Falling fuzzy hyper deductive systems of hyper residuated lattices

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Abstract. This paper introduces and applies the notion of fuzzy hyper deductive systems of hyper residuated lattices as a generalization of the notion of fuzzy deductive systems. It considers hyper deductive systems of hyper residuated lattices, and the relationship between hyper deductive systems and fuzzy hyper deductive systems. Several properties and characterizations of fuzzy hyper deductive systems are given. Based on the falling shadow theory, we establish a theoretical approach by means of exploring the relationships between hyper residuated lattices and probability spaces, and tend to ascertain a falling fuzzy hyper deductive system of a hyper residuated lattice as a generalization of a fuzzy hyper deductive system. We also show some conditions for a falling fuzzy hyper deductive system to be a falling fuzzy hyper implicative deductive system.

Keywords: hyper residuated lattice, falling fuzzy hyper deductive system, falling fuzzy hyper implicative deductive system.

1. Introduction

Residuated lattices as generalization of ideal lattices of rings with identity were introduced by Ward and Dilworth [1] in 1939. The algebraic counterparts of some non-classical logics satisfy residuation and those logics can be considered

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in a frame of residuated lattices. The main examples of residuated lattices related to logic are MV-algebras and BL-algebras. Apart from their logical interest, residuated lattices have interesting algebraic properties, they have long been considered of interest by algebraists starting from the classical example of the lattice-ordered monoid [2, 3, 4]. In order to provide a unified algebraic foundation for probabilities of fuzzy events in substructural logics, Zhao and He [5] enlarged the language of noncommutative residuated lattices by adding an internal state. Since various filters correspond to various sets of provable formulae from a logical point of view, they become an important tool in studying logical algebras and the completeness of non-classical logics [6, 7]. Some filters and their fuzzification of pseudo BCI-algebras [8, 9] and equality algebras [10] which closely related to residual lattices have been widely studied, and some important results have been published.

Algebraic hyperstructures were introduced in 1934 by Marty [11] at 8th Congress of Scandinavian Mathematicians, which represent a natural extension of classical algebraic structures. Nowadays, algebraic hyperstructure theory has a multiplicity of applications to other disciplines such as geometry, graphs and hypergraphs, binary relations, lattices, groups, fuzzy sets and rough sets, artificial intelligence and probability theory [12]. It seems that one of important applications is in logic where the logical operations is not uniquely determined, i.e., they give some (set) possibilities. This provides sufficient motivations for researchers to study hyperstructures of various logical algebras. Ghorbani et al. applied hyperstructures to MV-algebras and introduced the concept of hyper MV-algebras which is a generalization of MV-algebras [13]. Moreover, some types of hyper MV-ideals [14], deductive systems [15], fuzzy deductive systems [16] and the state theory [17] of hyper MV-algebras were introduced and investigated. It is well know that the class of MV-algebras, BL-algebras, and Heyting algebras are proper subclass of residuated lattices. As an application of hyperstructures to residuated lattices, Zahiri et al. put forward the notion of hyper residuated lattices, and constructed quotient hyper residuated lattices by the concept of a regular compatible congruence on hyper residuated lattices [18]. Continue the works in hyper residuated lattices, Borzooei et al. proposed the notions of hyper filters and hyper deductive systems in hyper residuated lattices, and obtained some properties of them [19]. hyperstructures were also applied other algebraic structures, and many kinds of hyper algebras were introduced in recent years, such as hyper EQ-algebras [20], hyper equality algebras [21] and hyper hoop-algebras [22].

The theory of falling shadows is an important tool in the theoretical developments and practical applications of fuzzy sets, and some of their properties and notions are displayed in [23]. On the basis of the theory of a falling shadow, Jun and Kang [24] gave a theoretical approach of a fuzzy positive implicative ideal of a BCK-algebra, and a fuzzy subalgebraic system was considered as the falling shadow of the cloud of the subalgebraic system. Inspired by the work in [25], Yang et al. investigated falling fuzzy (implicative) ideals of MV-algebras

[26] based on the theory of falling shadows and fuzzy sets, which provides a theoretical approach for the further studying of fuzzy ideals in MV-algebras. The falling shadow theory was also applied to study Gödel ideals of BL-algebras [27], hemirings [28] and prefilters of EQ-algebras [29]. These studies have presented a preliminary, but potentially interesting research direction.

Based on the aforementioned, in the present paper we further consider some characterizations of hyper deductive systems of hyper residuated lattices based on the fuzzy set theory and the falling shadow theory. The notion of fuzzy hyper deductive systems of hyper residuated lattices is introduced and some related properties are investigated. We also establish a theoretical approach by means of exploring the relationships between hyper residuated lattices and probability spaces, and tend to use hyper deductive systems of falling shadows to ascertain a falling fuzzy hyper deductive systems in an exceedingly hyper residuated lattices as a generalization of a fuzzy hyper hyper deductive system. Some relationships between fuzzy hyper deductive systems and falling fuzzy hyper deductive systems are offered, and conditions for a falling fuzzy hyper deductive system to be a falling fuzzy hyper implicative deductive system are presented. Related research methods in this paper may provide a useful tool for the research of fuzzy mathematics and hyper algebraic structures, and enrich the research contents of probability theory to some extent.

2. Preliminaries

To facilitate our discussion, we give some primary notions and previous results about MV-algebras which are necessary for the subsequent discussions.

Definition 2.1 ([18]). *A hyper residuated lattice is a non-empty set L endowed with four binary hyperoperations $\vee$, $\wedge$, $\otimes$, $\to$ and two constants 0 and 1 satisfying the following conditions:*

(1) $(L, \leq, \vee, \wedge, 0, 1)$ *is a bounded superlattice,*

(2) $(L, \otimes, 1)$ *is a commutative semihypergroup with 1 as the identity,*

(3) $a \otimes b \ll c$ *if and only if* $a \ll b \to c$.

where $A \ll B$ means that there exist $a \in A$ and $b \in B$ such that $a \leq b$, for any nonempty subsets A and B of L. We also define $x * A = \bigcup_{a \in A} x * a$, $A * x = \bigcup_{a \in A} a * x$, and $A * B = \bigcup_{a \in A, b \in B} a * b$, where $* \in \{\vee, \wedge, \otimes, \to\}$.

L is called nontrivial if $0 \neq 1$. An element $a \in L$ is called scalar if $|a \otimes x| = 1$, for any $x \in L$. In what follows, unless otherwise specified, we denote a hyper residuated lattice $(L, \vee, \wedge, \otimes, \to, 0, 1)$ by L. Some primary properties of hyper residuated lattices are collected in the following proposition.

Proposition 2.2 ([19]). *Given a hyper residuated lattice L, for any $x, y, z \in L$ and for any non-empty subsets A, B and C of L, the following hold:*

(1) *$x \le y$ implies $1 \in x \to y$, and if 1 is a scalar, the converse hold,*

(2) *$1 \in x \to x$, $1 \in x \to 1$, $1 \in 0 \to x$, $0 \in x \otimes 0$, $x \ll \neg\neg x$, where $\neg x = x \to 0$,*

(3) *$x \otimes y \ll y$, $x \otimes y \ll x$, $x \ll y \to (x \otimes y)$, $x \otimes (x \to y) \ll x$ and $x \otimes (x \to y) \ll y$,*

(4) *$x \to (y \to z) \ll (x \otimes y) \to z \ll x \to (y \to z)$,*

(5) *$x \le y$ implies that $x \otimes z \ll y \otimes z$, $z \to x \ll z \to y$ and $y \to z \ll x \to z$,*

(6) *$x \le y$ and $x \le z$ imply $x \ll y \wedge z$,*

(7) *$y \le x$ and $z \le x$ imply $y \vee z \ll x$,*

(8) *$A \ll B \to C$ if and only if $A \otimes B \ll C$,*

(9) *$A \ll x \ll B$ implies $A \ll B$.*

(10) *$A \cap B \ne \emptyset$ implies $A \ll B$ and $B \ll A$,*

(11) *$A \otimes B \ll A$, $A \otimes B \ll B$,*

(12) *$A \subseteq B$ implies $A \ll B$.*

Definition 2.3 ([19]). *A nonempty subset D of L is called a hyper deductive system if it satisfies the conditions: (d_1) $1 \in D$, (d_2) $D \ll x \to y$ and $x \in D$ imply $y \in D$ for any $x, y \in L$.*

Proposition 2.4 ([19]). *Let L be a hyper residuated lattice and D be a nonempty subset of L. Then D is a hyper deductive system of L if and only if it satisfying (d_1) and (d_3) $(x \to y) \cap D \ne \emptyset$ and $x \in D$ imply $y \in D$.*

Definition 2.5 ([19]). *Let L be a hyper residuated lattice and D be a nonempty subset D of L containing 1. Then D is called a hyper implicative deductive system of L if $x \to (y \to z) \cap D \ne \emptyset$ and $(x \to y) \cap D \ne \emptyset$ imply $(x \to z) \cap D \ne \emptyset$ for any $x, y, z \in L$.*

For purpose of convenience, let $\emptyset$ be a hyper deductive system or a hyper implicative deductive system of L in the the rest of the sections.

Let $(P, \le)$ be a partially ordered set and θ be an equivalence relation on P. Then θ is called regular if the set $P/\theta = \{[x] | x \in P\}$ can be ordered in such a way that the natural map $\pi : P \to P/\theta$ is order preserving. Let θ be an equivalence relation on L and $A, B \subseteq L$. Then

(1) $A\theta B$ means that there exist $a \in A$ and $b \in B$ such that $a\theta b$,

(2) $A\bar{\theta}B$ means that for all $a \in A$, there exists $b \in B$ such that $a\theta b$ and for all $b \in B$, there exists $a \in A$ such that $a\theta b$.

Definition 2.6 ([18]). *An equivalence relation θ on L is called a congruence relation if for all $x, y, w, v \in L$, $x\theta y$ and $w\theta v$ imply $(x * w)\bar{\theta}(y * v)$, where $* \in \{\vee, \wedge, \otimes, \rightarrow\}$.*

Let θ be a regular congruence relation on L, $L/\theta = \{[x]|x \in L\}$ and $\leq_\theta$ be the relation on L/θ (for more details, see [18]). For any $x, y \in L$,

$$[x]\bar{\vee}[y] = [x \vee y], [x]\bar{\wedge}[y] = [x \wedge y], [x]\bar{\otimes}[y] = [x \otimes y], [x] \rightsquigarrow [y] = [x \rightarrow y],$$

where $[A] = \{[a]|a \in A\}$, for all $A \subseteq L$.

A regular congruence relation θ on L if it satisfies the conditions: (i) $[x] \in [x]\bar{\vee}[y]$ iff $[x] \leq_\theta [y]$, (ii) $[x] \in [x]\bar{\wedge}[y]$ iff $[x] \leq_\theta [y]$, then it said to be a regular compatible congruence relation on L.

Theorem 2.7. *Let θ be a regular compatible congruence relation on L. Then $(L/\theta, \bar{\vee}, \bar{\wedge}, \bar{\otimes}, \rightsquigarrow, [0], [1])$ is a hyper residuated lattice.*

Definition 2.8 ([18]). *Let $(L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ and $(L', \vee', \wedge', \otimes', \rightarrow', 0', 1')$ be two hyper residuated lattices. A map $f : L \rightarrow L'$ is said to be a homomorphism if $f(0) = 0'$, $f(1) = 1'$ and $f(x * y) = f(x) *' (y)$ for any $x, y \in L$, where $* \in \{\vee, \wedge, \otimes, \rightarrow\}$.*

3. Fuzzy hyper deductive systems

In this section, we consider fuzzy hyper deductive systems of hyper residuated lattices and investigate some properties of them.

Theorem 3.1. *Let D be a nonempty subset of L. Then D is a hyper deductive system of L if and only if*

(1) *$x \leq y$ and $x \in D$ imply $y \in D$,*

(2) *$x \in D$ and $y \in D$ imply $x \otimes y \subseteq D$,*

for any $x, y \in L$.

Proof. $\Rightarrow$ Let $x, y \in L$ be such that $x \leq y$ and $x \in D$. Then $1 \in x \rightarrow y$. From $1 \in D$, we get that $D \ll x \rightarrow y$, and so $y \in D$ by hypothesis. Now let $x \in D$ and $y \in D$. For any $u \in x \otimes y$, we have $x \otimes y \ll u$, it follows that $x \ll y \rightarrow u$ by Proposition 2.2 (8). Since $x \in D$, then $D \ll y \rightarrow u$, and so $u \in D$ by hypothesis. Thus $x \otimes y \subseteq D$.

$\Leftarrow$ Let $x, y \in L$ be such that $D \ll x \rightarrow y$ and $x \in D$. Then there exist $u \in D$ and $v \in x \rightarrow y$ such that $u \leq v$, it follows that $v \in D$, and so $x \otimes v \subseteq D$. Notice that $v \in x \rightarrow y$, we have $v \ll x \rightarrow y$, that is, $v \otimes x \ll y$. Thus there exists $t \in v \otimes x \subseteq D$ such that $t \leq y$, therefore we get $y \in D$. □

Proposition 3.2. *Let D be a hyper deductive system of L. For any nonempty subset B of L, and $a \in D$, if $(a \rightarrow B) \cap D \neq \emptyset$, then $B \cap D \neq \emptyset$.*

Proof. Let $B \subseteq L$ and $a \in D$ be such that $(a \to B) \cap D \neq \emptyset$. Then there exists $x \in D$ such that $x \in a \to b$ for some $b \in B$. Thus $(a \to b) \cap D \neq \emptyset$. Since D is a hyper deductive system of L, it follows from Proposition 2.4 that $b \in D$, and so $B \cap D \neq \emptyset$. □

Proposition 3.3. *Let D be a hyper deductive system of L. Then the following hold: for any $x, y, z \in L$,*

(1) $(x \to y) \subseteq D$ *and* $x \in D$ *imply* $y \in D$,

(2) *if* $x, y \in D$ *and* $x \ll y \to z$, *then* $z \in D$,

(3) *if* $(x \to y) \subseteq D$ *and* $x \otimes z \subseteq D$, *then* $y \otimes z \subseteq D$.

Proof. (1) Let $x, y \in L$ be such that $(x \to y) \subseteq D$ and $x \in D$. Then $x \otimes (x \to y) = \bigcup_{u \in x \to y} x \otimes u \subseteq D$. On the other hand, from Proposition 2.2 (3), we know that $x \otimes (x \to y) \ll y$. Hence, there exists $v \in x \otimes (x \to y)$ such that $v \leq y$. Notice that $v \in D$, we get $y \in D$.

(2) Suppose that $x, y \in D$ and $x \ll y \to z$ for any $x, y, z \in L$, then $x \otimes y \subseteq D$ and $x \otimes y \ll z$. Then there exists $u \in x \otimes y \subseteq D$ such that $u \leq z$. By hypothesis that D is a hyper deductive system of L, we get $z \in D$.

(3) According to Proposition 2.2 (3), we have $x \otimes z \ll x, z$, then there exists $u \in x \otimes z \subseteq D$ such that $u \leq x$, and so $x \in D$. Similarly, $z \in D$. Now, since $x \in D$ and $(x \to y) \subseteq D$, then we get $y \in D$ by (1). From Theorem 3.1 and $y, z \in D$, it follows that $y \otimes z \subseteq D$. □

A fuzzy subset μ of L satisfies the sup-property if for any nonempty subset T of L there exists $x_0 \in T$ such that $\mu(x_0) = \sup_{x \in T} \mu(x)$. For any fuzzy subset μ of L the set $\mu_t := \{x \in L | \mu(x) \geq t\}$ is called a level set of μ. In order to facilitate the discussion, we shall use the notations "$\bigvee \mu(A)$" and "$\bigwedge \mu(A)$" to mean "$\sup_{a \in A} \mu(a)$" and "$\inf_{a \in A} \mu(a)$", respectively, for any nonempty set A. Moreover, for any $a, b \in A$, $a \wedge b$ and $a \vee b$ mean $\min\{a, b\}$ and $\max\{a, b\}$, respectively.

Definition 3.4. *A fuzzy set μ of L is called a fuzzy hyper deductive system if it satisfies: for any $x, y \in L$,*

(1) $\mu(1) \geq \mu(x)$,

(2) $\mu(y) \geq \bigvee \mu(x \to y) \wedge \mu(x)$.

For better understanding the definition of fuzzy hyper deductive systems, we illustrate it by the following example.

Example 3.5. Let $L = \{0, a, b, 1\}$ be a chain such that $0 < a < b < 1$. Consider hyperoperations $\vee$ and $\wedge$ as given in the following tables,

$\vee$	0	a	b	1
0	{0, a, b, 1}	{a, b, 1}	{b, 1}	{1}
a	{a, b, 1}	{a, b, 1}	{b, 1}	{1}
b	{b, 1}	{b, 1}	{b, 1}	{1}
1	{0, 1}	{1}	{1}	{1}

$\wedge$	0	a	b	1
0	{0}	{0}	{0}	{0}
a	{0}	{0, a}	{0, a}	{0,a}
b	{0}	{0, a}	{0, a, b}	{0, a, b}
1	{0}	{0, a}	{0, a, b}	{0, a, b, 1}

Then $(L, \vee, \wedge, 0, 1)$ is a bound hyper lattice. Define hyper hyperoperations $\otimes$ and $\rightarrow$ as follows,

$\otimes$	0	a	b	1
0	{0}	{0}	{0}	{0}
a	{0}	{0, a}	{a}	{a}
b	{0}	{a}	{b}	{b}
1	{0}	{a}	{b}	{1}

$\rightarrow$	0	a	b	1
0	{1}	{1}	{1}	{1}
a	{0, a}	{1}	{1}	{1}
b	{0}	{0, a}	{1}	{1}
1	{0}	{a}	{b}	{1}

it is easy to verify that $(L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ is a hyper residuated lattice [19]. Define a fuzzy set μ of L as follows:

$$\mu(x) = \begin{cases} 0.3, & x = 0, \\ 0.3, & x = a, \\ 0.5, & x = b, \\ 0.5, & x = 1. \end{cases}$$

Route calculations show that μ is a fuzzy hyper deductive system of L.

Proposition 3.6. *Let μ be a fuzzy hyper deductive system of L. Then $\emptyset \neq \mu_t$ is a hyper deductive system of L for any $t \in [0, 1]$.*

Proof. Let $t \in [0, 1]$ such that $\emptyset \neq \mu_t$. Then there exists $x \in \mu_t$, and so $\mu(x) \geq t$. Since μ is a fuzzy hyper deductive system of L, then $\mu(1) \geq \mu(x)$, so $1 \in \mu_t$. Let $(x \rightarrow y) \cap \mu_t \neq \emptyset$ and $x \in \mu_t$. Then $\mu(x) \geq t$ and there exists $w \in (x \rightarrow y) \cap \mu_t$, therefore $\bigvee \mu(x \rightarrow y) \geq \mu(w) \geq t$. It follows that $\mu(y) \geq \bigvee \mu(x \rightarrow y) \wedge \mu(x) \geq t$, hence $y \in \mu_t$. According to Proposition 2.4, we know that μ_t is a hyper deductive system of L. □

Proposition 3.7. *Let μ be a fuzzy subset of L with the sup-property. If $\emptyset \neq \mu_t$ is a hyper deductive system of L for any $t \in [0,1]$, then μ is a fuzzy hyper deductive system of L.*

Proof. For any $x \in L$, $x \in \mu_{\mu(x)}$. Since $\mu_{\mu(x)}$ is a hyper deductive system of L, then $1 \in \mu_{\mu(x)}$, and so $\mu(1) \geq \mu(x)$. For every $x, y \in L$, Now let $t_1 = \bigvee \mu(x \to y)$, $t_2 = \mu(x)$, and $t = t_1 \wedge t_2$, then $x \in \mu_t$. Notice that μ satisfies the sup-property, there exists $w \in x \to y$ such that $t_1 = \bigvee \mu(x \to y) = \mu(w)$, and so $w \in \mu_t$. It follows that $(x \to y) \cap \mu_t \neq \emptyset$ and $x \in \mu_t$, we get $y \in \mu_t$ by Proposition 2.4, hence $\mu(y) \geq t = \bigvee \mu(x \to y) \wedge \mu(x)$. Thus, μ is a fuzzy hyper deductive system of L. □

Corollary 3.8. *Let $\emptyset \subsetneq D \subseteq L$ and $\alpha \in (0,1]$. Define a fuzzy set $\mu_D : L \to [0,1]$ by*

$$\mu_D(x) = \begin{cases} \alpha, & x \in D, \\ 0, & x \notin D. \end{cases}$$

Then μ_D is a fuzzy hyper deductive system of L if and only if D is a hyper deductive system of L.

Proposition 3.9. *Let μ be a fuzzy hyper deductive system of L. Then for any $x, y \in L$, the following statements hold:*

(1) *if $x \leq y$, then $\mu(x) \leq \mu(y)$,*

(2) $\bigwedge \mu(x \otimes y) \geq \mu(x) \wedge \mu(y)$.

Proof. (1) Assume that $x \leq y$, then $1 \in x \to y$. Notice that μ is a fuzzy hyper deductive system of L, we get $\mu(1) \geq \mu(z)$ for any $z \in L$, and so, $\bigvee \mu(x \to y) = \mu(1)$. Moreover, $\mu(y) \geq \bigvee \mu(x \to y) \wedge \mu(x) = \mu(1) \wedge \mu(x) = \mu(x)$.

(2) For any $x, y \in L$, let $\mu(x) = t_1$, $\mu(y) = t_2$ and $t_1 \wedge t_2 = t$. It follows that $x, y \in \mu_t$, and μ_t is a hyper deductive system of L by Proposition 3.6. Therefore, $x \otimes y \subseteq \mu_t$, and so $\bigwedge \mu(x \otimes y) \geq t = \mu(x) \wedge \mu(y)$. □

Proposition 3.10. *Let μ be a fuzzy subset of L with the sup-property. If μ satisfies the following conditions: for any $x, y \in L$, (1) $x \leq y$ implies $\mu(x) \leq \mu(y)$, (2) $\bigwedge \mu(x \otimes y) \geq \mu(x) \wedge \mu(y)$, then μ is a fuzzy hyper deductive system of L.*

Proof. Let $t \in [0,1]$ and $\mu_t \neq \emptyset$. Let $x, y \in L$ be such that $x \leq y$ and $x \in \mu_t$. By assumption, we obtain that $t \leq \mu(x) \leq \mu(y)$, thus $y \in \mu_t$. For any $x, y \in \mu_t$, we have $\mu(x) \geq t$ and $\mu(y) \geq t$. It follows that $\bigwedge \mu(x \otimes y) \geq \mu(x) \wedge \mu(y) \geq t$. For any $w \in x \otimes y$, we get $\mu(w) \geq t$, therefore $w \in \mu_t$, and so $x \otimes y \subseteq \mu_t$. According to Proposition 3.7, we conclude that μ is a fuzzy hyper deductive system of L. □

Theorem 3.11. *Let μ be a fuzzy subset of L with the sup-property. Then μ is a fuzzy hyper deductive system of L if and only if $x \otimes y \ll z$ implies $\mu(z) \geq \bigwedge \mu(x \otimes y) \geq \mu(x) \wedge \mu(y)$, for any $x, y, z \in L$.*

Proof. The sufficiency is obviouse by Proposition 3.9, now we give the proof of the necessity. Since $x \otimes y \ll x$ for any $x, y \in L$, then $\bigwedge \mu(x \otimes y) \geq \mu(x) \wedge \mu(y)$. Notice that $x \otimes x \ll x \ll 1$, we get $x \otimes x \ll 1$, thus $\mu(1) \geq \mu(x) \wedge \mu(x) = \mu(x)$. If $x \leq y$, then $x \otimes 1 \ll x \ll y$, and so $x \otimes 1 \ll y$, it follows that $\mu(y) \geq \mu(x) \wedge \mu(1) = \mu(x)$. From Proposition 3.10, we get that μ is a fuzzy hyper deductive system of L. □

Proposition 3.12. *Let μ, ν be two fuzzy hyper deductive systems of L. Then we have*

(1) *$H_\mu := \{x \in L | \mu(x) = \mu(1)\}$ is a hyper deductive system of L,*

(2) *if $\nu \leq \mu$ and $\nu(1) = \mu(1)$, then $H_\mu \otimes H_\nu = \bigcup_{a \in H_\mu, b \in H_\nu} a \otimes b$ is a hyper deductive system of L, where $\nu \leq \mu$ is defined by $\nu(x) \leq \mu(x)$ for any $x \in L$.*

Proof. (1) Let $x, y \in L$ be such that $x \leq y$ and $x \in H_\mu$. Since μ is a fuzzy hyper deductive system of L, then $\mu(1) = \mu(x) \leq \mu(y)$, and so $\mu(y) = \mu(1)$, that is, $y \in H_\mu$. Now let $x \in H_\mu$ and $y \in H_\mu$. Then $\mu(x) = \mu(1)$ and $\mu(y) = \mu(1)$. For any $t \in x \otimes y$, $\mu(t) \geq \bigwedge \mu(x \otimes y) \geq \mu(x) \wedge \mu(y) = \mu(1)$ by Proposition 3.9. It follows that $\mu(t) = \mu(1)$, that is, $t \in H_\mu$, and so $x \otimes y \subseteq H_\mu$. According to the above discussion, we get that H_μ is a hyper deductive system of L by Theorem 3.1.

(2) Obviously, $1 \in H_\mu \otimes H_\nu$. Let $x, y \in L$ be such that $(x \to y) \cap (H_\mu \otimes H_\nu) \neq \emptyset$ and $x \in H_\mu \otimes H_\nu$. Then there exist $a \in H_\mu$ and $b \in H_\nu$ such that $x \in a \otimes b$. It follows that $\mu(a) = \mu(1)$, $\nu(b) = \nu(1) = \mu(1)$ and $\mu(y) \geq \bigvee \mu(x \to y) \wedge \mu(x) = \mu(1) \wedge \mu(x) = \mu(x) \geq \bigwedge \mu(a \otimes b) \geq \mu(a) \wedge \mu(b) = \mu(b) \geq \nu(b) = \mu(1)$. Therefore $\mu(y) = \mu(1)$, and so $y \in y \otimes 1 \subseteq H_\mu \otimes H_\nu$. Hence $H_\mu \otimes H_\nu$ is a hyper deductive system of L. □

Let $(L_1, \vee_1, \wedge_1, \otimes_1, \to_1, 0_1, 1_1)$ and $(L_2, \vee_2, \wedge_2, \otimes_2, \to_2, 0_2, 1_2)$ be two hyper residuated lattices. Then $L_1 \times L_2$ is also a hyper residuated lattice with respect to the point-wise hyper operations given by:

$$(a, b) * (u, v) = (a *_1 u, b *_2 v),$$

for any $(a, b), (u, v) \in L_1 \times L_2$, where $* \in \{\vee, \wedge, \otimes, \to\}$.

Definition 3.13. *Let μ and ν be fuzzy sets of L_1 and L_2, respectively, where L_1 and L_2 are two hyper residuated lattices. The cartesian product $\mu \times \nu$ of μ and ν is defined by*

$$(\mu \times \nu)(x, y) = \mu(x) \wedge \nu(y),$$

for any $(x, y) \in L_1 \times L_2$. Obviously, $\mu \times \nu$ is a fuzzy set of $L_1 \times L_2$.

Proposition 3.14. *Let L_1 and L_2 be two hyper residuated lattices. If μ and ν are fuzzy hyper deductive systems of L_1 and L_2, respectively, then $\mu \times \nu$ is a fuzzy hyper deductive system of $L_1 \times L_2$.*

Proof. For any $(x, y), (a, b) \in L_1 \times L_2$, we have $(\mu \times \nu)(x, y) = \mu(x) \wedge \nu(y) \leq \mu(1_1) \wedge \nu(1_2) = (\mu \times \nu)(1_1, 1_2)$. And

$$\begin{aligned}(\mu \times \nu)(a, b) &= \mu(a) \wedge \nu(b) \\ &\geq \Big(\bigvee \mu(x \to_1 a) \wedge \mu(x)\Big) \wedge \Big(\bigvee \nu(y \to_2 b) \wedge \nu(y)\Big) \\ &= \Big(\bigvee \mu(x \to_1 a) \wedge \bigvee \nu(y \to_2 b)\Big) \wedge (\mu(x) \wedge \nu(y)) \\ &= \bigvee (\mu \times \nu)\left((x, y) \to (a, b)\right) \wedge (\mu \times \nu)((x, y)).\end{aligned}$$

Therefore, $\mu \times \nu$ is a fuzzy hyper deductive system of $L_1 \times L_2$. □

Proposition 3.15. *Let μ be a fuzzy set of L. Then μ is a fuzzy hyper deductive system of L if and only if $\mu \times \mu$ is a fuzzy hyper deductive system of $L \times L$.*

Proof. The sufficiency is very clear by Proposition 3.14, we only need to give the proof of the necessity. We first show that $\mu(x) \leq \mu(1)$ for any $x \in L$. In fact that $\mu(x) = \mu(x) \wedge \mu(x) = (\mu \times \mu)(x, x) \leq (\mu \times \mu)(1, 1) = \mu(1) \wedge \mu(1) = \mu(1)$, hence $\mu(x) \leq \mu(1)$. Moreover, due to $1 \in 1 \to 1$, we get that $\bigvee \mu(1 \to 1) = \mu(1)$.

For $x, y \in L$, it follows that

$$\begin{aligned}\mu(y) &= \mu(y) \wedge \mu(1) \\ &= (\mu \times \mu)(y, 1) \\ &\geq \bigvee (\mu \times \mu)((x, 1) \to (y, 1)) \wedge (\mu \times \mu)(x, 1) \\ &= \bigvee (\mu \times \mu)(x \to y, 1 \to 1) \wedge (\mu \times \mu)(x, 1) \\ &= \bigvee \mu(x \to y) \wedge \bigvee \mu(1 \to 1) \wedge \mu(x) \wedge \mu(1) \\ &= \bigvee \mu(x \to y) \wedge \mu(x).\end{aligned}$$

Therefore, μ is a fuzzy hyper deductive system of L. □

Definition 3.16. *Let μ and ν be fuzzy sets of L. Then we define the fuzzy product $\mu \circledast \nu$ as*

$$(\mu \circledast \nu)(t) = \begin{cases} \bigvee_{t \in x \otimes y} (\mu(x) \wedge \nu(y)), & if\ \exists x, y \in L \text{ such that } t \in x \otimes y, \\ 0, & otherwise, \end{cases}$$

for any $t \in L$.

Theorem 3.17. *Let μ be a fuzzy set of L with the sup-property. If μ is order preserving, that is, $x \leq y$ implies $\mu(x) \leq \mu(y)$ for any $x, y \in L$, then μ is a fuzzy hyper deductive system of L if and only if $\mu \circledast \mu \leq \mu$.*

Proof. Let μ be a fuzzy hyper deductive system of L and $x \in L$. If there do not exist $y, z \in L$ such that $x \in y \otimes z$, then $(\mu \circledast \mu)(x) = 0 \leq \mu(x)$, and so

$\mu \circledast \mu \leq \mu$. Otherwise, notice that μ is a fuzzy hyper deductive system of L, we get that $\mu(x) \geq \bigwedge_{x\in y\otimes z} \mu(x) \geq \mu(y) \wedge \mu(z)$ by Proposition 3.9. Hence

$$(\mu \circledast \mu)(x) = \bigvee_{x\in y\otimes z} (\mu(y) \wedge \mu(z)) \leq \bigvee_{x\in y\otimes z} \mu(x) = \mu(x),$$

and so $\mu \circledast \mu \leq \mu$ is valid.

Conversely, assume that $\mu \circledast \mu \leq \mu$ hold. Then for every $x \in y \otimes z$, we have $\mu(x) \geq (\mu \circledast \mu)(x) = \bigvee_{x\in y\otimes z} (\mu(y) \wedge \nu(z)) \geq \mu(y) \wedge \nu(z)$. Thus $\bigwedge \mu(y \otimes z) = \bigwedge_{x\in y\otimes z} \mu(x) \geq \mu(y) \wedge \nu(z)$, from Proposition 3.10 we get that μ is a fuzzy hyper deductive system of L. □

Theorem 3.18. *Let L_1 and L_2 be two hyper residuated lattices, $f : L_1 \to L_2$ be a surjective homomorphism. Then the fuzzy set $\nu : L_2 \to [0,1]$ is a fuzzy hyper deductive system of L_2 if and only if $\nu_f : L_1 \to [0,1]$ is a fuzzy hyper deductive system of L_1, where $\nu_f : L_1 \to [0,1]$ is defined by $\nu_f(x) = \nu(f(x))$ for any $x \in L_1$.*

Proof. Suppose that ν is a fuzzy hyper deductive system of L_2, then

$$\nu_f(1_1) = \nu(f(1_1)) = \nu(1_2) \geq \nu(f(x)) = \nu_f(x).$$

For any $x, y \in L_1$,

$$\begin{aligned}
\nu_f(y) &= \nu(f(y)) \\
&\geq \bigvee \nu(f(x) \to_2 f(y)) \wedge \nu(f(x)) \\
&= \bigvee \nu(f(x \to_1 y)) \wedge \nu(f(x)) \\
&= \bigvee \nu_f(x \to_1 y) \wedge \nu_f(x).
\end{aligned}$$

Therefore, ν_f is a fuzzy hyper deductive system of L_1.

Conversely, assume that ν_f is a fuzzy hyper deductive system of L_1. Since f is surjective, then there exists $x \in L_1$ such that $f(x) = y$ for any $y \in L_2$. We get that

$$\nu(1_2) = \nu(f(1_1)) = \nu_f(1_1) \geq \nu_f(x) = \nu(f(x)) = \nu(y).$$

Let $a, b \in L_2$. Then there exists $x, y \in L_1$ such that $f(x) = a$ and $f(y) = b$. It follows that

$$\begin{aligned}
\nu(b) &= \nu(f(y)) \\
&= \nu_f(y) \\
&\geq \bigvee \nu_f(x \to_1 y) \wedge \nu_f(x) \\
&= \bigvee \nu(f(x \to_1 y)) \wedge \nu(f(x)) \\
&= \bigvee \nu(f(x) \to_2 f(y)) \wedge \nu(f(x)) \\
&= \bigvee \nu(a \to_2 b) \wedge \nu(a).
\end{aligned}$$

Hence, ν is a fuzzy hyper deductive system of L_2. □

Proposition 3.19. *Let L_1 and L_2 be two hyper residuated lattices, $f : L_1 \to L_2$ be a homomorphism, the fuzzy set $\nu : L_2 \to [0, 1]$ is a fuzzy hyper deductive system of L_2. Then we have*

(1) *if $x \in ker(f) := \{x \in L_1 | f(x) = 1_2\}$, then $\nu_f(x) \geq \nu(y)$ for any $y \in L_2$,*

(2) *if μ is a fuzzy hyper deductive system of L_1 and f is an epimorphism, then $f(\mu) : L_2 \to [0, 1]$ is a fuzzy hyper deductive system of L_2, where $f(\mu)(y) = \bigvee\{\mu(x)|f(x) = y\}$ for any $y \in L_2$.*

Proof. (1) For any $x \in ker(f)$, we get $f(x) = 1_2$. Notice that ν is a fuzzy hyper deductive system of L_2, then $\nu_f(x) = \nu(f(x)) = \nu(1_2) \geq \nu(y)$ for any $y \in L_2$.

(2) For any $y \in L_2$, we have

$$f(\mu)(y) = \bigvee\{\mu(x)|f(x) = y\} \leq \mu(1_1) = \bigvee\{\mu(x)|f(x) = 1_2\} = f(\mu)(1_2).$$

For any $a, b \in L_2$,

$$\begin{aligned}
&f(\mu)(a) \wedge f(\mu)(a \to_1 b)\\
&= \bigvee\{\mu(x)|f(x) = a, x \in L_1\} \wedge \bigvee\{\mu(x \to_1 y)|f(x \to_1 y) = a \to_2 b, x, y \in L_1\}\\
&= \bigvee\{\mu(x)|f(x) = a, x \in L_1\} \wedge \bigvee\{\mu(x \to y)|f(x) \to_2 f(y) = a \to_2 b, x, y \in L_1\}\\
&\geq \bigvee\{\mu(x) \wedge \bigvee \mu(x \to_1 y)|f(x) = a, f(y) = b, x, y \in L_1\}\\
&\geq \bigvee\{\mu(y)|f(y) = b, y \in L_1\}\\
&= f(\mu)(b),
\end{aligned}$$

thus $f(\mu)$ is a fuzzy hyper deductive system of L_2. □

Theorem 3.20. *Let μ be a fuzzy hyper deductive system of L. Then there exists a fuzzy hyper deductive system $\bar{\mu}$ of L/θ such that $\bar{\mu} \circ \pi \geq \mu$.*

$$\begin{array}{ccc}
L & \xrightarrow{\pi} & L/\theta \\
& \overset{\mu}{\searrow} & \downarrow \bar{\mu} \\
& & [0,1]
\end{array}$$

Proof. Define $\bar{\mu} : L/\theta \to [0, 1]$ by $\bar{\mu}([x]) = \bigvee_{t \in [x]} \mu(t)$, for any $x \in L$. Obviously, $\bar{\mu}$ is well defined. Let $x, y \in L$. Then for any $a \in [x]$ and $b \in [y]$, we get $\bigvee_{b \in [x]} \mu(b) \geq \mu(b) \geq \bigvee \mu(a \to b) \wedge \mu(a)$, and so

$$\begin{aligned}
\bar{\mu}([y]) &= \bigvee_{b \in [y]} \mu(b)\\
&= \bigvee_{a \in [x]} \bigvee_{b \in [y]} \mu(b)\\
&\geq \bigvee_{a \in [x]} \bigvee_{b \in [y]} \Big(\bigvee \mu(a \to b) \wedge \mu(a)\Big)
\end{aligned}$$

$$\geq \bigvee_{a\in[x]}\bigvee_{b\in[y]}\Big(\bigvee \mu(a\to b)\Big)\wedge \bigvee_{a\in[x]}\bigvee_{b\in[y]}\mu(a)$$
$$\geq \bigvee_{a\to b\in[x\to b]}\Big(\bigvee \mu(a\to b)\Big)\wedge \bigvee_{a\in[x]}\mu(a)$$
$$= \bigvee \bar{\mu}([a]\rightsquigarrow[b])\wedge\bar{\mu}([a]).$$

Moreover, $\bar{\mu}([1]) = \bigvee_{t\in[1]}\mu(t) = \mu(1) \geq \bigvee_{t\in[x]}\mu(t) = \bar{\mu}([x])$, thus $\bar{\mu}$ is a fuzzy hyper deductive system of L/θ and clearly $\bar{\mu}\circ\pi \geq \mu$. □

4. Hyper deductive systems based on falling shadows

Given an universal set U, and let $\mathscr{P}(U)$ denote the power set of U. For any $u \in U$ and $D \subseteq U$, let $\dot{u} = \{E|u \in E, E \subseteq U\}$ and $\dot{D} = \{\dot{u}|u \in D\}$. An order pair $(\mathscr{P}(U), \mathscr{B})$ is called to be a hyper-measurable structure on U if $\mathscr{B}$ is a σ-field in $\mathscr{P}(U)$, and $\dot{U} \subseteq \mathscr{B}$.

Let $(\Omega, \mathscr{A}, P)$ be a probability space and $(\mathscr{P}(U), \mathscr{B})$ be a hyper-measurable structure on U. If a map $\xi : \Omega \to \mathscr{P}(U)$ is a random set on U, and for any $C \in \mathscr{B}$,

$$\xi^{-1}(C) = \{\omega \in \Omega|\xi(\omega) \in C\} \in \mathscr{A},$$

then ξ is called a $\mathscr{A} \to \mathscr{B}$ measurable .

Given a random set ξ on U. Consider a function $\tilde{H}(x) := P\{\omega|x \in \xi(\omega)\}$ for any $x \in U$, then $\tilde{H}$ is a fuzzy set of U, and we call it a falling shadow of the random set ξ, and ξ is called a cloud of $\tilde{H}$.

Definition 4.1. *Consider a probability space $(\Omega, \mathscr{A}, P)$, and let $\xi : \Omega \to \mathscr{P}(L)$ be a random set. If $\xi(\omega)$ is a hyper deductive system of L for any $\omega \in \Omega$, then the falling shadow $\tilde{H}$ of the random set ξ, that is, for any $x \in L$,*

$$\tilde{H}(x) = P(\omega|x \in \xi(\omega)),$$

is called a falling fuzzy hyper deductive system of L.

Example 4.2. Let L be a hyper residuated lattice defined in Example 3.5, and $(\Omega, \mathscr{A}, P) = ([0, 1], \mathscr{A}, m)$, where $\mathscr{A}$ is a Borel field on $[0, 1]$ and m is the usual Lebesgue measure. The mapping $\xi : \Omega \to \mathscr{P}(L)$ is defined by

$$\xi(t) = \begin{cases} \{1\}, & t \in [0, 0.4), \\ \{b, 1\}, & t \in [0.4, 0.6), \\ \{0, a, b, 1\}, & t \in [0.6, 1], \end{cases}$$

then $\xi(t)$ is a hyper deductive system of L for any $t \in [0, 1]$. Thus $\tilde{H}$ is a falling fuzzy hyper deductive system of L, where $\tilde{H}(x) = P(t|x \in \xi(t))$ is represented as follows:

$$\tilde{H}(x) = \begin{cases} 0.4, & x = 0, \\ 0.4, & x = a, \\ 0.6, & x = b, \\ 1, & x = 0. \end{cases}$$

Proposition 4.3. *If we consider a probability space $(\Omega, \mathscr{A}, P) = ([0,1], \mathscr{A}, m)$, where $\mathscr{A}$ is a Borel field on $[0,1]$ and m is the usual Lebesgue measure, then every fuzzy hyper deductive system of a hyper residuated lattice is a falling fuzzy hyper deductive system.*

Proof. Let μ is a fuzzy hyper deductive system of L, then μ_t is a hyper deductive system of L for any $t \in [0,1]$ by Proposition 3.6. Define the random set $\xi : [0,1] \to \mathscr{P}(A)$ by $\xi(t) = \mu_t$ for any $t \in [0,1]$, then μ is a falling fuzzy hyper deductive system of L. □

Given a probability space $(\Omega, \mathscr{A}, P)$, and let $\tilde{H}$ be a falling shadow of a random set $\xi : \Omega \to \mathscr{P}(M)$. We define

$$\Omega(K;\xi) := \begin{cases} \{\omega \in \Omega | K \in \xi(\omega)\}, & K \in M, \\ \{\omega \in \Omega | K \cap \xi(\omega) \neq \emptyset\}, & K \subseteq M, \end{cases}$$

then $\Omega(K;\xi) \in \mathscr{A}$. In what follows, we give a number of equivalent conditions of falling fuzzy hyper deductive systems for further discussion.

Theorem 4.4. *Let $\xi : \Omega \to \mathscr{P}(L)$ be a random set and $\tilde{H}$ a falling shadow of ξ. Then $\tilde{H}$ is a falling fuzzy hyper deductive system of L if and only if for any $x, y \in L$,*

(1) $\Omega(x;\xi) \subseteq \Omega(1;\xi)$;

(2) $\Omega(x \to y;\xi) \cap \Omega(x;\xi) \subseteq \Omega(y;\xi)$.

Proof. Suppose that $\tilde{H}$ is a falling fuzzy hyper deductive system of L, then $\xi(\omega)$ is a hyper deductive system of L for any $\omega \in \Omega(x;\xi)$. It follows that $1 \in \xi(\omega)$, that is, $\omega \in \Omega(1;\xi)$, and so $\Omega(x;\xi) \subseteq \Omega(1;\xi)$. For any $\omega \in \Omega(x \to y;\xi) \cap \Omega(x;\xi)$, we get that $(x \to y) \cap \xi(\omega) \neq \emptyset$ and $x \in \xi(\omega)$. From Proposition 2.4 we have $y \in \xi(\omega)$, i.e., $\omega \in \Omega(y;\xi)$, therefore $\Omega(x \to y;\xi) \cap \Omega(x;\xi) \subseteq \Omega(y;\xi)$.

Conversely, assume that (1) and (2) hold. Let $\omega \in \Omega$ be such that $\xi(\omega) \neq \emptyset$. Then there exists $x \in \xi(\omega)$, that is, $\omega \in \Omega(x;\xi) \subseteq \Omega(1;\xi)$, and so $1 \in \xi(\omega)$. Let $x, y \in L$ be such that $(x \to y) \cap \xi(\omega) \neq \emptyset$ and $x \in \xi(\omega)$, that is, $\omega \in \Omega(x \to y;\xi)$ and $\omega \in \Omega(x;\xi)$. It follows that $\omega \in \Omega(x \to y;\xi) \cap \Omega(x;\xi) \subseteq \Omega(y;\xi)$, therefore $y \in \xi(\omega)$. According to the above arguments, we get that $\xi(\omega)$ is a hyper deductive system of L for any $\omega \in \Omega$, thus $\tilde{H}$ is a falling fuzzy hyper deductive system of L. □

Proposition 4.5. *Consider a falling shadow $\tilde{H}$ of a random set $\xi : \Omega \to \mathscr{P}(L)$. If $\tilde{H}$ is a falling fuzzy hyper deductive system of L, then the following statements hold: for any $x, y, z \in L$,*

(1) $\Omega(x;\xi) \cap \Omega(y;\xi) \subseteq \Omega(x \otimes y;\xi)$,

(2) $x \leq y$ *implies* $\Omega(x;\xi) \subseteq \Omega(y;\xi)$,

(3) $x \otimes y \ll z$ *implies* $\Omega(x;\xi) \cap \Omega(y;\xi) \subseteq \Omega(z;\xi)$,

(4) *for any nonempty subset A of L, if $x \ll A$, then* $\Omega(x;\xi) \subseteq \Omega(A;\xi)$,

(5) *for any nonempty subset A of L,* $\Omega(x \to A;\xi) \cap \Omega(x;\xi) \subseteq \Omega(A;\xi)$,

(6) $\Omega(x \to y;\xi) \cap \Omega(x;\xi) = \Omega(x;\xi) \cap \Omega(y;\xi)$,

(7) $\Omega(1 \to y;\xi) \cap \Omega(1;\xi) = \Omega(y;\xi)$,

(8) *if $x \leq y$, then* $\Omega(x \to y;\xi) \cap \Omega(x;\xi) = \Omega(x;\xi)$,

(9) *if $\Omega(x \to y;\xi) = \Omega(1;\xi)$, then* $\Omega(x;\xi) \subseteq \Omega(y;\xi)$,

Proof. (1) Assume that $\tilde{H}$ is a falling fuzzy hyper deductive system of L, then $\xi(\omega)$ is a hyper deductive system of L for any $\omega \in \Omega(x;\xi)$. For any $x, y \in L$, if $x \in \xi(\omega)$, $y \in \xi(\omega)$, then $x \otimes y \subseteq \xi(\omega)$ by Theorem 3.1, and so $(x \otimes y) \cap \xi(\omega) \neq \emptyset$, this yields $\omega \in \Omega(x \otimes y;\xi)$. Therefore, $\Omega(x;\xi) \cap \Omega(y;\xi) \subseteq \Omega(x \otimes y;\xi)$.

(2) Let $x \leq y$. For any $\omega \in \Omega(x;\xi)$, that is, $x \in \xi(\omega)$, then $y \in \xi(\omega)$. Hence $\omega \in \Omega(y;\xi)$, and so $\Omega(x;\xi) \subseteq \Omega(y;\xi)$.

(3) Let $x, y, z \in M$ be such that $x \otimes y \ll z$. For any $\omega \in \Omega(x;\xi) \cap \Omega(y;\xi)$, we have $x \in \xi(\omega)$ and $y \in \xi(\omega)$. By hypothesis that $\tilde{H}$ is a falling fuzzy hyper deductive system of L, we get that $\xi(\omega)$ is a hyper deductive system of L. From Theorem 3.1, it follows that $x \otimes y \subseteq \xi(\omega)$. Notice that $x \otimes y \ll z$, there exists $u \in x \otimes y \subseteq \xi(\omega)$ such that $u \leq z$, therefore $z \in \xi(\omega)$, that is, $\omega \in \Omega(z;\xi)$. Hence $\Omega(x;\xi) \cap \Omega(y;\xi) \subseteq \Omega(z;\xi)$.

(4) For any $\omega \in \Omega(x;\xi)$, we have $x \in \xi(\omega)$. Since $x \ll A$, then there exists $a \in A$ such that $x \leq a$. By hypothesis, $\xi(\omega)$ is a hyper deductive system of L, therefore $a \in \xi(\omega)$ by Theorem 3.1. Hence $A \cap \xi(\omega) \neq \emptyset$, and so $\omega \in \Omega(A;\xi)$, it follows that $\Omega(x;\xi) \subseteq \Omega(A;\xi)$.

(5) For any $\omega \in \Omega(x \to A;\xi) \cap \Omega(x;\xi)$, we get that $(x \to A) \cap \xi(\omega) \neq \emptyset$ and $x \in \xi(\omega)$. Then there exists $a \in A$ such that $(x \to a) \cap \xi(\omega) \neq \emptyset$, it follows from Proposition 2.4 that $a \in \xi(\omega)$. And therefore, $A \cap \xi(\omega) \neq \emptyset$, that is, $\omega \in \Omega(A;\xi)$, hence $\Omega(x \to A;\xi) \cap \Omega(x;\xi) \subseteq \Omega(A;\xi)$.

(6) From Theorem 4.4, we get $\Omega(x \to y;\xi) \cap \Omega(x;\xi) \subseteq \Omega(y;\xi)$, and so $\Omega(x \to y;\xi) \cap \Omega(x;\xi) \subseteq \Omega(x;\xi) \cap \Omega(y;\xi)$. To obtain the reverse inclusion, observe that $y \ll x \to y$, we get $\Omega(y;\xi) \subseteq \Omega(x \to y;\xi)$, moreover $\Omega(y;\xi) \cap \Omega(x;\xi) \subseteq \Omega(x \to y;\xi) \cap \Omega(x;\xi)$. And thus $\Omega(x \to y;\xi) \cap \Omega(x;\xi) = \Omega(x;\xi) \cap \Omega(y;\xi)$.

(7)–(9) The proofs directly follow from (6) and Theorem 4.4. □

Definition 4.6. *Give a probability space $(\Omega, \mathscr{A}, P)$, and let $\xi : \Omega \to \mathscr{P}(L)$ be a random set. For any $\omega \in \Omega$, if $(x \otimes y) \cap \xi(\omega) \neq \emptyset$ implies $x \otimes y \subseteq \xi(\omega)$ for any $x, y \in L$, then ξ is said to be a $S_{\otimes}$-reflexive random set.*

Proposition 4.7. *Let $\xi : \Omega \to \mathscr{P}(L)$ be a $S_{\otimes}$-reflexive random set, and $\tilde{H}$ be a falling shadow of ξ. If the following conditions are valid: for any $x, y, z \in L$,*

(1) $\Omega(x;\xi) \cap \Omega(y;\xi) \subseteq \Omega(x \otimes y;\xi)$,

(2) $x \leq y$ *implies* $\Omega(x;\xi) \subseteq \Omega(y;\xi)$,

then $\tilde{H}$ is a falling fuzzy hyper deductive system of L.

Proof. Assume that (1) and (2) hold. For any $\omega \in \Omega$, if $x \leq y$ and $x \in \xi(\omega)$, then $\omega \in \Omega(x;\xi) \subseteq \Omega(y;\xi)$, and so $y \in \xi(\omega)$. Now let $x \in \xi(\omega)$ and $y \in \xi(\omega)$. Then $\omega \in \Omega(x;\xi)$ and $\omega \in \Omega(y;\xi)$. By hypothesis, $\omega \in \Omega(x;\xi) \cap \Omega(y;\xi) \subseteq \Omega(x \otimes y;\xi)$, and so $(x \otimes y) \cap \xi(\omega) \neq \emptyset$. Due to the fact that ξ is a $S_{\otimes}$-reflexive random set, we get $x \otimes y \subseteq \xi(\omega)$. Thus $\xi(\omega)$ is a hyper deductive system of L for any $\omega \in \Omega$, so $\tilde{H}$ is a falling fuzzy hyper deductive system of L. □

In the following, we give the notion of falling fuzzy hyper implicative deductive systems, and present some properties of it.

Definition 4.8. *Let $(\Omega, \mathscr{A}, P)$ be a probability space and $\xi : \Omega \to \mathscr{P}(L)$ be a random set. If $\xi(\omega)$ is a hyper implicative deductive system of L for any $\omega \in \Omega$, then the falling shadow $\tilde{H}$ of the random set ξ, that is, for any $x \in L$,*

$$\tilde{H}(x) = P(\omega | x \in \xi(\omega)),$$

is called a falling fuzzy hyper implicative deductive system of L.

Example 4.9. Let $L = \{0, a, b, c, 1\}$ be a set such that $0 < a < b < 1$ and $0 < c < 1$. The hyperoperations $\otimes$ and $\to$ are given in the following Cayley tables.

⊗	0	a	b	c	1
0	{0}	{0}	{0}	{0}	{0}
a	{0}	{a}	{a}	{0}	{a}
b	{0}	{a}	{a, b}	{0}	{a, b}
c	{0}	{0}	{0}	{c}	{c}
1	{0}	{a}	{a, b}	{c}	{c}

→	0	a	b	c	1
0	{1}	{1}	{1}	{1}	{1}
a	{c}	{1}	{1}	{1}	{1}
b	{c}	{a, b, c}	{1}	{c}	{1}
c	{a, b}	{a, b}	{a, b}	{1}	{1}
1	{0}	{a}	{a, b}	{c}	{1}

Let $x \wedge y = \{u \in L | u \leq x, u \leq y\}$ and $x \vee y = \{u \in L | x \leq u, y \leq u\}$ for any $x, y \in L$. It is easy to check that $(L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ is a hyper residuated lattice [19]. Let $(\Omega, \mathscr{A}, P) = ([0,1], \mathscr{A}, m)$, where $\mathscr{A}$ is a Borel field on $[0,1]$ and m is the usual Lebesgue measure. The mapping $\xi : \Omega \rightarrow \mathscr{P}(L)$ is defined by

$$\xi(t) = \begin{cases} \{a, 1\}, & t \in [0, 0.2), \\ \{c, 1\}, & t \in [0.2, 0.6), \\ \{a, b, 1\}, & t \in [0.6, 0.7), \\ \{0, a, b, c, 1\}, & t \in [0.7, 1], \end{cases}$$

then $\xi(t)$ is a hyper implicative deductive system of L for any $t \in [0,1]$. Thus $\tilde{H}$ is a falling fuzzy hyper implicative deductive system of L, where $\tilde{H}(x) = P(t | x \in \xi(t))$ is represented as follows:

$$\tilde{H}(x) = \begin{cases} 0.3, & x = 0, \\ 0.6, & x = a, \\ 0.4, & x = b, \\ 0.7, & x = c, \\ 1, & x = 1. \end{cases}$$

Theorem 4.10. *Let $\xi : \Omega \rightarrow \mathscr{P}(L)$ be a random set and $\tilde{H}$ be a falling shadow of ξ. Then $\tilde{H}$ is a falling fuzzy hyper implicative deductive system of L if and only if for any $x, y, z \in L$,*

(1) $\Omega(x; \xi) \subseteq \Omega(1; \xi)$,

(2) $\Omega(x \rightarrow (y \rightarrow z); \xi) \cap \Omega(x \rightarrow y; \xi) \subseteq \Omega(x \rightarrow z; \xi)$.

Proof. Assume that $\tilde{H}$ is a falling fuzzy hyper implicative deductive system of L, then $\xi(\omega)$ is a hyper implicative deductive system of L for any $\omega \in \Omega(x; \xi)$. Therefore $1 \in \xi(\omega)$, that is, $\omega \in \Omega(1; \xi)$, and so $\Omega(x; \xi) \subseteq \Omega(1; \xi)$. For any $\omega \in \Omega(x \rightarrow (y \rightarrow z); \xi) \cap \Omega(x \rightarrow y; \xi)$, we have $(x \rightarrow (y \rightarrow z)) \cap \xi(\omega) \neq \emptyset$ and $(x \rightarrow y) \cap \xi(\omega) \neq \emptyset$. From the definition of hyper implicative deductive system, it follows that $(x \rightarrow z) \cap \xi(\omega) \neq \emptyset$, i.e., $\omega \in \Omega(x \rightarrow z; \xi)$, therefore $\Omega(x \rightarrow (y \rightarrow z); \xi) \cap \Omega(x \rightarrow y; \xi) \subseteq \Omega(x \rightarrow z; \xi)$.

Conversely, suppose that (1) and (2) hold. Let $\omega \in \Omega$ be such that $\xi(\omega) \neq \emptyset$. Then there exists $x \in \xi(\omega)$, that is, $\omega \in \Omega(x; \xi) \subseteq \Omega(1; \xi)$, and so $1 \in \xi(\omega)$. Let $x, y, z \in L$ be such that $x \rightarrow (y \rightarrow z) \cap \xi(\omega) \neq \emptyset$ and $(x \rightarrow y) \cap \xi(\omega) \neq \emptyset$, that is, $\omega \in \Omega(x \rightarrow (y \rightarrow z); \xi)$ and $\omega \in \Omega(x \rightarrow y; \xi)$. It follows that $\omega \in \Omega(x \rightarrow (y \rightarrow z); \xi) \cap \Omega(x \rightarrow y; \xi) \subseteq \Omega(x \rightarrow z; \xi)$, therefore $(x \rightarrow z) \cap \xi(\omega) \neq \emptyset$. Hence $\xi(\omega)$ is a hyper implicative deductive system of L for any $\omega \in \Omega$, thus $\tilde{H}$ is a falling fuzzy hyper implicative deductive system of L. □

Due to the fact that if a hyper implicative deductive system D of L is an upset, then D is a hyper deductive system [19], it is easy to obtain the following result.

Proposition 4.11. *Give a falling shadow $\tilde{H}$ of a random set $\xi : \Omega \to \mathscr{P}(L)$. If $\xi(\omega)$ is an upset for any $\omega \in \Omega$, then a falling fuzzy hyper implicative deductive system $\tilde{H}$ of L is a falling fuzzy hyper deductive system.*

For any nonempty subsets A, B of L, $A \leq B$ means that for any $a \in A$, there exists $b \in B$ such that $a \leq b$.

Lemma 4.12 ([19])**.** *Let L be a hyper residuated lattice. Then the following hold: for any $x, y, z \in L$,*

(1) $x \to (y \to z) \leq y \to (x \to z)$,

(2) $x \leq y$ *implies* $z \to x \leq z \to y$,

(3) $x \to y \leq (y \to z) \to (x \to z)$.

Proposition 4.13. *Let L be a hyper residuated lattice. Then we have:*

(1) $x \ll A$ *implies* $y \to x \leq y \to A$, *for any nonempty subset* $A \subseteq L$ *and* $x, y \in L$,

(2) *let D be a hyper deductive system of L. For any nonempty subsets A and B of L, if $A \cap D \neq \emptyset$ and $A \leq B$, then $B \cap D \neq \emptyset$.*

Proof. (1) Since $x \ll A$, then there exists $a \in A$ such that $x \leq a$. From Lemma 4.12 (2), we get that $y \to x \leq y \to a$, and so $y \to x \leq y \to A$.

(2) Let $A \cap D \neq \emptyset$. Then there exists $a \in A \cap D$, that is, $a \in A$ and $a \in D$. Notice that $A \leq B$, then there exists $b \in B$ such that $a \leq b$. It follows from Theorem 3.1 that $b \in D$, and therefore $B \cap D \neq \emptyset$. □

In the following, we give some conditions for a falling fuzzy hyper deductive system to be a falling fuzzy hyper implicative deductive system.

Theorem 4.14. *Let $\tilde{H}$ be a falling fuzzy hyper deductive system of L. Then the following statements are equivalent: for any $x, y, z \in L$,*

(1) *$\tilde{H}$ is a falling fuzzy hyper implicative deductive system of L,*

(2) $\Omega(y \to (y \to x); \xi) \subseteq \Omega(y \to x; \xi)$,

(3) $\Omega(z \to (y \to (y \to x)); \xi) \cap \Omega(z; \xi) \subseteq \Omega(y \to x; \xi)$.

Proof. (1) $\Rightarrow$ (2) We first show the fact that $\Omega(y \to y; \xi) = \Omega$. For any $\omega \in \Omega$, we get $1 \in \xi(\omega)$. In view that $1 \in y \to y$, we obtain $(y \to y) \cap \xi(\omega) \neq \emptyset$, that is, $\omega \in \Omega(y \to y; \xi)$. Therefore $\Omega \subseteq \Omega(y \to y; \xi)$, and so $\Omega(y \to y; \xi) = \Omega$. According to Theorem 4.10, we get that $\Omega(y \to (y \to x); \xi) = \Omega(y \to (y \to x); \xi) \cap \Omega(y \to y; \xi) \subseteq \Omega(y \to x; \xi)$.

(2) $\Rightarrow$ (3) Notice that $\tilde{H}$ is a falling fuzzy hyper deductive system of L, we get that $\Omega(z \to (y \to (y \to x)); \xi) \cap \Omega(z; \xi) \subseteq \Omega(y \to (y \to x); \xi) \subseteq \Omega(y \to x; \xi)$ by hypothesis and Proposition 4.5 (5).

(3) $\Rightarrow$ (1) Let $x, y, z \in L$ be such that $(x \to (y \to z)) \cap \xi(\omega) \neq \emptyset$ and $(x \to y) \cap \xi(\omega) \neq \emptyset$. Since $x \to (y \to z) \leq y \to (x \to z)$, then $y \to (x \to z) \cap \xi(\omega) \neq \emptyset$ by Proposition 4.13, and so there exists $t \in y \to (x \to z) \cap \xi(\omega)$. Moreover, $t \ll y \to (x \to z) \cap \xi(\omega)$, therefore $y \ll t \to (x \to z)$. Using Proposition 4.13, we get that $x \to y \leq x \to (t \to (x \to z) \leq t \to (x \to (x \to z)$. In view that $(x \to y) \cap \xi(\omega) \neq \emptyset$, we get that $(t \to (x \to (x \to z)) \cap \xi(\omega) \neq \emptyset$. By the above discussing, we know that $\omega \in \Omega(t \to (x \to (x \to z); \xi) \cap \Omega(t; \xi) \subseteq \Omega(x \to z; \xi)$. It follows that $\omega \in \Omega(x \to z; \xi)$, i.e. $(x \to z) \cap \xi(\omega) \neq \emptyset$. Hence we conclude that $(x \to (y \to z)) \cap \xi(\omega) \neq \emptyset$ and $(x \to y) \cap \xi(\omega) \neq \emptyset$ imply $(x \to z) \cap \xi(\omega) \neq \emptyset$, so $\xi(\omega)$ is a hyper implicative deductive system of L, and therefore $\tilde{H}$ is a falling fuzzy hyper implicative deductive system of L. □

Theorem 4.15. *Let $\tilde{H}$ be a falling fuzzy hyper deductive system of L. Then $\tilde{H}$ is a falling fuzzy hyper implicative deductive system of L if and only if $(x \to u) \cap \xi(\omega) \neq \emptyset$ for any $u \in x \otimes x$ and $\omega \in \Omega$.*

Proof. It is directly from Theorem 4.8 of [19]. □

Acknowledgements

The works described in this paper are partially supported by Higher Education Key Scientific Research Program Funded by Henan Province (No. 18A110008, 18A630001, 18A110010) and Research and Cultivation Fund Project of Anyang Normal University (No. AYNUKP-2018-B25, No. AYNUKP-2018-B26).

References

[1] M. Ward, R.P. Dilworth, *Residuated lattices*, Transactions of the American Mathematical Society, 45 (1939), 335-354.

[2] P. He, X. Xin, J. Zhan. *On derivations and their fixed point sets in residuated lattices*, Fuzzy Sets and Systems, 303 (2016), 97-117.

[3] Y. Yang, X. Xin, *On characterizations of BL-algebras via implicative ideals*, Italian Journal of Pure and Applied Mathematics, 37 (2017), 493-506.

[4] J. Qiao, B.Q. Hu, *Granular variable precision L-fuzzy rough sets based on residuated lattices*, Fuzzy Sets and Systems, 336 (2018), 148-166.

[5] B. Zhao, P. He, *On non-commutative residuated lattices with internal states*, IEEE Transactions on Fuzzy Systems, 26 (2018), 1387-1400.

[6] D. Buşneag, D. Piciu, *A new approach for classification of filters in residuated lattices*, Fuzzy Sets and Systems, 260 (2015), 121-130.

[7] A. Kadji, C. Lele, M. Tonga, *Fuzzy prime and maximal filters of residuated lattices*, Soft Computing, 21(8) (2017), 1913-1922.

[8] X.H. Zhang, *Fuzzy anti-grouped filters and fuzzy normal filters in pseudo-BCI algebras*, Journal of Intelligent and Fuzzy Systems, 33 (2017), 1767-1774.

[9] X.H. Zhang, C. Park, S.P. Wu, *Soft set theoretical approach to pseudo-BCI algebras*, Journal of Intelligent and Fuzzy Systems, 34 (2018), 559-568.

[10] X. Yu, X. Xin, J. Wang, *Fuzzy filters on equality algebras with applications*, Journal of Intelligent & Fuzzy Systems, 35(3) (2018), 3709-3719.

[11] F. Marty, *Sur une generalization de la notion de groupe*, 8th Congress Math. Scandinaves, Stockholm, (1934), 45-49.

[12] P. Corsini, V. Leoreanu-Fotea, *Applications of hyperstructure theory*, Kluwer, Dordrecht, 2003.

[13] S. Ghorbani, A. Hasankhani, E. Eslami, *Hyper MV-algebras*, Set-Valued Mathematics and Applications, 2 (2008), 205-222.

[14] L. Torkzadeh, A. Ahadpanah, *Hyper MV-ideals in hyper MV-algebras*, Mathematical Logic Quarterly, 56 (2010), 51-62.

[15] Y.B. Jun, M.S. Kang, H.S. Kim, *New types of hyper MV-deductive systems in hyper MV-algebras*, Mathematical Logic Quarterly, 56 (2010), 400-405.

[16] Y.B. Jun, M.S. Kang, H.S. Kim, *Fuzzy structures of hyper-MV-deductive systems in hyper-MV-algebras*, Computers & Mathematics with Applications, 59 (2010), 2982-2989.

[17] X. Xin, P. He, Y. Fu, *States on hyper MV-algebras*, Journal of Intelligent & Fuzzy Systems, 31 (2016), 1299-1309.

[18] O. Zahiri, R.A Borzooei, M. Bakhshi, *(Quotient) hyper residuated lattices*, Quasigroups and Related Systems, 20 (2012), 125-138.

[19] R.A Borzooei, Bakhshi M, O. Zahiri, *Filter theory on hyper residuated lattices*, Quasigroups and Related Systems, 22 (2014), 33-50.

[20] R.A Borzooei, B.G. Saffar, R. Ameri, *On hyper EQ-algebras*, Italian Journal of Pure and Applied Mathematics, 31 (2013), 77-96.

[21] X. Cheng, X. Xin, Y.B. Jun, *Hyper equality algebras*, Quantitative Logic and Soft Computing 2016, (2017), 415-428.

[22] H.R. Varasteh, R.A. Borzooei, *Fuzzy regular relation on hyper hoop-algebras*, Journal of Intelligent & Fuzzy Systems, 30 (2016), 1275-1282.

[23] P.Z. Wang, *Fuzzy sets and falling shadows of random sets*, Beijing Normal University Press, Beijing, 1985.

[24] Y.B. Jun, M.S. Kang, *Fuzzy positive implicative ideals of BCK-algebras based on the theory of falling shadows*, Computers & Mathematics with Applications, 61 (2011), 62-67.

[25] J. Zhan, Y.B. Jun, H.K. Kim, *Some types of falling fuzzy filters of BL-algebras and its applications*, Journal of Intelligent & Fuzzy Systems, 26 (2014), 1675-1685.

[26] Y. Yang, X. Xin, P. He, *Characterizations of MV-algebras based on the theory of falling shadows*, The Scientific World Journal, 2014 (2014), 1-11.

[27] B. Meng, X. Xin, *Falling fuzzy Gödel ideals of BL-algebras*, Italian Journal of Pure and Applied Mathematics, 37 (2017), 237-258.

[28] J. Zhan, B. Yu, V.E. Fotea, *Characterizations of two kinds of hemirings based on probability spaces*, Soft Computing, 20 (2016), 637-648.

[29] Y. Li, A.B. Saeid, J. Wang, *Characterization of prefilters of EQ-algebra by falling shadow*, Journal of Intelligent & Fuzzy Systems, 33 (2017), 3805-3818.

Accepted: 1.03.2019