, \tag{2.5}$$

where

$$\beta = \begin{bmatrix} \beta^1 \\ \beta^2 \\ \beta^3 \end{bmatrix} \quad ; \quad \beta^i = \frac{dx^i}{dx^0} \quad ; \quad i = 1, 2, 3, \tag{2.6}$$

is called *β-factor*. Eqn (2.5) expresses *proper Lorentz Boost* (LB) in M^4 [2] (p. 30) and E^4, correspondingly:

$$\Lambda_{L(\beta)} = \begin{bmatrix} \gamma_{(\beta)} & -\gamma_{(\beta)}\beta^T \\ -\gamma_{(\beta)}\beta & I_3 + \frac{\gamma_{(\beta)}-1}{\beta^T\beta}\beta\beta^T \end{bmatrix} \quad ; \quad X = \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}, \tag{2.7}$$

$$\Lambda^E_{L(\beta)} = \begin{bmatrix} \gamma_{(\beta)} & \mathrm{i}\gamma_{(\beta)}\beta^T \\ -\mathrm{i}\gamma_{(\beta)}\beta & I_3 + \frac{\gamma_{(\beta)}-1}{\beta^T\beta}\beta\beta^T \end{bmatrix} ; \; X = \begin{bmatrix} X^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\mathrm{i}}x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}, \tag{2.8}$$

where I_3 is the unitary 3×3 matrix and *Lorentz γ-factor* is

$$\gamma_{(\beta)} = \frac{1}{\sqrt{1 - \beta^T\beta}}. \tag{2.9}$$

The *typical proper LB* along x-axis in M^4 [2] (p. 21) and E^4 has, correspondingly:

$$\Lambda_{L(x)(\beta)} = \begin{bmatrix} \gamma_{(\beta)} & -\gamma_{(\beta)}\beta & 0 & 0 \\ -\gamma_{(\beta)}\beta & \gamma_{(\beta)} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; \; X = \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}, \tag{2.10}$$

$$\Lambda^E_{L(x)(\beta)} = \begin{bmatrix} \gamma_{(\beta)} & \mathrm{i}\gamma_{(\beta)}\beta & 0 & 0 \\ -\mathrm{i}\gamma_{(\beta)}\beta & \gamma_{(\beta)} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ; \; X = \begin{bmatrix} X^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\mathrm{i}}x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}. \tag{2.11}$$

It is noted that transformation matrices (2.8) and (2.11) are rotation matrices. So, LB in M^4 becomes rotation in E^4 (*Wick Rotation*) [3].

The physical content of the four-dimensional space is obtained by the replacement

$$x^0 = ct, \tag{2.12}$$

where

$$c = 299,792,458ms^{-1} \tag{2.13}$$

is the speed of light in vacuum. Then β-factor is called *velocity factor*

$$\beta^i = \frac{1}{c}\frac{dx^i}{dt} = \frac{\upsilon^i}{c}. \tag{2.14}$$

3. Derivation of the proper closed isometric linear transformation in four-dimensional space endowed with Euclidean or Lorentz metric

3.1 Motion in x-direction

We consider one unmoved observer (frame) $Oxyz$, who measures real space-time and another observer (frame) $O'x'y'z'$ with parallel spatial axes, moving to the right, along x-axis with velocity

$$\upsilon = \beta c \tag{3.1.1}$$

wrt the observer (frame) $Oxyz$ (Figure 1).

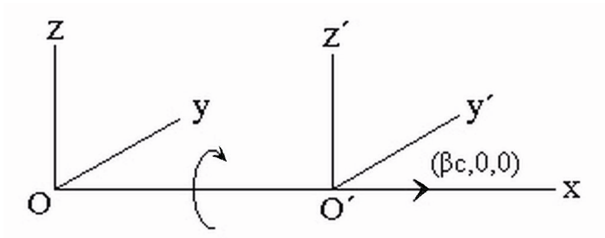

Figure 1. Two frames $Oxyz$ and $O'x'y'z'$, which initially coincide. The second is moving with velocity $(\beta c.0.0)$ wrt to $Oxyz$.

Supposing the next linear transformation:

$$ct' = bct + ax + ky + \nu z \tag{3.1.2}$$

$$x' = gct + fx + \delta y + \theta z \tag{3.1.3}$$

$$y' = g_1 ct + f_1 x + hy + \lambda z \tag{3.1.4}$$

$$z' = g_2 ct + f_2 x + \xi y + \mu z, \tag{3.1.5}$$

we determine the coefficients with the following conditions:

(i) Isotropy: We postulate the transformation to be invariant to the spatial rotation. Rotating the frame about x-axis, through one right angle (Figure 1), we correspond the new axes to the initial axes:

$$\begin{aligned} &ct \to ct\,;\, ct' \to ct'\,;\, x \to x\,;\, x' \to x'\,;\, y \to -z\,; \\ &y' \to -z'\,;\, z \to y\,;\, z' \to y'. \end{aligned} \tag{3.1.6}$$

Thus, from (3.1.2), we have

$$ct' = bct + ax - kz + \nu y. \tag{3.1.7}$$

Comparing (3.1.2) and (3.1.7), it emerges $k = \nu = 0$. Besides, from (3.1.3) we have

$$x' = gct + fx - \delta x + \theta y. \tag{3.1.8}$$

Comparing (3.1.3) and (3.1.8), it emerges $\delta = \theta = 0$. Besides, from (3.1.4) we have

$$-z' = g_1 ct + f_1 x - hz + \lambda y. \tag{3.1.9}$$

Comparing (3.1.5) and (3.1.9), it emerges $g_2 = -g_1$, $f_2 = -f_1$, $\xi = -\lambda$ and $\mu = h$. Besides, from (3.1.5), we have

$$y' = g_2 ct + f_2 x - \xi z + \mu y. \tag{3.1.10}$$

Comparing (3.1.4) and (3.1.10), it emerges $g_2 = g_1$, $f_2 = f_1$, $\xi = -\lambda$ and $\mu = h$. So, $k = \nu = \delta = \theta = g_1 = g_2 = f_1 = f_2 = 0$; $\xi = -\lambda$; $\mu = h$ and the transformation becomes:

$$ct' = bct + ax \tag{3.1.11}$$

$$x' = gct + fx \tag{3.1.12}$$

$$y' = hy + \lambda z \tag{3.1.13}$$

$$z' = -\lambda y + hz. \tag{3.1.14}$$

(ii) The frame $O'x'y'z'$ is moving with velocity $(\beta c, 0, 0)$ wrt to $Oxyz$: for $x' = 0$, it is $x = \beta ct$. Replacing these to (3.1.12), we obtain

$$0 = gct + f\beta ct, \tag{3.1.15}$$

for any value of t. This emerges

$$g = -\beta f \tag{3.1.16}$$

and the transformation becomes:

$$ct' = bct + ax \tag{3.1.17}$$

$$x' = -\beta fct + fx \tag{3.1.18}$$

$$y' = hy + \lambda z \tag{3.1.19}$$

$$z' = -\lambda y + hz. \tag{3.1.20}$$

(iii) Maintenance of Lorentz length (spacetime interval) $|\vec{X}| = S : S'^2 = S^2$. Thus,

$$x'^2 + y'^2 + z'^2 - c^2t'^2 = x^2 + y^2 + z^2 - c^2t^2, \tag{3.1.21}$$

or equivalently,

$$\begin{aligned} &(-\beta cft + fx)^2 + (hy + \lambda z)^2 + (-\lambda y + hz)^2 - (bct + ax)^2 \\ &= x^2 + y^2 + z^2 - c^2t^2. \end{aligned} \tag{3.1.22}$$

From the terms $x^2; y^2; z^2; c^2t^2; ctx$, we obtain:

$$f^2 - a^2 = 1 \tag{3.1.23}$$

$$h^2 + \lambda^2 = 1 \tag{3.1.24}$$

$$\beta^2 f^2 - b^2 = -1 \tag{3.1.25}$$

$$-\beta f^2 - ab = 0. \tag{3.1.26}$$

Combining (3.1.26) with (3.1.23), we have

$$b = \frac{-\beta a^2 - \beta}{a}. \tag{3.1.27}$$

The Combination of (3.1.25) with (3.1.23) and (3.1.27) gives

$$a = \pm\frac{\beta}{\sqrt{1-\beta^2}} = \pm\beta\gamma. \tag{3.1.28}$$

Replacing the above to (3.1.27), we obtain

$$b = \mp\frac{1}{\sqrt{1-\beta^2}} = \mp\gamma. \tag{3.1.29}$$

Now, we must choose the sign in the above equations. We observe that for $\beta = 0$ the *upper sign* ($\uparrow$) gives $a = 0$ and $b = -1$. This transforms (3.1.11) to $t' = -t$, producing *time inversion*. The *lower sign* ($\downarrow$) gives $t' = t$, corresponding to the *proper transformation* and we have:

$$a = -\frac{\beta}{\sqrt{1-\beta^2}} = -\beta\gamma \quad ; \quad b = \frac{1}{\sqrt{1-\beta^2}} = \gamma. \tag{3.1.30}$$

The replacement of the above to (3.1.23) gives

$$f = \pm\frac{1}{\sqrt{1-\beta^2}} = \pm\gamma. \tag{3.1.31}$$

We also observe that for $\beta = 0$ the *upper sign* $(\uparrow)$ gives $f = 1$. This transforms (3.1.18) to $x' = x$. Thus, the *proper transformation* has

$$f = \frac{1}{\sqrt{1-\beta^2}} = \gamma. \tag{3.1.32}$$

On the other hand the *lower sign* $(\downarrow)$ for $\beta = 0$, gives $f = -1$ and (3.1.18) is transformed to $x' = -x$, producing *space inversion*. So, the *proper transformation* $(\uparrow\downarrow)$ becomes:

$$ct' = \gamma(ct - \beta x) \tag{3.1.33}$$

$$x' = \gamma(-\beta ct + x) \tag{3.1.34}$$

$$y' = hy + \lambda z \tag{3.1.35}$$

$$z' = -\lambda y + hz, \tag{3.1.36}$$

with condition (3.1.24). Using matrices, LT (2.5) has

$$\Lambda_{B(x)} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & h & \lambda \\ 0 & 0 & -\lambda & h \end{bmatrix}. \tag{3.1.37}$$

Besides, the differential form of the transformation is

$$cdt' = \gamma(cdt - \beta dx) \tag{3.1.38}$$

$$dx' = \gamma(-\beta cdt + dx) \tag{3.1.39}$$

$$dy' = hdy + \lambda dz \tag{3.1.40}$$

$$dz' = -\lambda dy + hdz. \tag{3.1.41}$$

Thus, the velocities are related as following:

$$\upsilon'_x = \frac{-\beta c + u_x}{c - \beta u_x} c,\ \upsilon'_y = \frac{hu_y + \lambda u_z}{\gamma(c - \beta u_x)} c,\ \upsilon'_z = \frac{-\lambda u_y + hu_z}{\gamma(c - \beta u_x)} c. \tag{3.1.42}$$

3.2 General linear transformation (motion in random direction)

We consider an unmoved observer (frame) $Oxyz$ and another observer (frame) $O'x'y'z'$ with parallel spatial axes, moving with velocity $(\upsilon_x, \upsilon_y, \upsilon_z)$ wrt the observer (frame) $Oxyz$ (Figure 2).

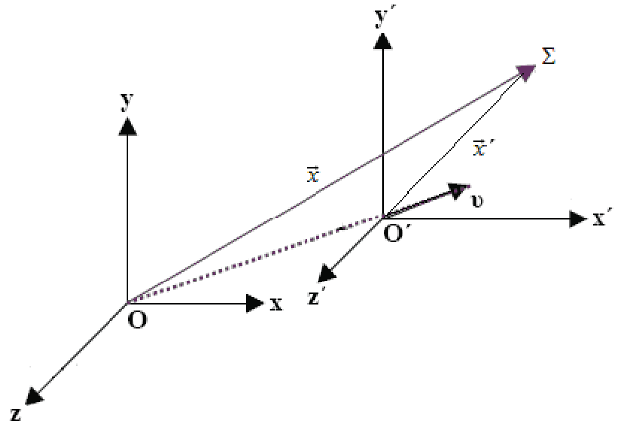

Figure 2. Two frames $Oxyz$ and $O'x'y'z'$, which initially coincide. The second is moving with random velocity $(\upsilon_x, \upsilon_y, \upsilon_z)$ wrt to $Oxyz$.

We rotate the initial frame $Oxyz$, in order to parallelize the unitary vector $\hat{x}$ to the velocity vector $\vec{\upsilon}$ of the observer $O'x'y'z'$. This is sequentially achieved as following: We firstly rotate the coordinate system $Oxyz$ about z-axis, through an angle θ $[O(\hat{x}, \hat{y}, \hat{z}) \;\rightarrow\; O(\hat{\imath}, \hat{\jmath}, \hat{k})]$. We then rotate the coordinate system $O(\hat{\imath}, \hat{\jmath}, \hat{k})$ about $\hat{\jmath}$ through an angle ω $[O(\hat{\imath}, \hat{\jmath}, \hat{k}) \;\rightarrow\; O(\hat{\imath}', \hat{\jmath}', \hat{k}')]$ (Figure 3). The corresponding matrices are:

$$(3.2.1) \qquad R_{xy(\theta)} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \;;\; R_{xz(\omega)} = \begin{bmatrix} \cos\omega & 0 & \sin\omega \\ 0 & 1 & 0 \\ -\sin\omega & 0 & \cos\omega \end{bmatrix}.$$

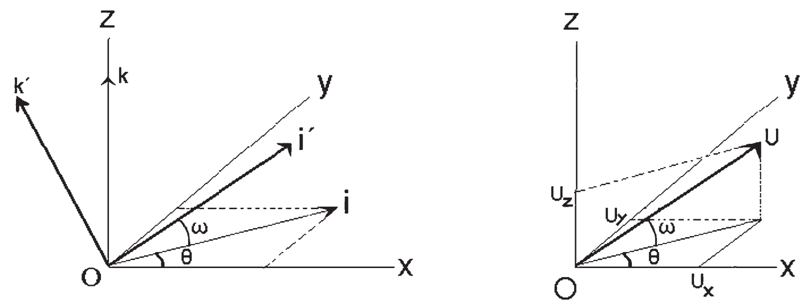

Figure 3. Rotation of the initial frame $Oxyz$, in order to achieve parallelization of vector $\hat{x}$ to the velocity vector $\vec{\upsilon}$ of the moving observer $O'x'y'z'$ $[O(\hat{x}, \hat{y}, \hat{z}) \;\rightarrow\; O(\hat{\imath}, \hat{\jmath}, \hat{k}) \;\rightarrow\; O(\hat{\imath}', \hat{\jmath}', \hat{k}')]$.

Thus, we have the transformation:

$$\begin{bmatrix} x_R \\ y_R \\ z_R \end{bmatrix} = \begin{bmatrix} \cos\omega\cos\theta & \cos\omega\sin\theta & \sin\omega \\ -\sin\theta & \cos\theta & 0 \\ -\sin\omega\cos\theta & -\sin\omega\sin\theta & \cos\omega \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \tag{3.2.2}$$

where (x_R, y_R, z_R) are the coordinates wrt the frame $O(\hat{i}', \hat{j}', \hat{k}')$ and

$$\sin\theta = \frac{\upsilon_y}{\sqrt{\upsilon_x^2 + \upsilon_y^2}} \quad ; \quad \cos\theta = \frac{\upsilon_x}{\sqrt{\upsilon_x^2 + \upsilon_y^2}}, \tag{3.2.3}$$

$$\sin\omega = \frac{\upsilon_z}{|\upsilon|} \quad ; \quad \cos\omega = \frac{\sqrt{\upsilon_x^2 + \upsilon_y^2}}{|\upsilon|}. \tag{3.2.4}$$

As a result, the above 3×3 matrix becomes

$$R = \begin{bmatrix} \frac{\beta_x}{|\beta|} & \frac{\beta_y}{|\beta|} & \frac{\beta_z}{|\beta|} \\ -\frac{\beta_y}{\sqrt{\beta_x^2+\beta_y^2}} & \frac{\beta_x}{\sqrt{\beta_x^2+\beta_y^2}} & 0 \\ -\frac{\beta_x\beta_z}{|\beta|\sqrt{\beta_x^2+\beta_y^2}} & -\frac{\beta_y\beta_z}{|\beta|\sqrt{\beta_x^2+\beta_y^2}} & \frac{\sqrt{\beta_x^2+\beta_y^2}}{|\beta|} \end{bmatrix} \tag{3.2.5}$$

and we define

$$\widetilde{R} = \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix}. \tag{3.2.6}$$

The unit means that time is not affected by the spatial rotation.

Moreover, the transformation $O(\hat{x}, \hat{y}, \hat{z}) \to O'(\hat{x}, \hat{y}, \hat{z})$ is analyzed to the following sequence of successive transformations:
$O(\hat{x}, \hat{y}, \hat{z}) \to O(\hat{i}', \hat{j}', \hat{k}'); O(\hat{i}', \hat{j}', \hat{k}') \to O'(\hat{i}', \hat{j}', \hat{k}'); O'(\hat{i}', \hat{j}', \hat{k}') \to O'(\hat{x}, \hat{y}, \hat{z})$.
The above simple transformations have active interpretations, respectively:

$$X_R = \widetilde{R}X \quad ; \quad X'_R = \Lambda_{B(x)} X_R \quad ; \quad X' = \widetilde{R}^T X'_R, \tag{3.2.7}$$

where $\widetilde{R}^T$ is the transpose matrix of $\widetilde{R}$.
Thus, the transformation $O(\hat{x}, \hat{y}, \hat{z}) \to O'(\hat{x}, \hat{y}, \hat{z})$ is actively interpreted:

$$X' = \widetilde{R}^T \Lambda_{B(x)(\beta)} \widetilde{R} X, \tag{3.2.8}$$

and LT (2.5) has

$$\Lambda_{B(\beta)} = \widetilde{R}^T \Lambda_{B(x)(\beta)} \widetilde{R}. \tag{3.2.9}$$

It is $\beta > 0$. So, $\beta = |\vec{\beta}|$ and we calculate:

$$\Lambda_{B(\beta)} = \widetilde{R}^T \begin{bmatrix} \gamma & -|\beta|\gamma & 0 & 0 \\ -|\beta|\gamma & \gamma & 0 & 0 \\ 0 & 0 & h & \lambda \\ 0 & 0 & -\lambda & h \end{bmatrix} \cdot$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\beta_x}{|\beta|} & \frac{\beta_y}{|\beta|} & \frac{\beta_z}{|\beta|} \\ 0 & -\frac{\beta_y}{\sqrt{\beta_x^2+\beta_y^2}} & \frac{\beta_x}{\sqrt{\beta_x^2+\beta_y^2}} & 0 \\ 0 & -\frac{\beta_x\beta_z}{|\beta|\sqrt{\beta_x^2+\beta_y^2}} & -\frac{\beta_y\beta_z}{|\beta|\sqrt{\beta_x^2+\beta_y^2}} & \frac{\sqrt{\beta_x^2+\beta_y^2}}{|\beta|} \end{bmatrix}, \tag{3.2.10}$$

or equivalently,

$$\Lambda_{B(\beta)} = \widetilde{R}^T \tag{3.2.11}$$

$$\cdot \begin{bmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma|\beta| & \gamma\frac{\beta_x}{|\beta|} & \gamma\frac{\beta_y}{|\beta|} & \gamma\frac{\beta_z}{|\beta|} \\ 0 & -\frac{\beta_y h}{\sqrt{\beta_x^2+\beta_y^2}} - \frac{\beta_x\beta_z\lambda}{|\beta|\sqrt{\beta_x^2+\beta_y^2}} & \frac{\beta_x h}{\sqrt{\beta_x^2+\beta_y^2}} - \frac{\beta_y\beta_z\lambda}{|\beta|\sqrt{\beta_x^2+\beta_y^2}} & \frac{\lambda\sqrt{\beta_x^2+\beta_y^2}}{|\beta|} \\ 0 & \frac{\beta_y\lambda}{\sqrt{\beta_x^2+\beta_y^2}} - \frac{\beta_x\beta_z h}{|\beta|\sqrt{\beta_x^2+\beta_y^2}} & -\frac{\beta_x\lambda}{\sqrt{\beta_x^2+\beta_y^2}} - \frac{\beta_y\beta_z h}{|\beta|\sqrt{\beta_x^2+\beta_y^2}} & \frac{h\sqrt{\beta_x^2+\beta_y^2}}{|\beta|} \end{bmatrix}.$$

Furthermore, we have

$$\Lambda_{B(\beta)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\beta_x}{|\beta|} & -\frac{\beta_y}{\sqrt{\beta_x^2+\beta_y^2}} & -\frac{\beta_x\beta_z}{|\beta|\sqrt{\beta_x^2+\beta_y^2}} \\ 0 & \frac{\beta_y}{|\beta|} & \frac{\beta_x}{\sqrt{\beta_x^2+\beta_y^2}} & -\frac{\beta_y\beta_z}{|\beta|\sqrt{\beta_x^2+\beta_y^2}} \\ 0 & \frac{\beta_z}{|\beta|} & 0 & \frac{\sqrt{\beta_x^2+\beta_y^2}}{|\beta|} \end{bmatrix} \cdot \tag{3.2.12}$$

$$\cdot \begin{bmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma|\beta| & \gamma\frac{\beta_x}{|\beta|} & \gamma\frac{\beta_y}{|\beta|} & \gamma\frac{\beta_z}{|\beta|} \\ 0 & -\frac{\beta_y h}{\sqrt{\beta_x^2+\beta_y^2}} - \frac{\beta_x\beta_z\lambda}{|\beta|\sqrt{\beta_x^2+\beta_y^2}} & \frac{\beta_x h}{\sqrt{\beta_x^2+\beta_y^2}} - \frac{\beta_y\beta_z\lambda}{|\beta|\sqrt{\beta_x^2+\beta_y^2}} & \frac{\lambda\sqrt{\beta_x^2+\beta_y^2}}{|\beta|} \\ 0 & \frac{\beta_y\lambda}{\sqrt{\beta_x^2+\beta_y^2}} - \frac{\beta_x\beta_z h}{|\beta|\sqrt{\beta_x^2+\beta_y^2}} & -\frac{\beta_x\lambda}{\sqrt{\beta_x^2+\beta_y^2}} - \frac{\beta_y\beta_z h}{|\beta|\sqrt{\beta_x^2+\beta_y^2}} & \frac{h\sqrt{\beta_x^2+\beta_y^2}}{|\beta|} \end{bmatrix}.$$

So, we obtain

$$\Lambda_{B(\beta)} = \begin{bmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & (\gamma-h)\frac{\beta_x^2}{|\beta|^2}+h & (\gamma-h)\frac{\beta_x\beta_y}{|\beta|^2}+\frac{\beta_z\lambda}{|\beta|} & (\gamma-h)\frac{\beta_x\beta_z}{|\beta|^2}-\frac{\beta_y\lambda}{|\beta|} \\ -\gamma\beta_y & (\gamma-h)\frac{\beta_x\beta_y}{|\beta|^2}-\frac{\beta_z\lambda}{|\beta|} & (\gamma-h)\frac{\beta_y^2}{|\beta|^2}+h & (\gamma-h)\frac{\beta_y\beta_z}{|\beta|^2}+\frac{\beta_x\lambda}{|\beta|} \\ -\gamma\beta_z & (\gamma-h)\frac{\beta_x\beta_z}{|\beta|^2}+\frac{\beta_y\lambda}{|\beta|} & (\gamma-h)\frac{\beta_y\beta_z}{|\beta|^2}-\frac{\beta_x\lambda}{|\beta|} & (\gamma-h)\frac{\beta_z^2}{|\beta|^2}+h \end{bmatrix} \tag{3.2.13}$$

and we have the transformation

(3.2.14)
$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & (\gamma-h)\frac{\beta_x^2}{|\beta|^2}+h & (\gamma-h)\frac{\beta_x\beta_y}{|\beta|^2}+\frac{\beta_z\lambda}{|\beta|} & (\gamma-h)\frac{\beta_x\beta_z}{|\beta|^2}-\frac{\beta_y\lambda}{|\beta|} \\ -\gamma\beta_y & (\gamma-h)\frac{\beta_x\beta_y}{|\beta|^2}-\frac{\beta_z\lambda}{|\beta|} & (\gamma-h)\frac{\beta_y^2}{|\beta|^2}+h & (\gamma-h)\frac{\beta_y\beta_z}{|\beta|^2}+\frac{\beta_x\lambda}{|\beta|} \\ -\gamma\beta_z & (\gamma-h)\frac{\beta_x\beta_z}{|\beta|^2}+\frac{\beta_y\lambda}{|\beta|} & (\gamma-h)\frac{\beta_y\beta_z}{|\beta|^2}-\frac{\beta_x\lambda}{|\beta|} & (\gamma-h)\frac{\beta_z^2}{|\beta|^2}+h \end{bmatrix} \cdot \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}.$$

3.3 The solution of proper closed isometric linear transformation (correlation of two perpendicular moving observers)

We consider one unmoved observer (frame $Oxyz$, another observer (frame $O'x'y'z'$ with parallel spatial axes, moving to the right, along x-axis with velocity $(\beta c, 0, 0)$ wrt $Oxyz$ and also a third observer (frame) $O''x''y''z''$ with parallel spatial axes, moving upward, along y-axis with velocity $(0, \beta c, 0)$ wrt $Oxyz$ (Figure 4).

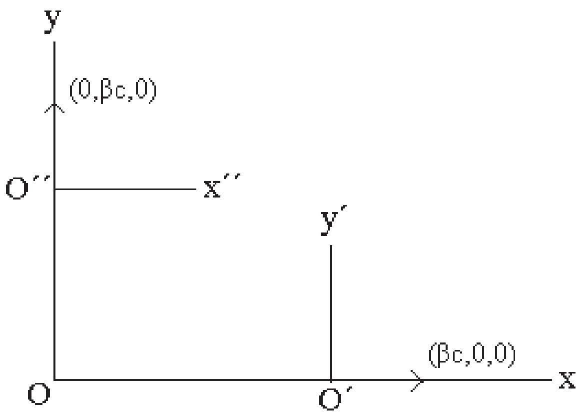

Figure 4. Two frames $O'x'y'z'$ and $O''x''y''z''$ moving with corresponding velocities $(\beta c, 0, 0)$ and $(0, \beta c, 0)$ wrt $Oxyz$.

Now, the transformation $O'(\hat{x}, \hat{y}, \hat{z}) \to O''(\hat{x}, \hat{y}, \hat{z})$ is analyzed to the following sequence: $O'(\hat{x}, \hat{y}, \hat{z}) \to O(\hat{x}, \hat{y}, \hat{z})$; $O(\hat{x}, \hat{y}, \hat{z}) \to O''(\hat{x}, \hat{y}, \hat{z})$. The above simple transformations have active interpretations, respectively:

(3.3.1) $$X = \Lambda^{-1}_{B(x)(\beta)} X' \quad ; \quad X'' = \Lambda_{B(y)(\beta)} X.$$

Thus, the transformation $O'(\hat{x}, \hat{y}, \hat{z}) \to O''(\hat{x}, \hat{y}, \hat{z})$ is actively interpreted:

(3.3.2) $$X'' = \Lambda_{B(y)(\beta)} \Lambda^{-1}_{B(x)(\beta)} X'$$

and LT (2.5) has

(3.3.3) $$\Lambda_3 = \Lambda_{B(y)(\beta)} \Lambda^{-1}_{B(x)(\beta)}.$$

According to (3.2.13), it is
(3.3.4)

$$\Lambda_{B(x)(\beta)} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & h & \lambda \\ 0 & 0 & -\lambda & h \end{bmatrix} ; \Lambda_{B(y)(\beta)} = \begin{bmatrix} \gamma & 0 & -\beta\gamma & 0 \\ 0 & h & 0 & -\lambda \\ -\beta\gamma & 0 & \gamma & 0 \\ 0 & \lambda & 0 & h \end{bmatrix}.$$

Besides, the inverse of matrix $\Lambda_{B(x)(\beta)}$ is

(3.3.5)
$$\Lambda^{-1}_{B(x)(\beta)} = \begin{bmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & h & -\lambda \\ 0 & 0 & \lambda & h \end{bmatrix}.$$

With that

(3.3.6)

$$\Lambda_3 = \begin{bmatrix} \gamma & 0 & -\beta\gamma & 0 \\ 0 & h & 0 & -\lambda \\ -\beta\gamma & 0 & \gamma & 0 \\ 0 & \lambda & 0 & h \end{bmatrix} \cdot \begin{bmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & h & -\lambda \\ 0 & 0 & \lambda & h \end{bmatrix} = \begin{bmatrix} \gamma^2 & \beta\gamma^2 & -\beta\gamma h & \beta\gamma\lambda \\ \beta\gamma h & \gamma h & -\lambda^2 & -h\lambda \\ -\beta\gamma^2 & -\beta^2\gamma^2 & \gamma h & -\gamma\lambda \\ \beta\gamma\lambda & \gamma\lambda & h\lambda & h^2 \end{bmatrix}.$$

Now, we calculate velocity factor $\vec{\beta}'_4$ of observer $O''x''y''z''$ wrt observer (frame) $O'x'y'x'$. Eqn (3.1.41) can be applied, because observer $O'x'y'z'$ is moving in x-direction and observer O'' can be considered as observed body:

(3.3.7)
$$\beta'_{4x} = \frac{-\beta c + 0}{c - 0} = -\beta$$

(3.3.8)
$$\beta'_{4y} = \frac{h\beta c + \lambda \cdot 0}{\gamma(c-0)} = \frac{h\beta}{\gamma}$$

(3.3.9)
$$\beta'_{4z} = \frac{-\lambda\beta c + 0}{\gamma(c-0)} = \frac{-\lambda\beta}{\gamma}.$$

As a result, it is

(3.3.10)
$$|\vec{\beta}'_4|^2 = \beta^2 + \beta^2 h^2 \frac{1}{\gamma^2} + \frac{\beta^2\lambda^2}{\gamma^2} = \frac{\beta^2\gamma^2 + \beta^2 h^2 + \beta^2\lambda^2}{\gamma^2},$$

or equivalently,

(3.3.11)
$$|\vec{\beta}'_4|^2 = \beta^2 \frac{\gamma^2 + h^2 + \lambda^2}{\gamma^2} = \beta^2 \frac{\gamma^2 + 1}{\gamma^2}.$$

Thus, it emerges

$$|\vec{\beta}'_4| = \beta\frac{\sqrt{\gamma^2+1}}{\gamma} \quad ; \quad \gamma_{(\vec{\beta}'_4)} = \gamma^2. \tag{3.3.12}$$

According to (3.2.13), the matrix corresponding to the velocity factor $\vec{\beta}'_4$ is

$$\Lambda_4 = \Lambda_{B(\vec{\beta}'_4)} = \begin{bmatrix} \gamma^2 & \beta\gamma^2 & -\beta\gamma h & \beta\gamma\lambda \\ \beta\gamma^2 & \cdot & \cdot & \cdot \\ -\beta\gamma h & \cdot & \cdot & \cdot \\ \beta\gamma\lambda & \cdot & \cdot & \cdot \end{bmatrix} \tag{3.3.13}$$

We postulate the transformation to be closed:

$$\Lambda_3 = \Lambda_{B(y)(\beta)}\Lambda^{-1}_{B(x)(\beta)} = \Lambda_{(\vec{\beta}'_4)} = \Lambda_4. \tag{3.3.14}$$

Comparing the matrices, element by element, we have:

$$(\Lambda_3)_{10} = (\Lambda_4)_{10}, \tag{3.3.15}$$

or equivalently,

$$h = \gamma. \tag{3.3.16}$$

Applying the foregoing equation in (3.1.24), we obtain

$$\lambda^2 = -\beta^2\gamma^2 \ ; \ \lambda = i\beta\gamma = i|\vec{\beta}|\gamma. \tag{3.3.17}$$

Thus, (3.3.16), (3.3.12ii), (3.3.17ii) and (3.3.12i) emerge

$$h_{(\vec{\beta}'_4)} = \gamma_{(\beta'_4)} = \gamma^2 \tag{3.3.18}$$

and

$$\lambda_{(\vec{\beta}'_4)} = i|\vec{\beta}'_4|\gamma_{(\vec{\beta}'_4)} = i\beta\frac{\sqrt{\gamma^2+1}}{\gamma}\gamma^2 = i\beta\gamma\sqrt{\gamma^2+1} = \lambda\sqrt{\gamma^2+1}. \tag{3.3.19}$$

Thus, the matrix (3.2.13) for the velocity factor $\vec{\beta}'_4$ [see also (3.3.13)] is written:

$$\Lambda_4 = \begin{bmatrix} \gamma^2 & \beta\gamma^2 & -\beta\gamma^2 & \beta\gamma\lambda \\ \beta\gamma^2 & h_{(\vec{\beta}'_4)} & \frac{\beta'_{4z}\lambda_{(\vec{\beta}'_4)}}{|\vec{\beta}'_4|} & -\frac{\beta'_{4y}\lambda_{(\vec{\beta}'_4)}}{|\vec{\beta}'_4|} \\ -\beta\gamma^2 & -\frac{\beta'_{4z}\lambda_{(\vec{\beta}'_4)}}{|\vec{\beta}'_4|} & h_{(\vec{\beta}'_4)} & +\frac{\beta'_{4x}\lambda_{(\vec{\beta}'_4)}}{|\vec{\beta}'_4|} \\ \beta\gamma\lambda & \frac{\beta'_{4y}\lambda_{(\vec{\beta}'_4)}}{|\vec{\beta}'_4|} & -\frac{\beta'_{4x}\lambda_{(\vec{\beta}'_4)}}{|\vec{\beta}'_4|} & h_{(\vec{\beta}'_4)} \end{bmatrix}. \tag{3.3.20}$$

Replacing only (3.3.16) to (3.3.8), we rewrite the velocity factor components

$$\beta'_{4x} = -\beta \quad ; \quad \beta'_{4y} = \beta \quad ; \quad \beta'_{4z} = -\frac{\lambda\beta}{\gamma}. \tag{3.3.21}$$

Now, we calculate the following quotients contained in matrix Λ_4, by using the above and also (3.3.12):

$$\frac{\beta'_{4x}}{|\vec{\beta}'_4|} = -\frac{\gamma}{\sqrt{\gamma^2+1}}\ ;\ \frac{\beta'_{4y}}{|\vec{\beta}'_4|} = \frac{\gamma}{\sqrt{\gamma^2+1}}\ ;\ \frac{\beta'_{4z}}{|\vec{\beta}'_4|} = -\frac{\lambda}{\sqrt{\gamma^2+1}}. \tag{3.3.22}$$

The replacement of the above, (3.3.18) and (3.3.19) to (3.3.20) gives

$$\Lambda_4 = \begin{bmatrix} \gamma^2 & \beta\gamma^2 & -\beta\gamma^2 & \beta\gamma\lambda \\ \beta\gamma^2 & \gamma^2 & -\lambda^2 & -\gamma\lambda \\ -\beta\gamma^2 & \lambda^2 & \gamma^2 & -\gamma\lambda \\ \beta\gamma\lambda & \gamma\lambda & \gamma\lambda & \gamma^2 \end{bmatrix}, \tag{3.3.23}$$

while from (3.3.6) it is

$$\Lambda_3 = \begin{bmatrix} \gamma^2 & \beta\gamma^2 & -\beta\gamma h & \beta\gamma\lambda \\ \beta\gamma h & \gamma h & -\lambda^2 & -h\lambda \\ -\beta\gamma^2 & -\beta^2\gamma^2 & \gamma h & -\gamma\lambda \\ \beta\gamma\lambda & \gamma\lambda & h\lambda & h^2 \end{bmatrix}. \tag{3.3.24}$$

We validate the equation of the matrices: $\Lambda_3 = \Lambda_4$, because of (3.3.16) and (3.3.17i).

Finally, we replace (3.3.16) and (3.3.17ii) to (3.2.14) and (3.2.13) and we obtain the *proper closed isometric LT*:

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \gamma_{(\beta)} \begin{bmatrix} 1 & -\beta_x & -\beta_y & -\beta_z \\ -\beta_x & 1 & i\beta_z & -i\beta_y \\ -\beta_y & -i\beta_z & 1 & i\beta_x \\ -\beta_z & i\beta_y & -i\beta_x & 1 \end{bmatrix} \cdot \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} \tag{3.3.25}$$

and the corresponding matrix

$$\Lambda_{B(\beta)} = \gamma_{(\beta)} \begin{bmatrix} 1 & -\beta_x & -\beta_y & -\beta_z \\ -\beta_x & 1 & i\beta_z & -i\beta_y \\ -\beta_y & -i\beta_z & 1 & i\beta_x \\ -\beta_z & i\beta_y & -i\beta_x & 1 \end{bmatrix}. \tag{3.3.26}$$

We have preferred the physical approach (spacetime) for the derivation of the *proper isometric LT* in M^4, because SR is the main application [4]. The pure mathematical approach is simply obtained, by replacing $ct \to x^0$, according to (2.11):

$$\begin{bmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{bmatrix} = \gamma_{(\beta)} \begin{bmatrix} 1 & -\beta^1 & -\beta^2 & -\beta^3 \\ -\beta^1 & 1 & i\beta^3 & -i\beta^2 \\ -\beta^2 & -i\beta^3 & 1 & i\beta^1 \\ -\beta^3 & i\beta^2 & -i\beta^1 & 1 \end{bmatrix} \cdot \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \tag{3.3.27}$$

and the corresponding matrix is
(3.3.28)
$$\Lambda_{B(\beta)} = \gamma_{(\beta)} \begin{bmatrix} 1 & -\beta^1 & -\beta^2 & -\beta^3 \\ -\beta^1 & 1 & i\beta^3 & -i\beta^2 \\ -\beta^2 & -i\beta^3 & 1 & i\beta^1 \\ -\beta^3 & i\beta^2 & -i\beta^1 & 1 \end{bmatrix} = \gamma_{(\beta)} \begin{bmatrix} 1 & -\beta^T \\ -\beta & I_3 + iA_{(\beta)} \end{bmatrix},$$
where
$$\beta = \begin{bmatrix} \beta^1 \\ \beta^2 \\ \beta^3 \end{bmatrix} \quad ; \quad A_{(\beta)} = \begin{bmatrix} 0 & \beta^3 & -\beta^2 \\ -\beta^3 & 0 & \beta^1 \\ \beta^2 & -\beta^1 & 0 \end{bmatrix}. \tag{3.3.29}$$
The matrix (Λ_B) of the *proper isometric LT* has determinant equal to the unit ($\det \Lambda_B = 1$). Besides, the *typical transformation* along x-axis, has
$$\Lambda_{B(x)(\beta)} = \gamma_{(\beta)} \begin{bmatrix} 1 & -\beta & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & i\beta \\ 0 & 0 & -i\beta & 1 \end{bmatrix} \tag{3.3.30}$$

It is noted that antisymmetric matrix $A_{(\beta)}$ is related to the *cross product (external product)* because:
$$\begin{aligned} A_{(\vec{e}_1,\vec{e}_2,\vec{e}_3)} = [\vec{e}_i \times \vec{e}_j] &= \begin{bmatrix} \vec{e}_1 \times \vec{e}_1 & \vec{e}_1 \times \vec{e}_2 & \vec{e}_1 \times \vec{e}_3 \\ \vec{e}_2 \times \vec{e}_1 & \vec{e}_2 \times \vec{e}_2 & \vec{e}_2 \times \vec{e}_3 \\ \vec{e}_3 \times \vec{e}_1 & \vec{e}_3 \times \vec{e}_2 & \vec{e}_3 \times \vec{e}_3 \end{bmatrix} \\ &= \begin{bmatrix} \vec{0} & \vec{e}_3 & -\vec{e}_2 \\ -\vec{e}_3 & \vec{0} & \vec{e}_1 \\ \vec{e}_2 & -\vec{e}_1 & \vec{0} \end{bmatrix}. \end{aligned} \tag{3.3.31}$$
So,
$$\begin{aligned} \vec{x} \times \vec{y} &= (x^1\vec{e}_1 + x^2\vec{e}_2 + x^3\vec{e}_3) \times (y^1\vec{e}_1 + y^2\vec{e}_2 + y^3\vec{e}_3) \\ &= [\vec{e}_1\ \vec{e}_2\ \vec{e}_3] \cdot \begin{bmatrix} 0 & -x^3 & x^2 \\ x^3 & 0 & -x^1 \\ -x^2 & x^1 & 0 \end{bmatrix} \cdot \begin{bmatrix} y^1 \\ y^2 \\ y^3 \end{bmatrix}, \end{aligned} \tag{3.3.32}$$
written in compact form:
$$\vec{x} \times \vec{y} = [\vec{e}_i] \cdot \lfloor A_{(-\vec{x})^i{}_j} \rfloor \cdot [y^j] = -[\vec{e}_i] \cdot \lfloor A_{(\vec{x})^i{}_j} \rfloor \cdot [y^j]. \tag{3.3.33}$$

On the other hand, the proper isometric transformation in E^4 is obtained as following: we initially divide (3.3.27) with i
$$\begin{bmatrix} \frac{x'^0}{i} \\ \frac{x'^1}{i} \\ \frac{x'^2}{i} \\ \frac{x'^3}{i} \end{bmatrix} = \gamma_{(\beta)} \begin{bmatrix} 1 & -\beta^1 & -\beta^2 & -\beta^3 \\ -\beta^1 & 1 & i\beta^3 & -i\beta^2 \\ -\beta^2 & -i\beta^3 & 1 & i\beta^1 \\ -\beta^3 & i\beta^2 & -i\beta^1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{x^0}{i} \\ \frac{x^1}{i} \\ \frac{x^2}{i} \\ \frac{x^3}{i} \end{bmatrix}. \tag{3.3.34}$$

This is equivalent to

$$\begin{bmatrix} \frac{x'^0}{i} \\ x'^1 \\ x'^2 \\ x'^3 \end{bmatrix} = \gamma_{(\beta)} \begin{bmatrix} 1 & i\beta^1 & i\beta^2 & i\beta^3 \\ -i\beta^1 & 1 & i\beta^3 & -i\beta^2 \\ -i\beta^2 & -i\beta^3 & 1 & i\beta^1 \\ -i\beta^3 & i\beta^2 & -i\beta^1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{x^0}{i} \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}. \tag{3.3.35}$$

The above is written by using (2.4):

$$\begin{bmatrix} X'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{bmatrix} = \gamma_{(\beta)} \begin{bmatrix} 1 & i\beta^1 & i\beta^2 & i\beta^3 \\ -i\beta^1 & 1 & i\beta^3 & -i\beta^2 \\ -i\beta^2 & -i\beta^3 & 1 & i\beta^1 \\ -i\beta^3 & i\beta^2 & -i\beta^1 & 1 \end{bmatrix} \cdot \begin{bmatrix} X^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix}. \tag{3.3.36}$$

So, the corresponding matrix in E^4 is

$$\begin{aligned} \Lambda^E_{B(\beta)} &= \gamma_{(\beta)} \begin{bmatrix} 1 & i\beta^1 & i\beta^2 & i\beta^3 \\ -i\beta^1 & 1 & i\beta^3 & -i\beta^2 \\ -i\beta^2 & -i\beta^3 & 1 & i\beta^1 \\ -i\beta^3 & i\beta^2 & -i\beta^1 & 1 \end{bmatrix} \\ &= \gamma_{(\beta)} \begin{bmatrix} 1 & i\beta^T \\ -i\beta & I_3 + iA_{(\beta)} \end{bmatrix}. \end{aligned} \tag{3.3.37}$$

This is rotation matrix, because it is orthogonal (unitary) matrix with determinant equal to the unit ($\det \Lambda^E_B = 1$). Besides, the *typical transformation* along x-axis, has

$$\Lambda^E_{B(x)(\beta)} = \gamma_{(\beta)} \begin{bmatrix} 1 & i\beta & 0 & 0 \\ -i\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & i\beta \\ 0 & 0 & -i\beta & 1 \end{bmatrix} \tag{3.3.38}$$

4. Improper isometric linear transformations in four-dimensional space endowed with Euclidean or Lorentz metric

In the derivation of proper closed isometric Linear Transformation (↓↑), we have chosen the *lower sign* (↓) in (3.1.28) and (3.1.29), but the *upper* (↑) in (3.1.31). So, they have remained three (3) improper non-closed isometric Linear Transformations (which does not contain the *identity transformation*):

(i) *Space inversion non-closed isometric Linear Transformation* (↓↓) [*lower sign* (↓) in (3.1.28) and (3.1.29) as well as *lower sign* (↓) in (3.1.31)] in M^4 and E^4 with corresponding matrices ($\det \Lambda_B = \det \Lambda^E_B = -1$):

$$\Lambda_{B(\beta)} = \gamma_{(\beta)} \begin{bmatrix} 1 & -\beta^T \\ \beta & -I_3 - iA_{(\beta)} \end{bmatrix} \; ; \; \Lambda^E_{B(\beta)} = \gamma_{(\beta)} \begin{bmatrix} 1 & i\beta^T \\ i\beta & -I_3 - iA_{(\beta)} \end{bmatrix}. \tag{4.1}$$

The respective *typical transformations* along x-axis, have

(4.2)

$$\Lambda_{B(x)(\beta)}=\gamma_{(\beta)}\begin{bmatrix}1 & -\beta & 0 & 0\\ \beta & -1 & 0 & 0\\ 0 & 0 & -1 & -i\beta\\ 0 & 0 & i\beta & -1\end{bmatrix};\Lambda^E_{B(\beta)}=\gamma_{(\beta)}\begin{bmatrix}1 & i\beta & 0 & 0\\ i\beta & -1 & 0 & 0\\ 0 & 0 & -1 & -i\beta\\ 0 & 0 & i\beta & -1\end{bmatrix}.$$

(ii) *Time inversion non-closed isometric Linear Transformation* (↑↑) [*upper sign* (↑) in (3.1.28) and (3.1.29) as well as *upper sign* (↑) in (3.1.31)] in M^4 and E^4 with corresponding matrices ($\det \Lambda_B = \det \Lambda_B^E = -1$):

$$\Lambda_{B(\beta)} = \gamma_{(\beta)}\begin{bmatrix}-1 & \beta^T\\ -\beta & I_3 + iA_{(\beta)}\end{bmatrix} \quad ; \quad \Lambda^E_{B(\beta)} = \gamma_{(\beta)}\begin{bmatrix}-1 & -i\beta^T\\ -i\beta & I_3 + iA_{(\beta)}\end{bmatrix}. \tag{4.3}$$

The respective *typical transformations* along x-axis, have

(4.4)

$$\Lambda_{B(x)(\beta)}=\gamma_{(\beta)}\begin{bmatrix}-1 & \beta & 0 & 0\\ -\beta & 1 & 0 & 0\\ 0 & 0 & 1 & i\beta\\ 0 & 0 & -i\beta & 1\end{bmatrix};\Lambda^E_{B(\beta)}=\gamma_{(\beta)}\begin{bmatrix}-1 & -i\beta & 0 & 0\\ -i\beta & 1 & 0 & 0\\ 0 & 0 & 1 & i\beta\\ 0 & 0 & -i\beta & 1\end{bmatrix}.$$

(iii) *Spacetime inversion closed isometric Linear Transformation* (↑↓) [*upper sign* (↑) in (3.1.28) and (3.1.29), but *lower sign* (↓) in (3.1.31)] in M^4 and E^4 with corresponding matrices ($\det \Lambda_B = \det \Lambda_B^E = 1$):

(4.5)

$$\Lambda_{B(\beta)} = \gamma_{(\beta)}\begin{bmatrix}-1 & \beta^T\\ \beta & -I_3 - iA_{(\beta)}\end{bmatrix} \quad ; \quad \Lambda^E_{B(\beta)} = \gamma_{(\beta)}\begin{bmatrix}-1 & -i\beta^T\\ i\beta & -I_3 - iA_{(\beta)}\end{bmatrix}.$$

The respective *typical transformations* along x-axis, have

(4.6)

$$\Lambda_{B(x)(\beta)}=\gamma_{(\beta)}\begin{bmatrix}-1 & \beta & 0 & 0\\ \beta & -1 & 0 & 0\\ 0 & 0 & -1 & -i\beta\\ 0 & 0 & i\beta & 1\end{bmatrix};\Lambda^E_{B(\beta)}=\gamma_{(\beta)}\begin{bmatrix}-1 & -i\beta & 0 & 0\\ i\beta & -1 & 0 & 0\\ 0 & 0 & -1 & -i\beta\\ 0 & 0 & i\beta & -1\end{bmatrix}.$$

These matrices are exactly the opposite of the corresponding *proper closed isometric LT*.

In case of *Lorentz Boost*, we have [2] (pp. 30-31):

(i) *Space inversion Lorentz Boost* in M^4 and E^4 with corresponding matrices ($\det \Lambda_L = \det \Lambda_L^E = -1$):

(4.7)

$$\Lambda_{L(\beta)}=\begin{bmatrix}\gamma_{(\beta)} & \gamma_{(\beta)}\beta^T\\ -\gamma_{(\beta)}\beta & -I_3 - \frac{\gamma_{(\beta)}-1}{\beta^T\beta}\beta\beta^T\end{bmatrix};\Lambda^E_{L(\beta)}=\begin{bmatrix}\gamma_{(\beta)} & -i\gamma_{(\beta)}\beta^T\\ -i\gamma_{(\beta)}\beta & -I_3 - \frac{\gamma_{(\beta)}-1}{\beta^T\beta}\beta\beta^T\end{bmatrix}.$$

The respective *typical transformations* along x-axis, have

$$\Lambda_{L(x)(\beta)}=\begin{bmatrix} \gamma_{(\beta)} & \gamma_{(\beta)}\beta & 0 & 0 \\ -\gamma_{(\beta)}\beta & -\gamma_{(\beta)} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix};$$

$$\text{Ł}^E_{L(x)(\beta)}=\begin{bmatrix} \gamma_{(\beta)} & -i\gamma_{(\beta)}\beta & 0 & 0 \\ -i\gamma_{(\beta)}\beta & -\gamma_{(\beta)} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{4.8}$$

(ii) *Time inversion Lorentz Boost* in M^4 and E^4 with corresponding matrices ($\det \Lambda_L = \det \Lambda_L^E = 1$):

$$\Lambda_{L(\beta)}=\begin{bmatrix} -\gamma_{(\beta)} & \gamma_{(\beta)}\beta^T \\ \gamma_{(\beta)}\beta & I_3-\frac{\gamma_{(\beta)}+1}{\beta^T\beta}\beta\beta^T \end{bmatrix};$$

$$\Lambda^E_{L(\beta)}=\begin{bmatrix} -\gamma_{(\beta)} & -i\gamma_{(\beta)}\beta^T \\ i\gamma_{(\beta)}\beta & I_3-\frac{\gamma_{(\beta)}+1}{\beta^T\beta}\beta\beta^T \end{bmatrix}. \tag{4.9}$$

The respective *typical transformations* along x-axis, have

$$\Lambda_{L(x)(\beta)}=\begin{bmatrix} -\gamma_{(\beta)} & \gamma_{(\beta)}\beta & 0 & 0 \\ \gamma_{(\beta)}\beta & -\gamma_{(\beta)} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

$$L^E_{L(x)(\beta)}=\begin{bmatrix} -\gamma_{(\beta)} & -i\gamma_{(\beta)}\beta & 0 & 0 \\ i\gamma_{(\beta)}\beta & -\gamma_{(\beta)} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.10}$$

(iii) *Spacetime inversion Lorentz Boost* in M^4 and E^4 with corresponding matrices ($\det \Lambda_L = \det \Lambda_L^E = -1$):

$$\Lambda_{L(\beta)}=\begin{bmatrix} -\gamma_{(\beta)} & -\gamma_{(\beta)}\beta^T \\ \gamma_{(\beta)}\beta & -I_3+\frac{\gamma_{(\beta)}+1}{\beta^T\beta}\beta\beta^T \end{bmatrix};$$

$$\Lambda^E_{L(\beta)}=\begin{bmatrix} -\gamma_{(\beta)} & i\gamma_{(\beta)}\beta^T \\ i\gamma_{(\beta)}\beta & -I_3+\frac{\gamma_{(\beta)}+1}{\beta^T\beta}\beta\beta^T \end{bmatrix}. \tag{4.11}$$

The respective *typical transformations* along x-axis, have

$$\varLambda_{L(x)(\beta)} = \begin{bmatrix} -\gamma_{(\beta)} & -\gamma_{(\beta)}\beta & 0 & 0 \\ \gamma_{(\beta)}\beta & \gamma_{(\beta)} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix};$$

$$\text{\L}^{E}_{L(x)(\beta)} = \begin{bmatrix} -\gamma_{(\beta)} & i\gamma_{(\beta)}\beta & 0 & 0 \\ i\gamma_{(\beta)}\beta & \gamma_{(\beta)} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{4.12}$$

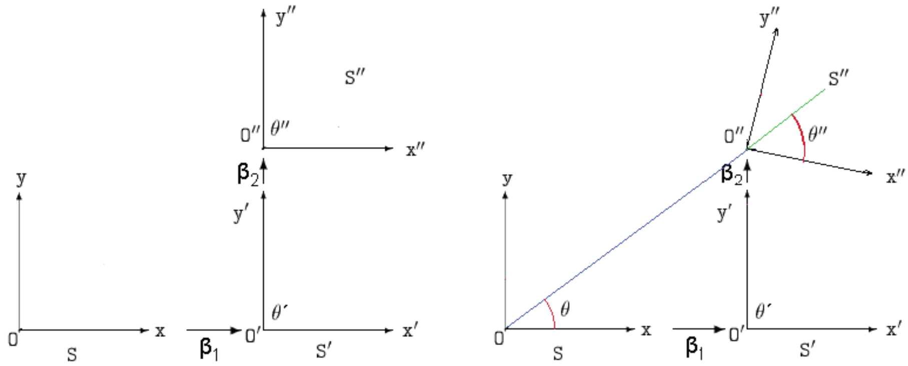

Figure 5. Correlation of three successive observers (frames), by using Lorentz Boost. The frame $O'x'y'z'$ has parallel axes to the corresponding of frame $Oxyz$, moving with velocity $(\beta_1 c, 0, 0)$ wrt $Oxyz$. The frame $O''x''y''z''$ has parallel axes to the corresponding of frame $O'x'y'z'$, moving with velocity $(0, \beta_2 c, 0)$ wrt $O'x'y'z'$. The correlation of the observers, by using Lorentz Boost, cancels the absolute character of parallelism. Thus, the axes of frame $O''x''y''z''$ are not parallel to the corresponding of frame $Oxyz$ (Thomas Rotation).

5. Conclusions

The closed isometric linear transformation which maintains *Lorentz length* (S^2) is represented by a matrix ($\varLambda_B$) containing real and imaginary numbers. Under this transformation, the real spacetime of the initial rest observer (frame) is transformed to real time and complex space for one moving observer (frame) ($R^4 \rightarrow R \times C^3$). Subsequently, the axes rotation (*Thomas Rotation* [5]) that happens from the correlation of three observers related by using *Lorentz Boost* (Figure 5) [2] (pp. 177-183), in this approach is avoided. Thus, the validation of the transitive attribute in parallelism of unmoved straight lines (which is equivalent to the 5$^{\text{th}}$ Euclidean postulate), is extended to the case moving straight lines (for any observer). This is achieved, by working in the domain of complex numbers, validating one more time, the words of J. Hadamard: "*It has been* *written that the shortest and best way between two truths of the real domain often passes through the imaginary one*" [6] (p. 123).

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Accepted: 03.04.2020