Closed linear transformations of complex space-time endowed with Euclidean or Lorentz metric

Elias Vossos  
Spyridon Vossos  
Christos G. Massouros  
National and Kapodistrian University of Athens  
Core department  
Euripus Campus, GR 34400  
Psahna, Euboia  
Greece  
evossos@uoa.gr  
svososs@uoa.gr  
ChrMas@uoa.gr

Abstract. Linear transformations (LTs) are essential for the development of Relativity Theory. Special Relativity is based on Lorentz Boost (LB). This cancels the transitive attribute in parallelism (which is equivalent to the 5th Euclidean Postulate), when three observers are related (successive transformations), because LB is not closed LT. So, LB is combined with Euclidean spatial rotation, in order to obtain Lorentz transformation (which is closed LT) and the corresponding Lorentz group. In this paper, a new closed isometric LT in spaces ($V^4$) of dimension four ($n = 4$), with Euclidean or Lorentz metric (Minkowski Space), is presented (correlating frames with parallel spatial axes). This LT is represented by a matrix ($A_L$) containing real and imaginary numbers. Thus, $V^4$ is based on the field of complex numbers ($C$), by using real 0-(temporal) and complex 1, 2, 3- (spatial) coordinates. 

Keywords: 5th Euclidean postulate, Euclidean metric, Euclidean space, linear transformation, spacetime, Lorentz Boost, Lorentz Transformation, Minkowski space, special relativity, Thomas rotation.

Abbreviations and annotations

$E^3$ ▶ three-dimensional Euclidean  
$E^4$ ▶ four-dimensional Euclidean Space  
$M^4$ ▶ Minkowski Space  
$LB$ ▶ Lorentz Boost  
$LT$ ▶ Linear Transformation  
$RT$ ▶ Relativity Theory  
$SR$ ▶ Special Relativity  
$V^4$ ▶ Four-dimensional Space

1. Introduction

Linear transformations (LTs) are essential for the development of Relativity Theory (RT), Quantum Mechanics (QMs) and generally in Modern Physics. In...
Special Relativity (SR), Hermann Minkowski combined [1] (pp. 39-53) the one-dimensional time (T) with the Three-dimensional Euclidean space \((E^3)\) endowed with Euclidean metric [2] (p. 14):

\[
g_{E^3} = \text{diag}(1,1,1) \tag{1.1}
\]

and he produced a four-dimensional space \(M^4\) (since known as Minkowski spacetime), endowed with Lorentz metric [2] (p. 8):

\[
g_L = \text{diag}(-1,1,1,1). \tag{1.2}
\]

In this space, the position four-vector is written as

\[
\vec{X} = x^0 \vec{e}_0 + x^1 \vec{e}_1 + x^2 \vec{e}_2 + x^3 \vec{e}_3 = \vec{e}_\mu x^\mu, \tag{1.3}
\]

where

\[
[\vec{e}_\mu] = [\vec{e}_0 \vec{e}_1 \vec{e}_2 \vec{e}_3]; \quad X = \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \tag{1.4}
\]

are the basis of \(M^4\) and the coordinates of the position four-vector, respectively. The Einstein’s summation convention [2] (p. 3) has been used in (1.3ii) and the following equations. Besides the Lorentz length \(|\vec{X}|\) of the position four-vector is defined [2] (p. 17):

\[
|\vec{X}|^2 = x^\mu g_L{}_{\mu\nu} x^\nu = -(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2. \tag{1.5}
\]

Correspondingly in \(E^3\), we have

\[
\vec{X} = x \vec{e}_1 + y \vec{e}_2 + z \vec{e}_3 = x^1 \vec{e}_1 + x^2 \vec{e}_2 + x^3 \vec{e}_3 = \vec{e}_i x^i. \tag{1.6}
\]

The common choice is the usage of real coordinates

\[
x^1, x^2, x^3 \in R, \tag{1.7}
\]

in order to be easily perceived by human senses.

We shall see that this field is not enough, in case that we wish to produce closed isometric LTs in Four-dimensional Spaces \((V^4)\). So, we prefer complex coordinates

\[
x^1, x^2, x^3 \in C. \tag{1.8}
\]

For simplicity reasons, wherever we write \(i\) (the imaginary unit), we mean \(\pm i\):

\[
i \to \pm i; \quad -i \to \mp i. \tag{1.9}
\]
Besides, $V^4$ endowed with *Euclidean metric* [2] (p. 8):

(1.10) \[ g_{E^4} = \text{diag}(1, 1, 1, 1), \]

is called *Euclidean Four-dimensional Space* ($E^4$). In this space, the position four-vector is written as

(1.11) \[ \vec{X} = X^0 \vec{E}_0 + x^1 \vec{e}_1 + x^2 \vec{e}_2 + x^3 \vec{e}_3 = \vec{e}_\mu x^\mu, \]

where

(1.12) \[ \begin{bmatrix} \vec{e}_\mu \end{bmatrix} = \begin{bmatrix} \vec{E}_0 \end{bmatrix} \begin{bmatrix} 1 \vec{e}_1 \vec{e}_2 \vec{e}_3 \end{bmatrix} ; \quad X = \begin{bmatrix} X^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \]

are the basis of $E^4$ and the coordinates of the position four-vector, respectively. Moreover, the *Euclidean length* of the position four-vector is

(1.13) \[ |\vec{X}|^2 = x_\mu g_{E^4 \mu \nu} x^\nu = (X^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2. \]

### 2. Connection between spaces endowed with Euclidean and Lorentz metric and their transformations

From (1.2) and (1.10), we respectively have

(2.1) \[ \vec{e}_0 \cdot \vec{e}_0 = -1 ; \quad \vec{E}_0 \cdot \vec{E}_0 = 1, \]

where dot “.” is *Euclidean inner product* [2] (p. 7). Thus, we understand that

(2.2) \[ \vec{E}_0 = i \vec{e}_0. \]

After replacing the above to (1.11), we have

(2.3) \[ \vec{X} = i X^0 \vec{e}_0 + x^1 \vec{e}_1 + x^2 \vec{e}_2 + x^3 \vec{e}_3 = \vec{e}_\mu x^\mu. \]

Comparing this to (1.3), we obtain

(2.4) \[ X^0 = \frac{1}{i} x^0. \]

The above procedure shows that the difference between *Euclidean* and *Lorentz metric* is caused by the different 0-four-vector of the used basis: $\vec{E}_0$ and $\vec{e}_0$, respectively. So, $E^4$ and $M^4$ are related via (2.2) and (2.4).

The corresponding LTs are also easily related. For instance, the *active interpretation* of LT [2] (p. 6) is

(2.5) \[ X' = \Lambda_{(\beta)} X, \]
where
\[
\beta = \begin{bmatrix}
\beta^1 \\
\beta^2 \\
\beta^3
\end{bmatrix}; \quad \beta^i = \frac{dx^i}{dx^0}; \quad i = 1, 2, 3,
\]
is called \(\beta\)-factor. Eqn (2.5) expresses proper Lorentz Boost (LB) in \(M^4\) \([2]\) (p. 30) and \(E^4\), correspondingly:
\[
A_{L(\beta)} = \begin{bmatrix}
\gamma(\beta) & -\gamma(\beta)\beta^T \\
-\gamma(\beta)\beta & I_3 + \frac{\gamma(\beta)-1}{\beta^T\beta}\beta\beta^T
\end{bmatrix}; \quad X = \begin{bmatrix}
x^0 \\
x^1 \\
x^2 \\
x^3
\end{bmatrix},
\]
\[
A_{E(\beta)}^L = \begin{bmatrix}
\gamma(\beta) & -i\gamma(\beta)\beta^T \\
-i\gamma(\beta)\beta & I_3 + \frac{\gamma(\beta)-1}{\beta^T\beta}\beta\beta^T
\end{bmatrix}; \quad X = \begin{bmatrix}
x^0 \\
x^1 \\
x^2 \\
x^3
\end{bmatrix} = \begin{bmatrix}
\frac{1}{\gamma}x^0 \\
x^1 \\
x^2 \\
x^3
\end{bmatrix},
\]
where \(I_3\) is the unitary \(3 \times 3\) matrix and Lorentz \(\gamma\)-factor is
\[
\gamma(\beta) = \frac{1}{\sqrt{1 - \beta^T\beta}}.
\]
The typical proper LB along \(x\)-axis in \(M^4\) \([2]\) (p. 21) and \(E^4\) has, correspondingly:
\[
A_{L(x)(\beta)} = \begin{bmatrix}
\gamma(\beta) & -\gamma(\beta)\beta & 0 & 0 \\
-\gamma(\beta)\beta & \gamma(\beta) & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}; \quad X = \begin{bmatrix}
x^0 \\
x^1 \\
x^2 \\
x^3
\end{bmatrix},
\]
\[
A_{E(x)(\beta)}^L = \begin{bmatrix}
\gamma(\beta) & i\gamma(\beta)\beta & 0 & 0 \\
-i\gamma(\beta)\beta & \gamma(\beta) & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}; \quad X = \begin{bmatrix}
x^0 \\
x^1 \\
x^2 \\
x^3
\end{bmatrix} = \begin{bmatrix}
\frac{1}{\gamma}x^0 \\
x^1 \\
x^2 \\
x^3
\end{bmatrix}.
\]
It is noted that transformation matrices (2.8) and (2.11) are rotation matrices. So, LB in \(M^4\) becomes rotation in \(E^4\) (Wick Rotation) \([3]\).

The physical content of the four-dimensional space is obtained by the replacement
\[
x^0 = ct,
\]
where
\[
c = 299,792,458\text{ms}^{-1}
\]
is the speed of light in vacuum. Then \(\beta\)-factor is called velocity factor
\[
\beta^i = \frac{1}{c} \frac{dx^i}{dt} = \frac{v^i}{c},
\]
3. Derivation of the proper closed isometric linear transformation in four-dimensional space endowed with Euclidean or Lorentz metric

3.1 Motion in $x$-direction

We consider one unmoved observer (frame) $Oxyz$, who measures real space-time and another observer (frame) $O'x'y'z'$ with parallel spatial axes, moving to the right, along $x$-axis with velocity

\[ v = \beta c \]

wrt the observer (frame) $Oxyz$ (Figure 1).

Supposing the next linear transformation:

\begin{align*}
(3.1.2) & \quad ct' = bc t + ax + k y + \nu z \\
(3.1.3) & \quad x' = gc t + f x + \delta y + \theta z \\
(3.1.4) & \quad y' = g_1 ct + f_1 x + h y + \lambda z \\
(3.1.5) & \quad z' = g_2 ct + f_2 x + \xi y + \mu z,
\end{align*}

we determine the coefficients with the following conditions:

(i) Isotropy: We postulate the transformation to be invariant to the spatial rotation. Rotating the frame about $x$-axis, through one right angle (Figure 1), we correspond the new axes to the initial axes:

\[ ct \rightarrow ct'; x \rightarrow x'; x' \rightarrow x'; y \rightarrow -z; \]

\[ y' \rightarrow -z'; z \rightarrow y; z' \rightarrow y'. \]

Thus, from (3.1.2), we have

\[ ct' = bc t + ax - kz + \nu y. \]
Comparing (3.1.2) and (3.1.7), it emerges $k = \nu = 0$. Besides, from (3.1.3) we have

$$x' = gct + fx - \delta x + \theta y.$$  

(3.1.8)

Comparing (3.1.3) and (3.1.8), it emerges $\delta = \theta = 0$. Besides, from (3.1.4) we have

$$-z' = g_1 ct + f_1 x - h z + \lambda y.$$  

(3.1.9)

Comparing (3.1.5) and (3.1.9), it emerges $g_2 = -g_1$, $f_2 = -f_1$, $\xi = -\lambda$ and $\mu = h$. Besides, from (3.1.5), we have

$$y' = g_2 ct + f_2 x - \xi z + \mu y.$$  

(3.1.10)

Comparing (3.1.4) and (3.1.10), it emerges $g_2 = g_1$, $f_2 = f_1$, $\xi = -\lambda$ and $\mu = h$. So, $k = \nu = \delta = \theta = g_1 = g_2 = f_1 = f_2 = 0$; $\xi = -\lambda$; $\mu = h$ and the transformation becomes:

$$ct' = bct + ax.$$  

(3.1.11)

$$x' = gct + fx.$$  

(3.1.12)

$$y' = hy + \lambda z.$$  

(3.1.13)

$$z' = -\lambda y + h z.$$  

(3.1.14)

(ii) The frame $O'x'y'z'$ is moving with velocity $(\beta c, 0, 0)$ wrt to $Oxyz$: for $x' = 0$, it is $x = \beta ct$. Replacing these to (3.1.12), we obtain

$$0 = gct + f \beta ct,$$  

(3.1.15)

for any value of $t$. This emerges

$$g = -\beta f.$$  

(3.1.16)

and the transformation becomes:

$$ct' = bct + ax.$$  

(3.1.17)

$$x' = -\beta f ct + fx.$$  

(3.1.18)

$$y' = hy + \lambda z.$$  

(3.1.19)

$$z' = -\lambda y + h z.$$  

(3.1.20)
(iii) Maintenance of Lorentz length (spacetime interval) \(|\vec{X}| = S : S'^2 = S^2\).

Thus,

\[(3.1.21) \quad x^2 + y^2 + z^2 - c^2 t^2 = x'^2 + y'^2 + z'^2 - c^2 t'^2,\]

or equivalently,

\[(3.1.22) \quad (-\beta c ft + fx)^2 + (hy + \lambda z)^2 + (-\lambda y + h z)^2 - (bct + ax)^2 = x^2 + y^2 + z^2 - c^2 t^2.\]

From the terms \(x^2; y^2; z^2; c^2 t^2; ctx\), we obtain:

\[(3.1.23) \quad f^2 - a^2 = 1\]
\[(3.1.24) \quad h^2 + \lambda^2 = 1\]
\[(3.1.25) \quad \beta^2 f^2 - b^2 = -1\]
\[(3.1.26) \quad -\beta f^2 - ab = 0.\]

Combining (3.1.26) with (3.1.23), we have

\[(3.1.27) \quad b = -\frac{\beta a^2 - \beta}{a}.\]

The Combination of (3.1.25) with (3.1.23) and (3.1.27) gives

\[(3.1.28) \quad a = \pm \frac{\beta}{\sqrt{1 - \beta^2}} = \pm \beta \gamma.\]

Replacing the above to (3.1.27), we obtain

\[(3.1.29) \quad b = \mp \frac{1}{\sqrt{1 - \beta^2}} = \mp \gamma.\]

Now, we must choose the sign in the above equations. We observe that for \(\beta = 0\) the upper sign (↑) gives \(a = 0\) and \(b = -1\). This transforms (3.1.11) to \(t' = -t\), producing time inversion. The lower sign (↓) gives \(t' = t\), corresponding to the proper transformation and we have:

\[(3.1.30) \quad a = -t \frac{\beta}{\sqrt{1 - \beta^2}} = -\beta \gamma ; \quad b = \frac{1}{\sqrt{1 - \beta^2}} = \gamma.\]

The replacement of the above to (3.1.23) gives

\[(3.1.31) \quad f = \pm \frac{1}{\sqrt{1 - \beta^2}} = \pm \gamma.\]
We also observe that for $\beta = 0$ the upper sign (↑) gives $f = 1$. This transforms (3.1.18) to $x' = x$. Thus, the proper transformation has

$$f = \frac{1}{\sqrt{1 - \beta^2}} = \gamma.$$  

On the other hand the lower sign (↓) for $\beta = 0$, gives $f = -1$ and (3.1.18) is transformed to $x' = -x$, producing space inversion. So, the proper transformation (↑↓) becomes:

$$ct' = \gamma(ct - \beta x)$$  

(3.1.33)

$$x' = \gamma(-\beta ct + x)$$  

(3.1.34)

$$y' = hy + \lambda z$$  

(3.1.35)

$$z' = -\lambda y + h z,$$  

(3.1.36)

with condition (3.1.24). Using matrices, LT (2.5) has

$$A_{B(x)} = \begin{bmatrix} \gamma & -\beta \gamma & 0 & 0 \\ -\beta \gamma & \gamma & 0 & 0 \\ 0 & 0 & h & \lambda \\ 0 & 0 & -\lambda & h \end{bmatrix}. $$  

(3.1.37)

Besides, the differential form of the transformation is

$$cdt' = \gamma(cdt - \beta dx)$$  

(3.1.38)

$$dx' = \gamma(-\beta cdt + dx)$$  

(3.1.39)

$$dy' = hdy + \lambda dz$$  

(3.1.40)

$$dz' = -\lambda dy + h dz.$$  

(3.1.41)

Thus, the velocities are related as following:

$$v'_x = \frac{-\beta c + u_x}{c - \beta u_x} c, \quad v'_y = \frac{hu_y + \lambda u_z}{\gamma(c - \beta u_x)} c, \quad v'_z = \frac{-\lambda u_y + hu_z}{\gamma(c - \beta u_x)} c.$$  

(3.1.42)
3.2 General linear transformation (motion in random direction)

We consider an unmoved observer (frame) $Oxyz$ and another observer (frame) $O'x'y'z'$ with parallel spatial axes, moving with velocity $(v_x, v_y, v_z)$ wrt the observer (frame) $Oxyz$ (Figure 2).

![Figure 2. Two frames Oxyz and O’x’y’z’, which initially coincide. The second is moving with random velocity (v_x, v_y, v_z) wrt to Oxyz.](image)

We rotate the initial frame $Oxyz$, in order to parallelize the unitary vector $\hat{x}$ to the velocity vector $\vec{v}$ of the moving observer $O'x'y'z'$. This is sequentially achieved as following: We firstly rotate the coordinate system $Oxyz$ about $z$-axis, through an angle $\theta$ [$O(\hat{x}, \hat{y}, \hat{z}) \rightarrow O(\hat{i}, \hat{j}, \hat{k})$]. We then rotate the coordinate system $O(\hat{i}, \hat{j}, \hat{k})$ about $\hat{j}$ through an angle $\omega$ [$O(\hat{i}, \hat{j}, \hat{k}) \rightarrow O(\hat{i}', \hat{j}', \hat{k}')$] (Figure 3). The corresponding matrices are:

\begin{equation}
R_{xy}(\theta) = \begin{bmatrix}
\cos \theta & \sin \theta & 0 \\
-\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{bmatrix};
R_{xz}(\omega) = \begin{bmatrix}
\cos \omega & 0 & \sin \omega \\
0 & 1 & 0 \\
-\sin \omega & 0 & \cos \omega
\end{bmatrix}.
\end{equation}

![Figure 3. Rotation of the initial frame Oxyz, in order to achieve parallelization of vector $\hat{x}$ to the velocity vector $\vec{v}$ of the moving observer O’x’y’z’ [O(\hat{x}, \hat{y}, \hat{z}) \rightarrow O(\hat{i}, \hat{j}, \hat{k}) \rightarrow O(\hat{i}', \hat{j}', \hat{k}')]](image)
Thus, we have the transformation:

\[
\begin{bmatrix}
    x_R \\
y_R \\
z_R
\end{bmatrix} =
\begin{bmatrix}
    \cos \omega \cos \theta & \cos \omega \sin \theta & \sin \omega \\
    -\sin \theta & \cos \theta & 0 \\
    -\sin \omega \cos \theta & -\sin \omega \sin \theta & \cos \omega
\end{bmatrix} \cdot
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix},
\]

where \((x_R, y_R, z_R)\) are the coordinates with respect to the frame \(O(\hat{i}', \hat{j}', \hat{k}')\) and

\[
\sin \theta = \frac{v_y}{\sqrt{v_x^2 + v_y^2}}; \quad \cos \theta = \frac{v_x}{\sqrt{v_x^2 + v_y^2}}.
\]

\[
\sin \omega = \frac{v_z}{|v|}; \quad \cos \omega = \frac{\sqrt{v_x^2 + v_y^2}}{|v|}.
\]

As a result, the above \(3 \times 3\) matrix becomes

\[
R = \begin{bmatrix}
    -\frac{\beta_y}{\sqrt{\beta_x^2 + \beta_y^2}} & \frac{\beta_x}{\sqrt{\beta_x^2 + \beta_y^2}} & \beta_z \\
    \frac{\beta_x}{\sqrt{\beta_x^2 + \beta_y^2}} & -\frac{\beta_y}{\sqrt{\beta_x^2 + \beta_y^2}} & 0 \\
    \frac{\beta_x \beta_y}{|\beta| \sqrt{\beta_x^2 + \beta_y^2}} & \frac{\beta_y \beta_z}{|\beta| \sqrt{\beta_x^2 + \beta_y^2}} & \frac{\beta_z \beta_x}{|\beta| \sqrt{\beta_x^2 + \beta_y^2}}
\end{bmatrix}
\]

and we define

\[
\tilde{R} = \begin{bmatrix} 1 & 0 \\ 0 & R \end{bmatrix}.
\]

The unit means that time is not affected by the spatial rotation.

Moreover, the transformation \(O(\hat{x}, \hat{y}, \hat{z}) \rightarrow O'(\hat{x}, \hat{y}, \hat{z})\) is analyzed to the following sequence of successive transformations:

\[O(\hat{x}, \hat{y}, \hat{z}) \rightarrow O(\hat{i}', \hat{j}', \hat{k}'); O(\hat{i}', \hat{j}', \hat{k}'); O'(\hat{i}', \hat{j}', \hat{k}'); O'(\hat{i}', \hat{j}', \hat{k}'); O'(\hat{x}, \hat{y}, \hat{z})\].

The above simple transformations have active interpretations, respectively:

\[
X' = \tilde{R} X; \quad X'_R = A_{B(x)} X_R; \quad X' = \tilde{R}^T X'_R,
\]

where \(\tilde{R}^T\) is the transpose matrix of \(\tilde{R}\).

Thus, the transformation \(O(\hat{x}, \hat{y}, \hat{z}) \rightarrow O'(\hat{x}, \hat{y}, \hat{z})\) is actively interpreted:

\[
X' = \tilde{R}^T A_{B(x)}(\beta) \tilde{R} X,
\]

and LT (2.5) has

\[
A_{B(\beta)} = \tilde{R}^T A_{B(x)}(\beta) \tilde{R}.
\]
It is $\beta > 0$. So, $\beta = |\tilde{\beta}|$ and we calculate:

$$A_{B(\beta)} = \tilde{R}^T \begin{bmatrix} \gamma & -|\tilde{\beta}|\gamma & 0 & 0 \\ -|\tilde{\beta}|\gamma & \gamma & 0 & 0 \\ 0 & 0 & h & \lambda \\ 0 & 0 & -\lambda & h \end{bmatrix}.$$  

(3.2.10)

or equivalently,

$$A_{B(\beta)} = \tilde{R}^T \begin{bmatrix} \gamma & -\gamma \beta_x & -\gamma \beta_y & -\gamma \beta_z \\ -\gamma |\beta| & \gamma \beta_x |\beta| & \gamma \beta_y |\beta| & \gamma \beta_z |\beta| \\ 0 & -\beta_y h \sqrt{\beta_x^2 + \beta_y^2} & -\beta_x \beta_y h \sqrt{\beta_x^2 + \beta_y^2} & -\beta_x h \sqrt{\beta_x^2 + \beta_y^2} \\ 0 & -\beta_y \lambda \sqrt{\beta_x^2 + \beta_y^2} & -\beta_x \beta_y \lambda \sqrt{\beta_x^2 + \beta_y^2} & -\beta_x \beta_y h \lambda \sqrt{\beta_x^2 + \beta_y^2} \end{bmatrix}.$$  

(3.2.11)

Furthermore, we have

$$A_{B(\beta)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \beta_x |\beta| & -\beta_y h \sqrt{\beta_x^2 + \beta_y^2} & -\beta_x h \sqrt{\beta_x^2 + \beta_y^2} \\ 0 & \beta_y |\beta| & -\beta_x \beta_y h \sqrt{\beta_x^2 + \beta_y^2} & -\beta_x \beta_y h \lambda \sqrt{\beta_x^2 + \beta_y^2} \\ 0 & \beta_x |\beta| & -\beta_y \lambda \sqrt{\beta_x^2 + \beta_y^2} & -\beta_x \beta_y \lambda \sqrt{\beta_x^2 + \beta_y^2} \end{bmatrix}.$$  

(3.2.12)

So, we obtain

$$A_{B(\beta)} = \begin{bmatrix} \gamma & -\gamma \beta_x (\gamma - h) \frac{\beta_x^2}{|\beta|^2} + h & -\gamma \beta_y (\gamma - h) \frac{\beta_y^2}{|\beta|^2} + \beta_x \lambda (\gamma - h) \frac{\beta_x \beta_y}{|\beta|^2} - \beta_y \lambda \frac{\beta_y}{|\beta|^2} + h & -\gamma \beta_z (\gamma - h) \frac{\beta_z^2}{|\beta|^2} - \beta_y \lambda \frac{\beta_z}{|\beta|^2} + h \\ -\gamma \beta_x (\gamma - h) \frac{\beta_x^2}{|\beta|^2} + h & -\gamma \beta_y (\gamma - h) \frac{\beta_y^2}{|\beta|^2} + \beta_x \lambda (\gamma - h) \frac{\beta_x \beta_y}{|\beta|^2} - \beta_y \lambda \frac{\beta_y}{|\beta|^2} + h & -\gamma \beta_z (\gamma - h) \frac{\beta_z^2}{|\beta|^2} - \beta_y \lambda \frac{\beta_z}{|\beta|^2} + h \\ -\gamma \beta_y (\gamma - h) \frac{\beta_y^2}{|\beta|^2} + \beta_x \lambda (\gamma - h) \frac{\beta_x \beta_y}{|\beta|^2} - \beta_y \lambda \frac{\beta_y}{|\beta|^2} + h & -\gamma \beta_y (\gamma - h) \frac{\beta_y^2}{|\beta|^2} + \beta_x \lambda (\gamma - h) \frac{\beta_x \beta_y}{|\beta|^2} - \beta_y \lambda \frac{\beta_y}{|\beta|^2} + h & -\gamma \beta_z (\gamma - h) \frac{\beta_z^2}{|\beta|^2} - \beta_y \lambda \frac{\beta_z}{|\beta|^2} + h \\ -\gamma \beta_z (\gamma - h) \frac{\beta_z^2}{|\beta|^2} - \beta_y \lambda \frac{\beta_z}{|\beta|^2} + h & -\gamma \beta_z (\gamma - h) \frac{\beta_z^2}{|\beta|^2} - \beta_y \lambda \frac{\beta_z}{|\beta|^2} + h & -\gamma \beta_z (\gamma - h) \frac{\beta_z^2}{|\beta|^2} - \beta_y \lambda \frac{\beta_z}{|\beta|^2} + h \end{bmatrix}.$$  

(3.2.13)
and we have the transformation

\[
\begin{bmatrix}
  c t' \\
  x' \\
  y' \\
  z'
\end{bmatrix}
= \begin{bmatrix}
  \gamma & -\gamma \beta_x & -\gamma \beta_y & -\gamma \beta_z \\
  -\gamma \beta_x & (\gamma - h) \frac{\beta_x^2}{|\beta|^2} + h & (\gamma - h) \frac{\beta_x \beta_y}{|\beta|^2} + \beta_x \beta_z & (\gamma - h) \frac{\beta_x \beta_z}{|\beta|^2} - \beta_x \beta_y \\
  -\gamma \beta_y & (\gamma - h) \frac{\beta_y^2}{|\beta|^2} - \beta_y \beta_z & (\gamma - h) \frac{\beta_y \beta_z}{|\beta|^2} + \beta_y \beta_x & (\gamma - h) \frac{\beta_y \beta_x}{|\beta|^2} + \beta_y \beta_z \\
  -\gamma \beta_z & (\gamma - h) \frac{\beta_z^2}{|\beta|^2} + \beta_z \beta_x & (\gamma - h) \frac{\beta_z \beta_y}{|\beta|^2} - \beta_z \beta_y & (\gamma - h) \frac{\beta_z \beta_y}{|\beta|^2} - \beta_z \beta_x
\end{bmatrix}
\begin{bmatrix}
  c t \\
  x \\
  y \\
  z
\end{bmatrix}.
\]

3.3 The solution of proper closed isometric linear transformation
(correlation of two perpendicular moving observers)

We consider one unmoved observer (frame \(Oxyz\), another observer (frame \(O'x'y'z'\) with parallel spatial axes, moving to the right, along \(x\)-axis with velocity \((\beta c, 0, 0)\) wrt \(Oxyz\) and also a third observer (frame \(O''x''y''z''\) with parallel spatial axes, moving upward, along \(y\)-axis with velocity \((0, \beta c, 0)\) wrt \(Oxyz\) (Figure 4).

\[X = A_{B(x)(\beta)}^{-1}X'\; ; \; \; X'' = A_{B(y)(\beta)}X.\]

Thus, the transformation \(O'(\hat{x}, \hat{y}, \hat{z}) \rightarrow O''(\hat{x}, \hat{y}, \hat{z})\) is actively interpreted:

\[X'' = A_{B(y)(\beta)}A_{B(x)(\beta)}^{-1}X'.\]

and LT (2.5) has

\[A_3 = A_{B(y)(\beta)}A_{B(x)(\beta)}^{-1}.\]
According to (3.2.13), it is

\[
A_{B(x)(\beta)} = \begin{bmatrix}
\gamma & -\beta \gamma & 0 & 0 \\
-\beta \gamma & \gamma & 0 & 0 \\
0 & 0 & h & \lambda \\
0 & 0 & -\lambda & h
\end{bmatrix};
A_{B(y)(\beta)} = \begin{bmatrix}
\gamma & 0 & -\beta \gamma & 0 \\
0 & h & 0 & -\lambda \\
-\beta \gamma & 0 & \gamma & 0 \\
0 & \lambda & 0 & h
\end{bmatrix}.
\]

Besides, the inverse of matrix \(A_{B(x)(\beta)}\) is

\[
A_{B(x)(\beta)}^{-1} = \begin{bmatrix}
\gamma & \beta \gamma & 0 & 0 \\
\beta \gamma & \gamma & 0 & 0 \\
0 & 0 & h & -\lambda \\
0 & 0 & \lambda & h
\end{bmatrix}.
\]

With that

\[
A_3 = \begin{bmatrix}
\gamma & 0 & -\beta \gamma & 0 \\
0 & h & 0 & -\lambda \\
-\beta \gamma & 0 & \gamma & 0 \\
0 & \lambda & 0 & h
\end{bmatrix} \begin{bmatrix}
\gamma & \beta \gamma & 0 & 0 \\
\beta \gamma & \gamma & 0 & 0 \\
0 & 0 & h & -\lambda \\
0 & 0 & \lambda & h
\end{bmatrix} = \begin{bmatrix}
\gamma^2 & \beta \gamma^2 & -\beta \gamma h & \beta \gamma \lambda \\
\beta \gamma h & \gamma h & -\lambda^2 & -h \lambda \\
-\beta \gamma^2 & -\beta^2 \gamma^2 & \gamma h & -\gamma \lambda \\
\beta \gamma \lambda & \gamma \lambda & h \lambda & h^2
\end{bmatrix}.
\]

Now, we calculate velocity factor \(\vec{\beta}_4\) of observer \(O''x''y''z''\) wrt observer (frame) \(O'x'y'x'\). Eqn (3.1.41) can be applied, because observer \(O'x'y'z'\) is moving in \(x\)-direction and observer \(O''\) can be considered as observed body:

\[
\beta_{4x}' = \frac{-\beta c + 0}{c - 0} = -\beta
\]

\[
\beta_{4y}' = \frac{h \beta c + \lambda \cdot 0}{\gamma (c - 0)} = \frac{h \beta}{\gamma}
\]

\[
\beta_{4z}' = \frac{-\lambda \beta c + 0}{\gamma (c - 0)} = \frac{-\lambda \beta}{\gamma}.
\]

As a result, it is

\[
|\vec{\beta}_4|^2 = \beta^2 + \beta^2_h^2 \frac{1}{\gamma^2} + \frac{\beta^2 \lambda^2}{\gamma^2} = \frac{\beta^2 \gamma^2 + \beta^2 h^2 + \beta^2 \lambda^2}{\gamma^2},
\]

or equivalently,

\[
|\vec{\beta}_4|^2 = \beta^2 \gamma^2 + h^2 + \lambda^2 = \frac{\beta^2 \gamma^2 + 1}{\gamma^2}.
\]
Thus, it emerges

\[
|\vec{\beta}_4'| = \beta \frac{\sqrt{\gamma^2 + 1}}{\gamma}; \quad \gamma(\vec{\beta}_4') = \gamma^2. \tag{3.3.12}
\]

According to (3.2.13), the matrix corresponding to the velocity factor \(\vec{\beta}_4'\) is

\[
A_4 = A_{B(\vec{\beta}_4')} = \begin{bmatrix}
\gamma^2 & \beta \gamma^2 & -\beta \gamma h & \beta \gamma \lambda \\
\beta \gamma^2 & -\beta \gamma h & \beta \gamma \lambda & \cdot \\
-\beta \gamma h & \beta \gamma \lambda & \cdot & \cdot \\
\beta \gamma \lambda & \cdot & \cdot & \cdot
\end{bmatrix} \tag{3.3.13}
\]

We postulate the transformation to be closed:

\[
A_3 = A_{B(y(\beta))}A_{B(x(\beta))}^{-1} = A_{(\vec{\beta}_4')} = A_4. \tag{3.3.14}
\]

Comparing the matrices, element by element, we have:

\[
(A_3)_{10} = (A_4)_{10}, \tag{3.3.15}
\]

or equivalently,

\[
h = \gamma. \tag{3.3.16}
\]

Applying the foregoing equation in (3.1.24), we obtain

\[
\lambda^2 = -\beta^2 \gamma^2; \quad \lambda = i \beta \gamma = i |\vec{\beta}| \gamma. \tag{3.3.17}
\]

Thus, (3.3.16), (3.3.12ii), (3.3.17ii) and (3.3.12i) emerge

\[
h(\vec{\beta}_4') = \gamma(\vec{\beta}_4') = \gamma^2 \tag{3.3.18}
\]

and

\[
(\vec{\beta}_4')_1 = i |\vec{\beta}_4'| \gamma(\vec{\beta}_4') = i \beta \frac{\sqrt{\gamma^2 + 1}}{\gamma} \gamma^2 = i \beta \gamma \sqrt{\gamma^2 + 1} = \lambda \sqrt{\gamma^2 + 1}. \tag{3.3.19}
\]

Thus, the matrix (3.2.13) for the velocity factor \(\vec{\beta}_4'\) [see also (3.3.13)] is written:

\[
A_4 = \begin{bmatrix}
\gamma^2 & \beta \gamma^2 & -\beta \gamma h & \beta \gamma \lambda \\
\beta \gamma^2 & -\beta \gamma h & \beta \gamma \lambda & \cdot \\
-\beta \gamma h & \beta \gamma \lambda & \cdot & \cdot \\
\beta \gamma \lambda & \cdot & \cdot & \cdot
\end{bmatrix} \tag{3.3.20}
\]

Replacing only (3.3.16) to (3.3.8), we rewrite the velocity factor components

\[
\beta_{4x}' = -\beta; \quad \beta_{4y}' = \beta; \quad \beta_{4z}' = -\frac{\lambda \beta}{\gamma}. \tag{3.3.21}
\]
Now, we calculate the following quotients contained in matrix $A_4$, by using the above and also (3.3.12):

$$(3.3.22) \quad \frac{\beta_{lx}}{|\beta'_{4}|} = -\frac{\gamma}{\sqrt{\gamma^2 + 1}} ; \quad \frac{\beta_{ly}}{|\beta'_{4}|} = \frac{\gamma}{\sqrt{\gamma^2 + 1}} ; \quad \frac{\beta_{lz}}{|\beta'_{4}|} = -\frac{\lambda}{\sqrt{\gamma^2 + 1}} $$

The replacement of the above, (3.3.18) and (3.3.19) to (3.3.20) gives

$$(3.3.23) \quad A_4 = \begin{bmatrix} \frac{\gamma^2}{\lambda^2} & -\beta\gamma^2 & -\beta\lambda \gamma^2 & -\beta\lambda \lambda^2 \\ \beta\gamma^2 & \frac{\gamma^2}{\lambda^2} & -\lambda \gamma & -\gamma \lambda \\ -\beta^2\gamma & \lambda^2 \gamma & \lambda^2 & -\gamma \lambda \\ \beta\lambda \gamma & \gamma \lambda & \lambda \gamma & \gamma^2 \end{bmatrix} ,$$

while from (3.3.6) it is

$$(3.3.24) \quad A_3 = \begin{bmatrix} \frac{\gamma^2}{\lambda^2} & -\beta\gamma^2 & -\beta\lambda \gamma^2 & -\beta\lambda \lambda^2 \\ \beta\gamma^2 & \frac{\gamma^2}{\lambda^2} & -\lambda \gamma & -\gamma \lambda \\ -\beta^2\gamma & \lambda^2 \gamma & \lambda^2 & -\gamma \lambda \\ \beta\lambda \gamma & \gamma \lambda & \lambda \gamma & \gamma^2 \end{bmatrix} .$$

We validate the equation of the matrices: $A_3 = A_4$, because of (3.3.16) and (3.3.17i). Finally, we replace (3.3.16) and (3.3.17ii) to (3.2.14) and (3.2.13) and we obtain the proper closed isometric LT:

$$(3.3.25) \quad \begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \gamma(\beta) \begin{bmatrix} 1 & -\beta_x & -\beta_y & -\beta_z \\ -\beta_x & 1 & i\beta_z & -i\beta_y \\ -\beta_y & -i\beta_z & 1 & i\beta_x \\ -\beta_z & i\beta_y & -i\beta_x & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} ,$$

and the corresponding matrix

$$(3.3.26) \quad A_{B(\beta)} = \gamma(\beta) \begin{bmatrix} 1 & -\beta_x & -\beta_y & -\beta_z \\ -\beta_x & 1 & i\beta_z & -i\beta_y \\ -\beta_y & -i\beta_z & 1 & i\beta_x \\ -\beta_z & i\beta_y & -i\beta_x & 1 \end{bmatrix} .$$

We have preferred the physical approach (spacetime) for the derivation of the proper isometric LT in $M^4$, because SR is the main application [4]. The pure mathematical approach is simply obtained, by replacing $ct \rightarrow x^0$, according to (2.11):

$$(3.3.27) \quad \begin{bmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{bmatrix} = \gamma(\beta) \begin{bmatrix} 1 & -\beta^1 & -\beta^2 & -\beta^3 \\ -\beta^1 & 1 & i\beta^3 & -i\beta^2 \\ -\beta^2 & -i\beta^3 & 1 & i\beta^1 \\ -\beta^3 & i\beta^2 & -i\beta^1 & 1 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} .$$
and the corresponding matrix is
\[(3.3.28)\]
\[
A_{B(\beta)} = \gamma(\beta) \begin{bmatrix}
1 & -\beta^1 & -\beta^2 & -\beta^3 \\
-\beta^1 & 1 & i\beta^3 & -i\beta^2 \\
-\beta^2 & -i\beta^3 & 1 & i\beta^1 \\
-\beta^3 & i\beta^2 & -i\beta^1 & 1
\end{bmatrix} = \gamma(\beta) \begin{bmatrix}
1 & -\beta^T \\
-\beta & I_3 + iA(\beta)
\end{bmatrix},
\]
where
\[(3.3.29)\]
\[
\beta = \begin{bmatrix}
\beta^1 \\
\beta^2 \\
\beta^3
\end{bmatrix};
A(\beta) = \begin{bmatrix}
0 & \beta^3 & -\beta^2 \\
-\beta^3 & 0 & \beta^1 \\
\beta^2 & -\beta^1 & 0
\end{bmatrix}.
\]
The matrix \((A_B)\) of the proper isometric LT has determinant equal to the unit \((\det A_B = 1)\). Besides, the typical transformation along \(x\)-axis, has
\[(3.3.30)\]
\[
A_{B(x)(\beta)} = \gamma(\beta) \begin{bmatrix}
1 & -\beta & 0 & 0 \\
-\beta & 1 & 0 & 0 \\
0 & 0 & 1 & i\beta \\
0 & 0 & -i\beta & 1
\end{bmatrix}
\]
It is noted that antisymmetric matrix \(A(\beta)\) is related to the cross product (external product) because:
\[(3.3.31)\]
\[
A(e_1,e_2,e_3) = [e_i \times e_j] = \begin{bmatrix}
e_1 \times e_1 & e_1 \times e_2 & e_1 \times e_3 \\
e_2 \times e_1 & e_2 \times e_2 & e_2 \times e_3 \\
e_3 \times e_1 & e_3 \times e_2 & e_3 \times e_3
\end{bmatrix}
\]
\[
= \begin{bmatrix}
0 & e_3 & -e_2 \\
e_3 & 0 & e_1 \\
e_2 & -e_1 & 0
\end{bmatrix}.
\]
So,
\[(3.3.32)\]
\[
\vec{x} \times \vec{y} = (x^1 \vec{e}_1 + x^2 \vec{e}_2 + x^3 \vec{e}_3) \times (y^1 \vec{e}_1 + y^2 \vec{e}_2 + y^3 \vec{e}_3)
\]
\[
= [e_1 \times e_2 \ e_2 \times e_3 \ e_3 \times e_1] \cdot \begin{bmatrix}
0 & x^3 & x^2 \\
x^3 & 0 & -x^1 \\
x^2 & -x^1 & 0
\end{bmatrix} \cdot \begin{bmatrix}
y^1 \\
y^2 \\
y^3
\end{bmatrix},
\]
written in compact form:
\[(3.3.33)\]
\[
\vec{x} \times \vec{y} = [e_i] \cdot [A(-\vec{x})^T] \cdot [y^j] = -[e_i] \cdot [A(\vec{x})^T] \cdot [y^j].
\]
On the other hand, the proper isometric transformation in \(E^4\) is obtained as following: we initially divide \((3.3.27)\) with \(i\)
\[(3.3.34)\]
\[
\begin{bmatrix}
x_0^0 \\
x_1^1 \\
x_2^2 \\
x_3^3
\end{bmatrix} = \gamma(\beta) \begin{bmatrix}
1 & -\beta^1 & -\beta^2 & -\beta^3 \\
-\beta^1 & 1 & i\beta^3 & -i\beta^2 \\
-\beta^2 & -i\beta^3 & 1 & i\beta^1 \\
-\beta^3 & i\beta^2 & -i\beta^1 & 1
\end{bmatrix} \cdot \begin{bmatrix}
x_0^0 \\
x_1^1 \\
x_2^2 \\
x_3^3
\end{bmatrix}.
\]
This is equivalent to

\[
\begin{bmatrix}
\frac{x_0}{x_1^2} \\
x_1' \\
x_2' \\
x_3'
\end{bmatrix} = \gamma(\beta) \begin{bmatrix}
1 & i\beta^1 & i\beta^2 & i\beta^3 \\
-i\beta^1 & 1 & i\beta^3 & -i\beta^2 \\
-i\beta^2 & -i\beta^3 & 1 & i\beta^1 \\
i\beta^3 & i\beta^2 & -i\beta^1 & 1
\end{bmatrix} \cdot
\begin{bmatrix}
\frac{x_0}{x_1^2} \\
x_1' \\
x_2' \\
x_3'
\end{bmatrix}
\]

The above is written by using (2.4):

\[
\begin{bmatrix}
x_0' \\
x_1' \\
x_2' \\
x_3'
\end{bmatrix} = \gamma(\beta) \begin{bmatrix}
1 & i\beta^1 & i\beta^2 & i\beta^3 \\
-i\beta^1 & 1 & i\beta^3 & -i\beta^2 \\
-i\beta^2 & -i\beta^3 & 1 & i\beta^1 \\
i\beta^3 & i\beta^2 & -i\beta^1 & 1
\end{bmatrix} \cdot
\begin{bmatrix}
x_0' \\
x_1' \\
x_2' \\
x_3'
\end{bmatrix}
\]

So, the corresponding matrix in $E^4$ is

\[
A^E_{B(\beta)} = \gamma(\beta) \begin{bmatrix}
1 & i\beta^1 & i\beta^2 & i\beta^3 \\
-i\beta^1 & 1 & i\beta^3 & -i\beta^2 \\
-i\beta^2 & -i\beta^3 & 1 & i\beta^1 \\
i\beta^3 & i\beta^2 & -i\beta^1 & 1
\end{bmatrix}
\]

\[
= \gamma(\beta) \begin{bmatrix}
1 & i\beta^T & iA_{(\beta)} \\
-i\beta & I_3 + iA_{(\beta)}
\end{bmatrix}
\]

This is rotation matrix, because it is orthogonal (unitary) matrix with determinant equal to the unit ($\det A^E_B = 1$). Besides, the typical transformation along $x$-axis, has

\[
A^E_{B(x)(\beta)} = \gamma(\beta) \begin{bmatrix}
1 & i\beta & 0 & 0 \\
-i\beta & 1 & 0 & 0 \\
0 & 0 & 1 & i\beta \\
0 & 0 & -i\beta & 1
\end{bmatrix}
\]

### 4. Improper isometric linear transformations in four-dimensional space endowed with Euclidean or Lorentz metric

In the derivation of proper closed isometric Linear Transformation ($\downarrow\uparrow$), we have chosen the lower sign ($\downarrow$) in (3.1.28) and (3.1.29), but the upper ($\uparrow$) in (3.1.31). So, they have remained three (3) improper non-closed isometric Linear Transformations (which does not contain the identity transformation):

(i) **Space inversion non-closed isometric Linear Transformation** ($\downarrow\downarrow$) [lower sign ($\downarrow$) in (3.1.28) and (3.1.29) as well as lower sign ($\downarrow$) in (3.1.31)] in $M^4$ and $E^4$ with corresponding matrices ($\det A_B = \det A^E_B = -1$):

\[
A^E_{B(\beta)} = \gamma(\beta) \begin{bmatrix}
1 & -\beta^T \\
\beta & -I_3 - iA_{(\beta)}
\end{bmatrix} \quad ; \quad A^E_{B(x)(\beta)} = \gamma(\beta) \begin{bmatrix}
1 & i\beta^T \\
i\beta & -I_3 - iA_{(\beta)}
\end{bmatrix}
\]
The respective typical transformations along $x$-axis, have

\begin{equation}
A_{B(x)} = \gamma(\beta) \begin{bmatrix}
1 & -\beta & 0 & 0 \\
\beta & -1 & 0 & 0 \\
0 & 0 & -1 & -i\beta \\
0 & 0 & i\beta & -1
\end{bmatrix} ; 
A_{B(\bar{x})}^E = \gamma(\beta) \begin{bmatrix}
1 & i\beta & 0 & 0 \\
i\beta & -1 & 0 & 0 \\
0 & 0 & -1 & -i\beta \\
0 & 0 & i\beta & -1
\end{bmatrix}.
\end{equation}

(ii) Time inversion non-closed isometric Linear Transformation $(\uparrow \downarrow)$ [upper sign $(\uparrow)$ in (3.1.28) and (3.1.29) as well as upper sign $(\uparrow)$ in (3.1.31)] in $M^4$ and $E^4$ with corresponding matrices $(\det A_B = \det A_B^E = -1)$:

\begin{equation}
A_B(\beta) = \gamma(\beta) \begin{bmatrix}
-1 & \beta^T \\
-\beta & I_3 + iA(\beta)
\end{bmatrix} ; 
A_B^E(\beta) = \gamma(\beta) \begin{bmatrix}
-1 & -i\beta^T \\
i\beta & I_3 + iA(\beta)
\end{bmatrix}.
\end{equation}

The respective typical transformations along $x$-axis, have

\begin{equation}
A_{B(x)} = \gamma(\beta) \begin{bmatrix}
-1 & \beta & 0 & 0 \\
-\beta & 1 & 0 & 0 \\
0 & 0 & 1 & i\beta \\
0 & 0 & -i\beta & 1
\end{bmatrix} ; 
A_{B(\bar{x})}^E = \gamma(\beta) \begin{bmatrix}
-1 & i\beta & 0 & 0 \\
i\beta & 1 & 0 & 0 \\
0 & 0 & 1 & i\beta \\
0 & 0 & -i\beta & 1
\end{bmatrix}.
\end{equation}

(iii) Spacetime inversion closed isometric Linear Transformation $(\uparrow \downarrow)$ [upper sign $(\uparrow)$ in (3.1.28) and (3.1.29), but lower sign $(\downarrow)$ in (3.1.31)] in $M^4$ and $E^4$ with corresponding matrices $(\det A_B = \det A_B^E = 1)$:

\begin{equation}
A_B(\beta) = \gamma(\beta) \begin{bmatrix}
-1 & \beta^T \\
\beta & -I_3 - iA(\beta)
\end{bmatrix} ; 
A_B^E(\beta) = \gamma(\beta) \begin{bmatrix}
-1 & -i\beta^T \\
i\beta & -I_3 - iA(\beta)
\end{bmatrix}.
\end{equation}

The respective typical transformations along $x$-axis, have

\begin{equation}
A_{B(x)} = \gamma(\beta) \begin{bmatrix}
-1 & \beta & 0 & 0 \\
\beta & -1 & 0 & 0 \\
0 & 0 & -1 & -i\beta \\
0 & 0 & i\beta & 1
\end{bmatrix} ; 
A_{B(\bar{x})}^E = \gamma(\beta) \begin{bmatrix}
-1 & i\beta & 0 & 0 \\
i\beta & -1 & 0 & 0 \\
0 & 0 & -1 & -i\beta \\
0 & 0 & i\beta & -1
\end{bmatrix}.
\end{equation}

These matrices are exactly the opposite of the corresponding proper closed isometric LT.

In case of Lorentz Boost, we have [2] (pp. 30-31):

(i) Space inversion Lorentz Boost in $M^4$ and $E^4$ with corresponding matrices $(\det A_L = \det A_L^E = -1)$:

\begin{equation}
A_L(\beta) = \begin{bmatrix}
\gamma(\beta) & \gamma(\beta)\beta^T \\
-\gamma(\beta)\beta^T & -I_3 - \frac{\gamma(\beta)}{\beta^2} \beta\beta^T
\end{bmatrix} ; 
A_L^E(\beta) = \begin{bmatrix}
\gamma(\beta) & -i\gamma(\beta)\beta^T \\
i\gamma(\beta)\beta^T & -I_3 - \frac{\gamma(\beta)}{\beta^2} \beta\beta^T
\end{bmatrix}.
\end{equation}
The respective *typical transformations* along $x$-axis, have

$$A_{L(x)(\beta)} = \begin{bmatrix} \gamma(\beta) & \gamma(\beta)\beta & 0 & 0 \\ -\gamma(\beta)\beta & -\gamma(\beta) & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix};$$

$$L_{E_{L(x)(\beta)}} = \begin{bmatrix} \gamma(\beta) & -i\gamma(\beta)\beta & 0 & 0 \\ -i\gamma(\beta)\beta & -\gamma(\beta) & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$  

(4.8)

(ii) *Time inversion Lorentz Boost* in $M^4$ and $E^4$ with corresponding matrices $(\det A_L = \det A_{E_L}^T = 1)$:

$$A_{L(\beta)} = \begin{bmatrix} -\gamma(\beta) & \gamma(\beta)\beta T \\ \gamma(\beta)\beta & I_3 - \frac{\gamma(\beta)+1}{\beta T}\beta T \\ \end{bmatrix};$$

$$A_{E_L(\beta)} = \begin{bmatrix} -\gamma(\beta) & -i\gamma(\beta)\beta T \\ i\gamma(\beta)\beta & I_3 - \frac{\gamma(\beta)+1}{\beta T}\beta T \\ \end{bmatrix}.$$  

(4.9)

The respective *typical transformations* along $x$-axis, have

$$A_{L(x)(\beta)} = \begin{bmatrix} -\gamma(\beta) & \gamma(\beta)\beta & 0 & 0 \\ \gamma(\beta)\beta & -\gamma(\beta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

$$L_{E_{L(x)(\beta)}} = \begin{bmatrix} -\gamma(\beta) & -i\gamma(\beta)\beta & 0 & 0 \\ i\gamma(\beta)\beta & -\gamma(\beta) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$  

(4.10)

(iii) *Spacetime inversion Lorentz Boost* in $M^4$ and $E^4$ with corresponding matrices $(\det A_L = \det A_{E_L}^T = -1)$:

$$A_{L(\beta)} = \begin{bmatrix} -\gamma(\beta) & -\gamma(\beta)\beta T \\ \gamma(\beta)\beta & -I_3 + \frac{\gamma(\beta)+1}{\beta T}\beta T \\ \end{bmatrix};$$

$$A_{E_L(\beta)} = \begin{bmatrix} -\gamma(\beta) & i\gamma(\beta)\beta T \\ i\gamma(\beta)\beta & -I_3 + \frac{\gamma(\beta)+1}{\beta T}\beta T \\ \end{bmatrix}.$$  

(4.11)
The respective typical transformations along $x$-axis, have

\[
A_{\Lambda L(x,\beta)} = \begin{bmatrix}
-\gamma(\beta) & -\gamma(\beta)\beta & 0 & 0 \\
\gamma(\beta)\beta & \gamma(\beta) & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix};
\]

(4.12)

\[
A_{\Lambda L^{E}(x,\beta)} = \begin{bmatrix}
-\gamma(\beta) & i\gamma(\beta)\beta & 0 & 0 \\
i\gamma(\beta)\beta & \gamma(\beta) & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix}.
\]

Figure 5. Correlation of three successive observers (frames), by using Lorentz Boost. The frame $O'x'y'z'$ has parallel axes to the corresponding of frame $Oxyz$, moving with velocity $(\beta_1 c, 0, 0)$ wrt $Oxyz$. The frame $O''x''y''z''$ has parallel axes to the corresponding of frame $O'x'y'z'$, moving with velocity $(0, \beta_2 c, 0)$ wrt $O'x'y'z'$. The correlation of the observers, by using Lorentz Boost, cancels the absolute character of parallelism. Thus, the axes of frame $O''x''y''z''$ are not parallel to the corresponding of frame $Oxyz$ (Thomas Rotation).

5. Conclusions

The closed isometric linear transformation which maintains Lorentz length ($S^2$) is represented by a matrix ($\Lambda_B$) containing real and imaginary numbers. Under this transformation, the real spacetime of the initial rest observer (frame) is transformed to real time and complex space for one moving observer (frame) ($R^4 \rightarrow R \times C^3$). Subsequently, the axes rotation (Thomas Rotation [5]) that happens from the correlation of three observers related by using Lorentz Boost (Figure 5) [2] (pp. 177-183), in this approach is avoided. Thus, the validation of the transitive attribute in parallelism of unmoved straight lines (which is equivalent to the 5th Euclidean postulate), is extended to the case moving straight lines (for any observer). This is achieved, by working in the domain of complex numbers, validating one more time, the words of J. Hadamard: “It has been...
written that the shortest and best way between two truths of the real domain often passes through the imaginary one” [6] (p. 123).

References


Accepted: 03.04.2020