Numerical solution of fractional order differential equation with different methods

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Abstract. Various motion phenomena in natural environment and engineering is regular, but it is often impossible to describe them accurately with linear theory, which promotes the development of nonlinear theory. Nonlinear differential equation is one of them. Solving equations is an important part of theory. In this study, the exponential function expansion method, the first integral method and the wavelet operator matrix algorithm were introduced firstly. Then the steps of the three methods were explained. Finally, the concrete equations were solved by the three methods respectively. The fractional order STO equation was solved by the exponential function expansion method and the fractional order cahn-allen equation was solved by the first integral method. The wavelet operator matrix algorithm was used to solve the fractional order differential equations. The results showed that the exponential function expansion method could obtain the exact solution of STO smoothly and easily, but the amount of calculation was determined by the equilibrium principle. The first integral method could obtain the exact solution of cahn-allen equation smoothly and effectively, and there were many kinds of exact solutions when m was equal to 1. The wavelet operator matrix algorithm could get the exact solution easily and quickly, and with the increase of the value of m', the absolute error between the numerical solution and the exact solution decreased.

Keywords: fractional order differential equation, fractional order STO equation, fractional order cahn-allen equation, numerical solution.

1. Introduction

Different natural phenomena show regular forms, but they often can not be described by linear mathematics, which promotes the emergence of nonlinear theory [1]. Nonlinear partial differential equation is a part of nonlinear theory. Mathematical modeling based on nonlinear partial differential equation can effectively help analyze the mathematical essence of nonlinear natural phenomena [2]. Integral order differential equation has a wide application, and there are relevant studies. Albzeirat et al. [3] proposed a solution which was based on reproducing kernel Hilbert space tool for the numerical solution of second-order integer and fractional differential equations under fuzzy conditions and verified the effectiveness and flexibility of the method using an example. Saadatmandi et al. [4] put forward a method based on Tau method and Legendre polynomial to solve a class of time fractional telegraph equations. The Legendre matrix of fractional derivatives was used to appropriately represent the solution, and the numerical was simplified to the solution of a set of linear algebraic equations. However, integer differential equations often have several problems in simulating complex systems and phenomena [5]: model parameters setting is out of practice, model is susceptible to external interference, and the solution is complex. Therefore, fractional differential equations was proposed. There is no universal solution for nonlinear differential equations at present, so there are many exact solutions or approximate solutions. Muthukumar et al. [6] studied the numerical solution of fractional order delay differential equations and verified its applicability by several examples like a mathematical model of houseflies and a model based on the effect of noise on light that reflected from laser to mirror. Ray et al. [7] calculate the numerical solutions of parabolic fractional partial differential equation (PDE) and nonlinear time fractional partial differential equation with two-dimensional Legendre wavelet method and two-dimensional Haar wavelet method respectively and compared them with the exact solutions obtained by homotopy perturbation method. In this paper, the exponential function expansion method, the first integral method and the wavelet operator matrix algorithm were introduced, and then the steps of the three methods were explained. Finally, the exponential function expansion method was used to solve the fractional order STO equation, the first integral method was used to solve the fractional order cahn-allen equation, and the wavelet operator matrix algorithm was used to solve the fractional order differential equation.

2. Fractional order partial differential equation

Nonlinear partial differential equations include integer-order differential equations and fractional order differential equations, in which fractional order differential equations can describe complex nonlinear systems concisely and accurately; hence it is very important to solve the theoretical and numerical solutions of fractional-order differential equations. The related definitions of fractional differential equations are as follows.

Definition 1 ([8])**.** $f(x)$ is real function and $x > 0$, and moreover $f(x)$ is contained in space $C_\alpha(\alpha \in R)$. If $\exists k \in R$ and $k > \alpha$, making $f_1(x)$ contained in $C \in [0, \infty)$ and $f(x) = x^k f_1(x)$, then it can be considered that $f(x) \in C_a^b$ when and only when $f^{(b)}$ belongs to space C_α and $b \in N$.

Definition 2. β-order R-L fractional calculus is defined as:

$$\left(I^\beta f\right)(x) = \begin{cases} f(x), & \beta = 0 \\ \frac{1}{\Gamma(\beta)} \int_0^x (x-a)^{\beta-1} f(a) da, & \beta > 0, x > 0 \end{cases} \tag{1}$$

Its derivative is:

$$\left(D_l^\beta f\right)(x) = \begin{cases} \frac{d^n f(x)}{dt^n}, & \beta = n \in N \\ \frac{d^n}{dt^n}\frac{1}{\Gamma(m-\beta)}\int_\beta^x \frac{f(y)}{(x-y)^{\beta-m+1}}dy, & 0 \le n-1 < \beta < n \end{cases} \tag{2}$$

Definition 3: caputo frictional order calculus is defined as:

$$ {}_\beta^C D_x^p[f(x)] = \frac{1}{\Gamma(n-p)}\int_\beta^x (x-y)^{n-p-1} f^{(n)}(y)dy, \tag{3}$$
$$n = \lfloor \beta \rfloor + 1, n-1 < \beta \le n, x > \beta.$$

3. Two solutions of fractional order differential equation

3.1 Exponential function expansion method

Exponential function expansion method is used for solving integer order differential equation initially. The equation is processed by travelling wave exchange, and then nonlinear ordinary differential equation is introduced. The forms of the solution are diverse.

$$\Phi'(\delta) = g\left(e^{-\Phi(\delta)}\right) + h\left(e^{\Phi(\delta)}\right) + o \tag{4}$$

was selected as the nonlinear ordinary differential equation [9].

The general form of fractional order partial differential equation is:

$$f\left(s, s_y, s_x, D_x^\beta s, D_y^\beta s, D_y^\gamma s, D_x^\beta D_x^\gamma s, D_y^\beta D_y^\gamma s, \cdots\right) = 0, 0 < \beta, \gamma < 1. \tag{5}$$

The solution steps of exponential function expansion method are as follows.

Firstly a new variable which is composed of y and x was set, and original function

$u(y,x)$ is transformed to new function $U(\delta)$ [10], then

$$s(y,x) = S(\delta), \tag{6}$$

$$, \delta = \frac{ky^\gamma}{\Gamma(1+\gamma)} \pm \frac{dx^\beta}{\Gamma(1+\beta)} \tag{7}$$

where k and c are constants. Equation (6) and (7) are substituted to equation (5). A new nonlinear ordinary differential equation with regard to U and derivatives of U can be obtained based on the nature of R-L fractional order derivative.

$$G\left(S, S', S'', \cdots\right) = 0. \tag{8}$$

Then the form of the solution of equation (8) [11] is supposed as:

$$S(\delta) = \sum_{i=0}^{N} B_i \left(e^{-\Phi(\delta)}\right)^i, B_N \ne 0, \tag{9}$$

where B_i is an undetermined constant and Φ satisfied equation (4).

The third step is to solve the value of N according to principle of equilibrium [12] and high-order derivative and the highest order nonlinear items in equation (8). The last step is to substitute to equation (4) and (9) to equation (8), then the right side of the equal sign of equation (8) remains to be 0, and the left side is a $e^{-\Phi(\delta)}$ related polynomial. The coefficients of different powers of $e^{-\Phi(\delta)}$ in the polynomial are set as 0, and then equation (8) becomes an algebraic equation set. Unknown numbers in equation (8) including B_i, k, g, h, o, d can be calculated using mathematica software [13].

Nonlinear ordinary differential equation (4) needs to be used in the solution of equation (5), i.e., the solution of equation (5) is related to the solution of equation (4). But the solution of equation (4) has several special circumstances.

Special circumstance 1: $g = 1$

$$\Phi(\delta) = \begin{cases} \ln\left(-\dfrac{\sqrt{o^2-4h}\tanh\left\{0.5\sqrt{o^2-4h}\,(\delta+\delta_0)\right\}}{2h} - \dfrac{o}{2h}\right), \\ h \neq 0, o^2-4h > 0, \\ \ln\left(-\dfrac{\sqrt{o^2-4h}\tan\left\{0.5\sqrt{o^2-4h}\,(\delta+\delta_0)\right\}}{2h} - \dfrac{o}{2h}\right), \\ h \neq 0, o^2-4h < 0, \\ -\ln\left(\dfrac{o}{e^{o(\delta+\delta_0)}-1}\right), h = 0, o \neq 0, o^2-4h > 0, \\ \ln\left(\dfrac{2\left(o(\delta+\delta_0)+2\right)}{o^2(\delta+\delta_0)}\right), h \neq 0, o \neq 0, o^2-4h = 0. \end{cases} \tag{10}$$

Special circumstance 2: $o = 0$

$$\Phi(\delta) = \begin{cases} \ln\left(\sqrt{\dfrac{g}{h}}\tan\left[\sqrt{gh}\,(\delta+\delta_0)\right]\right), g > 0, h > 0 \\ \ln\left(-\sqrt{\dfrac{g}{h}}\tan\left[\sqrt{gh}\,(\delta-\delta_0)\right]\right), g < 0, h < 0 \\ \ln\left(\sqrt{\dfrac{-g}{h}}\tanh\left[\sqrt{-gh}\,(\delta-\delta_0)\right]\right), g > 0, h < 0 \\ \ln\left(-\sqrt{\dfrac{-g}{h}}\tanh\left[\sqrt{-gh}\,(\delta+\delta_0)\right]\right), g < 0, h > 0. \end{cases} \tag{11}$$

Special circumstance 3: $h = 0$ and $o = 0, \Phi(\delta) = \ln\{g(\delta+\delta_0)\}$ (12). δ_0 in equation (10) $\sim$ (12) is integration constant. The exact solution of equation (5) can be obtained according to the above steps and equation (7), (9) and (10) $\sim$ (12).

3.2 First integral method

First integral method [14] is also one of the methods for solving nonlinear differential equations. Similar to the exponential function expansion method, it is applied to solving integer order differential equations initially. In order to better adapt to the solution of fractional differential equations, the method was a little different. In order to better adapt to the solution of fractional differential equations. The theoretical basis of first integral method is exact division of polynomial, i.e., binary polynomial $F(x, y)$ and $G(x, y)$ are both in complex number field and $F(x, y)$ is irreducible. If $G(x, y)$ is 0 at the zero point of $F(x, y)$, then there must have $H(x, y)$ in complex number field, and $G(x, y) = F(x, y)H(x, y)$ holds.

According to the theory of polynomial integral division [15], first integral method was used to solve equation (5), and its procedures are as follows:

The first step was the same with the step of exponential expansion function. New variable δ which is composed of y and x is introduced. Function $U(\delta)$ after transformation is substituted into equation (5) and transformed into non-linear differential equation (8) based on the properties of R-L fractional order derivative. If every item in equation (8) has derivative, then the integral of variable δ is taken, and the integration constant is set as 0.

Then the solution of equation (8) is supposed as

$$S(\delta) = X(\delta). \tag{12}$$

Moreover new variable is introduced to construct an equation set with equation (13).

$$\begin{cases} X'(\delta) = Y(\delta) \\ Y'(\delta) = H(X(\delta), Y(\delta)) \end{cases} . \tag{13}$$

The third step is to suppose that polynomial $X(\delta)$ and $Y(\delta)$ in complex number field satisfying the following equation according to first integral method.

$$H(X(\delta), Y(\delta)) = \sum_{i=0}^{m} b_i(X(\delta))Y(\delta)^i = 0. \tag{14}$$

where $b_i(X)$ is the polynomial of X and $b_m(X) \neq 0$. According to the principle of polynomial integral division, there must be polynomial $p(X) + q(X)Y$ in complex number field, which can make tenable.

$$\frac{dH}{d\delta} = \frac{\partial H}{\partial X}\frac{dX}{d\delta} + \frac{\partial H}{\partial Y}\frac{dY}{d\delta} = (p(X) + q(X)Y)\sum_{i=0}^{m} b_i(X)Y^i. \tag{15}$$

Then $b_i(X)$ in equation (16) is obtained using method of undetermined coefficients [16] and substituted into $H(X, Y) = \sum_{i=0}^{m} b_i(X)Y^i = 0$; then $X(\delta)$ is obtained, i.e., $S(\delta)$. $S(\delta)$ is substituted into equation (6) and (7), and then the solution of equation (5) is obtained.

The following equation is obtained in the process of solution:

$$S'(\delta) = c_0 + c_1S(\delta) + c_2S^2(\delta). \tag{16}$$

The solution of equation (5) is related to equation (16). c_0, c_1, c_2 in equation (16) are constants, and their relationships will affect the solution of equation (5) and (16).

Relationship 1: when $c_1^2 - 4c_0c_2 > 0$,

$$S(\delta) = \begin{cases} -\frac{\sqrt{c_1^2-4c_0c_2}}{2c_2} \tanh\left(\frac{\sqrt{c_1^2-4c_0c_2}}{2}(\delta+\delta_0)\right) - \frac{c_1}{2c_2} \\ -\frac{\sqrt{c_1^2-4c_0c_2}}{2c_2} \coth\left(\frac{\sqrt{c_1^2-4c_0c_2}}{2}(\delta+\delta_0)\right) - \frac{c_1}{2c_2} \end{cases}. \tag{17}$$

Relationship 2: When $c_1^2 - 4c_0c_2 < 0$,

$$S(\delta) = \begin{cases} \frac{\sqrt{-(c_1^2-4c_0c_2)}}{2c_2} \tan\left(\frac{\sqrt{-(c_1^2-4c_0c_2)}}{2}(\delta+\delta_0)\right) - \frac{c_1}{2c_2} \\ -\frac{\sqrt{-(c_1^2-4c_0c_2)}}{2c_2} \cot\left(\frac{\sqrt{-(c_1^2-4c_0c_2)}}{2}(\delta+\delta_0)\right) - \frac{c_1}{2c_2} \end{cases}. \tag{18}$$

Relationship 3: when $c_1^2 - 4c_0c_2 = 0$,

$$S(\delta) = -\frac{1}{c_2(\delta+\delta_0)} - \frac{c_1}{2c_2}. \tag{19}$$

3.3 Wavelet operator matrix solution method

Wavelet operator matrix solution method is based on wavelet transform and wavelet function. Wavelet transform was first proposed by an engineer who engaged in processing oil signal in 1974. At first, it was an empirical formula derived from the intuitive actual signal, but it was gradually improved in the later research and solved the problem that Fourier transform is difficult to solve. The basic principle of wavelet function is to represent $f(t)$ which needs to be solved using multiple approximated smooth expressions and regard $f(t)$ as an image, and the expression group is the resolution of the image. As a new mathematical branch, wavelet analysis has been widely applied in fields such as data compression, signal recognition and solution of differential equation.

he computation of fractional differential equation is large and complex, and the property of wavelet analysis which decomposes function into multi-resolution expression group can reduce the dimension of matrix in fractional differential equation, and the reduced expression group can be flexibly expressed according to different resolution requirements, which greatly reduces the computational difficulty. Its specific solving principle is shown below.

The second-class Chebyshev wavelet is defined as:

$$\psi_{nm}(x) = \begin{cases} 2^{\frac{k}{2}} \widetilde{S}_m\left(2^k x - 2n + 1\right), & x \in \left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right), \\ 0, & \text{otherwise}, \end{cases} \tag{20}$$

$$\widetilde{S}_m(x) = \sqrt{\frac{2}{\pi}} S_m(x), \tag{21}$$

where $n = 1, \cdots, 2^{k-1}, (k \in N^+)$ and $S_m(x)$ is the m-order second-class Chebyshev polynomia

The fractional integral operator matrix is composed of:

$$\left(I^\beta \Psi\right)(x) \approx \mathbf{Q}^\beta_{m'\times m'} \Psi(x), \tag{22}$$

$$\mathbf{Q}^\beta_{m'\times m'} = \mathbf{\Phi}_{m'\times m'} \mathbf{F}^\beta \mathbf{\Phi}^{-1}_{m'\times m'}, \tag{23}$$

$$\Phi_{m'\times m'} = \left[\Psi\left(\frac{1}{2m'}\right), \Psi\left(\frac{3}{2m'}\right), \cdots, \Psi\left(\frac{2m'-1}{2m'}\right)\right], \tag{24}$$

$$\mathbf{F}^\beta = \frac{1}{m'^\beta} \frac{1}{\Gamma(\beta+2)} \begin{bmatrix} 1 & \delta_1 & \delta_2 & \cdots & \delta_{m'-1} \\ 0 & 1 & \delta_1 & \cdots & \delta_{m'-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 & \delta_1 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}, \tag{25}$$

where I^β refers to fractional order integral operator and $\varphi_{m'\times m'}$ is Chebyshev wavelet matrix of the second class; the expression of δ_k in equation (25) is $\delta_k = (k+1)^{\beta+1} - 2k^{\beta+1} + (k-1)^{\beta+1}$

The solution steps of fractional order differential equation are as follows.

Suppose $\frac{\partial^2 s}{\partial y \partial x} \approx \Psi^T(y)\mathbf{S}\psi(x)$, then

$$\frac{s(y,x) \approx \psi^T(y)\left[Q^1_{m'\times m'}\right]^T S Q^1_{m'\times m'} \psi(x) + \int_0^y \varphi_2(y)\,dy + s(0,x) =}{\psi^T(y)\left[Q^1_{m'\times m'}\right]^T S Q^1_{m'\times m'} \psi(x) + \int_0^y \varphi_2(y)\,dy + \varepsilon_1(x)}. \tag{26}$$

It is obtained from equation (26) that

$$\begin{cases} \dfrac{\partial^\beta s}{\partial y^\beta} = I^{1-\beta}\left(\dfrac{\partial s}{\partial y}\right) \approx I^{1-\beta}\left(\psi^T(y) S Q^1_{m'\times m'} \psi(x) + \varphi_2(y)\right), \\ \dfrac{\partial^y s}{\partial x^y} = I^{1-y}\left(\dfrac{\partial s}{\partial x}\right) \approx I^{1-y}\left(\psi^T(y)\left[Q^1_{m'\times m'}\right]^T S\psi(x) + \varphi_1(x)\right). \end{cases} \tag{27}$$

Equation (27) is substituted into $f(x) = \sum_{n=0}^\infty \sum_{m\in Z} d_{nm}\psi_{mm}$, then

$$\begin{aligned} &\psi^T(y)[Q^{1-\beta}_{m'\times m'}] S Q^1_{m'\times m'} \psi(x) + \psi^T(y)[Q^1_{m'\times m'}]^T S [Q^{1-y}_{m'\times m'}]\psi(x) = g(y,x) \\ &= f(y,x) - I^{1-\beta}\varphi_2(y) - I^{1-y}\varphi_1(x) \approx \psi^T(y) G \psi(x), \end{aligned} \tag{28}$$

where $G = [g_{ij}]_{m\times n}$. Then collocation point (y_i, x_j) is substituted to equation (28), and i and j are both positive integer and the largest value of them is m'. Then we have

$$\left[\mathbf{Q}^{-1}_{m'\times m'}\right]^T \left[\mathbf{Q}^{1-\beta}_{m'\times m'}\right]^T \mathbf{S} + \mathbf{S}\mathbf{Q}^{1-\gamma}_{m'\times m} \mathbf{Q}^{-1}_{m'\times m'} = \left[\mathbf{Q}^{-1}_{m'\times m'}\right] \mathbf{G} \mathbf{Q}^{-1}_{m'\times m'}. \tag{29}$$

s is obtained after equation (29) is solved. The numerical solution of $s(y,x)$ can be calculated according to equation (26).

4. Application of different solutions

4.1 Solution of fractional order STO equation based on Exponential expansion

The object equation is:

$$D_x^{\beta} s + 3bs_y^2 + 3bs^2 s_y + 3bss_{yy} + bs_{yyy} = 0, x > 0, 0 < \beta \leq 1. \tag{30}$$

Solution: According to the above steps, equation (30) is transformed firstly. $s(y,x)$ is replaced by $S(\delta), \delta = k\left(y + \frac{dx^{\beta}}{\Gamma(1+\beta)}\right)$, where k and d are constants. Then equation (30) transformed from a fractional order differential equation into an integral differential equation. As there is derivative in every item, both sides of the equation is integrated and integration constant is set as 0. Finally the following equation is obtained.

$$dS + 3bkSS' + bS^3 + bk^2 S'' = 0. \tag{31}$$

According to principle of equilibrium, N = 1. Therefore the form of solution of equation (31) is set as

$$S(\delta) = B_0 + B_1 e^{-\Phi(\delta)} \tag{32}$$

according to equation (9). The solution is substituted into equation (31), and an algebraic equation set with regard to B_i, k, g, h, o, d is obtained:

$$\begin{cases} -3bkgB_1^2 + bB_1^3 + 2bk^2g^2B_1 = 0, \\ -3bk\left(gB_0B_1 + oB_1^2\right) + 3bB_0B_1^2 + 3bogk^2B_1 = 0, \\ dB_1 - 3bk\left(oB_0B_1 + hA_1^2\right) + 3bB_0^2B_1 + 2bghk^2B_1 + bk^2o^2B_1 = 0, \\ dB_0 - 3bkhB_0B_1 + bB_0^3 + bk^2hoB_1 = 0. \end{cases} \tag{33}$$

The equation set is solved using mathematica. k, g, h, o are constants. Then the groups of solution is obtained:

Solution 1:

$$\begin{cases} B_0 = \frac{ko}{2}, \\ B_1 = kg, \\ k = k, \\ d = \frac{bk^2\left(4gh - o^2\right)}{4}. \end{cases} \tag{34}$$

Solution 2:

$$\begin{cases} B_0 = \frac{k\left(o \pm \sqrt{o^2 - 4gh}\right)}{2}, \\ B_1 = kg, \\ k = k, \\ d = bk^2\left(4gh - o^2\right). \end{cases} \tag{35}$$

According to solution 1, equation (32), the initial equation transformation and the value of $\Phi(\delta)$ given in section 3.1, the solution of the fractional order STO equation under solution 1 is as follows.

When $g = 1$, then

$$s\left(y,x\right)=\begin{cases}\dfrac{k_0}{2}-\dfrac{2kh}{\sqrt{o^2-4h}\tanh\left(0.5\sqrt{o^2-4h}\left(\delta+\delta_0\right)+o\right)},\\ h\neq 0, o^2-4h>0\\ \dfrac{k_0}{2}+\dfrac{2kh}{\sqrt{4h-o^2}\tan\left(0.5\sqrt{4h-o^2}\left(\delta+\delta_0\right)-o\right)},\\ h\neq 0, o^2-4h<0\\ \dfrac{k_0}{2}+\dfrac{k_0}{e^{o(\delta+\delta_0)}-1}, h=0, o^2-4h>0\\ \dfrac{k_0}{2}+\dfrac{ko^2\left(\delta+\delta_0\right)}{2o\left(\delta+\delta_0\right)+4}, h\neq 0, o\neq 0, o^2-4h=0\end{cases}, \tag{36}$$

where $\delta = ky + \frac{bk^3x^\beta\left(4h-o^2\right)}{4\Gamma(1+\beta)}$; when $o = 0$, then

$$s\left(y,x\right)=\begin{cases}k\sqrt{gh}\cot\left\lfloor\sqrt{gh}\left(\delta+\delta_0\right)\right\rfloor, g>0, h>0,\\ k\sqrt{gh}\cot\left\lfloor\sqrt{gh}\left(\delta-\delta_0\right)\right\rfloor, g<0, h<0,\\ k\sqrt{-gh}\coth\left\lfloor\sqrt{-gh}\left(\delta-\delta_0\right)\right\rfloor, g>0, h<0,\\ k\sqrt{-gh}\coth\left\lfloor\sqrt{-gh}\left(\delta+\delta_0\right)\right\rfloor, g<0, h>0,\end{cases} \tag{37}$$

where $\delta = ky + \frac{bk^3ghx^8}{\Gamma(1+\beta)}$;

When $h = 0, o = 0$,

$$s(y,x)=\frac{k}{ky+\delta_0}. \tag{38}$$

The solution of the fractional order STO equation under solution 2 is as follows.

When $g = 1$,

$$s(y,x)=\begin{cases}k(\dfrac{o\pm\sqrt{o^2-4h}}{2})\\ \qquad -\dfrac{2h}{\sqrt{o^2-4h}\tanh(0.5\sqrt{o^2-4h}(\delta+\delta_0)+o)},\\ h\neq 0, o^2-4h>0\\ k(\dfrac{o\pm\sqrt{4h-o^2}}{2})\\ \qquad -\dfrac{2h}{\sqrt{4h-o^2}\tanh(0.5\sqrt{o^2-4h}(\delta+\delta_0)-o)},\\ h\neq 0, o^2-4h<0\\ k_0+\dfrac{k_0}{e^{o(\delta+\delta_0)}-1}, h=0, o^2-4h>0\\ \dfrac{k_0}{2}-\dfrac{ko^2(\delta+\delta_0)}{2o(\delta+\delta_0)+4}, h\neq 0, o\neq 0, o^2-4h=0\end{cases} \tag{39}$$

where $\delta = ky + \frac{bk^3x^\beta(4h-o^2)}{4\Gamma(1+\beta)}$;

When $o = 0$,

$$s(y,x) = \begin{cases} k\sqrt{gh}\left(\pm 1 + \cot\left\lfloor\sqrt{gh}\,(\delta+\delta_0)\right\rfloor\right), g > 0, h > 0, \\ k\sqrt{gh}\left(\pm 1 + \cot\left\lfloor\sqrt{gh}\,(\delta-\delta_0)\right\rfloor\right), g < 0, h < 0, \\ k\sqrt{-gh}\left(\pm 1 + \coth\left\lfloor\sqrt{-gh}\,(\delta-\delta_0)\right\rfloor\right), g > 0, h < 0, \\ k\sqrt{-gh}\left(\pm 1 + \coth\left\lfloor\sqrt{-gh}\,(\delta+\delta_0)\right\rfloor\right), g < 0, h > 0, \end{cases} \tag{40}$$

where $\delta = ky + \frac{bk^3ghx^\beta}{\Gamma(1+\beta)}$;

When $h = 0, o = 0$, it is the same with equation (38).

4.2 Solution of fractional order cahn-allen equation with first integral method

Object equation

$$D_x^\beta s - s_{yv} + s^3 - s = 0, (x > 0, 0 < \beta \leq 1) \tag{41}$$

is solved. It is usually used for describing fluid dynamics and reaction diffusion, and its energy has dissipativity.

Solution: According to the procedures of section 3.2, the equation is processed by complex transformation firstly. $s(y,x)$ is replaced by $S(\delta)$, where $\delta = ky - \frac{dx^\beta}{\Gamma(1+\beta)}$. After the simplification of equation (41), new variables $X = S(\delta)$ and $Y = S'(\delta)$ are introduced, and an equation set is formed by the new variables and the simplified equation:

$$\begin{cases} X'(\delta) = Y(\delta) \\ Y'(\delta) = \frac{X^3(\delta) - X(\delta) - dY(\delta)}{k^2} \end{cases}. \tag{42}$$

According to the procedures, suppose that equation (33) is tenable in complex number field:

$$H(X,Y) = \sum_{i=0}^{m} b_i(X)Y^i = 0. \tag{43}$$

Moreover $b_i(X)$ and $H(X,Y)$ satisfy the condition of polynomial integral division theorem; therefore

$$\frac{dH}{d\delta} = \frac{\partial H}{\partial X}\frac{dX}{d\delta} + \frac{\partial H}{\partial Y}\frac{dY}{d\delta} = (P(X) + q(Y)Y)\sum_{i=0}^{m} b_i(X)Y^i \tag{44}$$

is tenable.

Let m = 1, $b_0(X)$ and $b_1(X)$ are obtained using method of undetermined coefficients and substituted into equation (33), and the following algebraic equations are obtained:

$$\begin{cases} \frac{1}{k^2} - \frac{B_1^2}{2} = 0 \\ \frac{3B_0B_1}{2} + \frac{dB_1}{k^2} = 0 \\ \frac{1}{k^2} + C_0B_1 + B_0^2 + \frac{dB_0}{k^2} = 0 \\ B_0C_0 = 0 \end{cases} \tag{45}$$

The following groups of solutions are obtained:

(1) When $B_0 = d = 0$, $\begin{cases} C_0 = \frac{1}{\sqrt{2}k}, B_1 = -\frac{\sqrt{2}}{k} \\ C_0 = -\frac{1}{\sqrt{2}k}, B_1 = \frac{\sqrt{2}}{k} \end{cases}$;

(2) When $C_0 = 0, B_0 = -\frac{\sqrt{2}}{k}, d = \frac{3k}{\sqrt{2}}, B_1 = \pm\frac{\sqrt{2}}{k}$;

(3) When $C_0 = 0, B_0 = \frac{\sqrt{2}}{k}, d = \frac{3k}{\sqrt{2}}, B_1 = \pm\frac{\sqrt{2}}{k}$.

According to (1) and equation (43) and (42), an exact solution can be obtained:

$$s_1(y,x) = \begin{cases} \pm\tanh\left[\frac{\sqrt{2}(ky+\delta_0)}{2k}\right], \\ \pm\coth\left[\frac{\sqrt{2}(ky+\delta_0)}{2k}\right]. \end{cases} \tag{46}$$

In the same way, according to (2) and equation (43) and (42), an exact solution can be obtained:

$$s_2(y,x) = \begin{cases} \pm\frac{\tanh\left[\frac{\sqrt{2}}{4k}\left(ky-\frac{3k}{\sqrt{2}\Gamma(1+\beta)}\right)x^\beta+\delta_0\right]-1}{2}, \\ \pm\frac{\coth\left[\frac{\sqrt{2}}{4k}\left(ky-\frac{3k}{\sqrt{2}\Gamma(1+\beta)}\right)x^\beta+\delta_0\right]-1}{2}. \end{cases} \tag{47}$$

In the same way, according to (3) and equation (43) and (42), an exact solution can be obtained:

$$s_3(y,x) = \begin{cases} \pm\frac{\tanh\left[\frac{\sqrt{2}}{4k}\left(ky+\frac{3k}{\sqrt{2}\Gamma(1+\beta)}x^\beta+\delta_0\right)\right]+1}{2}, \\ \pm\frac{\coth\left[\frac{\sqrt{2}}{4k}\left(ky+\frac{3k}{\sqrt{2}\Gamma(1+\beta)}x^\beta+\delta_0\right)\right]+1}{2}. \end{cases} \tag{48}$$

4.3 The solution of fractional order differential equation based on wavelet operator matrix

The object equation is $\frac{\partial^\beta s}{\partial y^\beta} + \frac{\partial^\beta s}{\partial x^\beta} = f(y,x), (0 \le y, x \le 1)$ (17).

Solution: As $\frac{\partial s}{\partial x}\Big|_{y=0} = 2x$ and $\frac{\partial s}{\partial y}\Big|_{x=0} = 2y$, $\begin{cases} s(0,x) = x^2+1 \\ s(y,0) = y^2+1 \end{cases}$. Moreover as $f(y,x) = \frac{\Gamma(3)y^{2-\beta}\left(x^2+1\right)}{\Gamma(3-\beta)} + \frac{\Gamma(3)\left(y^2+1\right)x^{2-\gamma}}{\Gamma(3-\gamma)}$, the exact solution of equation (17) is $s(y,x) = \left(y^2+1\right)\left(x^2+1\right)$.

Suppose $\beta = \frac{1}{2}, \gamma = \frac{1}{3}$, then the changes of absolute error under different values of m' are shown in Figure 1. The absolute error between the numerical solution and exact solution decreased with the increase of m'.

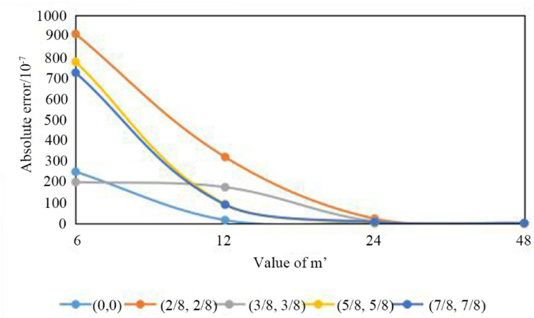

Figure 1: The absolute error between the numerical solution and exact solution under different values of m'

5. Conclusion

Fractional calculus is derived from integer calculus. At present, most of the solutions of fractional differential equations are derived from the solutions of integer differential equations. Because of the variety of nonlinear systems, the forms of nonlinear fractional calculus are various, so there is no general solution to solve all fractional differential equations. In this paper, the exponential function expansion method, first integral method and wavelet operator matrix method were introduced. Then the solving steps of the three methods were explained. Finally, the concrete equations were solved by the three methods respectively. Fractional order STO equation was solved by the exponential function expansion method, fractional cahn-allen equation was solved by the first integral method, and the fractional differential equation was solved by wavelet operator matrix method. Two groups of solutions were obtained in the solution of fractional order STO equation based on exponential expansion method. There was no special limitation in the solution. But suppose that N in $S(\delta) = \sum_{i=0}^{N} B_i \left(e^{-\Phi(\delta)}\right)^i, B_N \neq 0$ is determined by principle of equilibrium, the calculation quantity will be large if the value of N is too large. Three groups of solutions were obtained in the solution of fractional order cahn-allen equation using the first integral method, two solutions each group; the value of m was set as 1 for the convenience of calculation, and the number of solutions was rich. The exact solution of the equation was obtained by wavelet operator matrix algorithm, and the absolute error decreased with the increase of m' value.

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Accepted: 14.03.2019