On orbit reflexive tuple of operators and weak orbit reflexivity

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Abstract. In this paper we give a various conditions for which the tuple $\mathcal{T} = (T_1, T_2, \ldots, T_n)$ of commuting bounded linear operators on an infinite dimensional (real, complex) Banach space X is orbit reflexive. After we introduce the notion of weak orbit reflexive operator and we show some results.

Keywords: tuple of operators, orbit, orbit reflexive, weak orbit reflexive, C_0-semi group.

1. Introduction

By an n-tuple of operators we mean a finite sequence of length n of commuting bounded linear operators on a Banach space X. The orbit of a point $x \in X$ under an operator $T \in B(X)$ is the sequence,

$$Orb(T, x) := \{T^n x : n = 0, 1, \ldots\}.$$

The notion of orbit-reflexive operators on a Hilbert space was introduced and studied in [4]. While the reflexivity of operators is connected to the invariant subspace problem, its natural analogue of orbit reflexivity is in the same way connected to the problem of existence of closed invariant subsets.

Recall that if X is a Banach space and $B(X)$ the set of all bounded linear operators acting on X, an operator $T \in B(X)$ is said to be reflexive if every operator $A \in B(X)$, if $Ax \in \overline{\{p(T)x : p\,\text{polynomial}\}}$ (the norm closure of the set $\{p(T) : p\,\text{polynomial}\}$) for each $x \in X$, then A belongs to $\overline{\{p(T) : p \text{ polynomial}\}}^{SOT}$ (the closure in the strong operator topology).

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Similarly an operator T is said to be orbit reflexive if every operator $A \in B(X)$, if $Ax \in \overline{Orb(T,x)}$ for each $x \in X$ (or equivalently, such that every closed subset of X invariant for T is invariant for A), then A belongs to $\overline{Orb(T)}^{SOT}$.

The orbit reflexivity of many classes of Hilbert space operators was shown in [4], e.g. for normal operators, contractions, algebraic operators, weighted shifts and compact operators. Among others, each operator whose spectrum does not intersect the unit circle is orbit reflexive, and a various conditions which insure that an operator acting on a Banach space is orbit reflexive are given in [10]. Other authors are introduce a similar notions (Null-orbit reflexive operators [5], $\mathbb{C}$-orbit reflexive operators [6], $\mathbb{R}-$orbit Reflexive Operators [7]).

In this work we focus on the orbit reflexivity.

2. Main results

2.1 Operator orbit reflexive

In this part, we will introduce the conditions for which the adjoint and the power of an orbit reflexive operator are also orbit reflexive. We will answer the open question posed in the thesis of Jan Versovsky under the direction of Vladimir Muller supported in 2013 [11] in relation to the reflexivity of the orbit of the inverse of an orbit reflexive operator.

Theorem 2.1 ([4, Theorem 5])**.** *Let H be a Hilbert space and $T \in B(H)$. Then T is orbit reflexive in any of the following cases hold:*

1. *there is a non empty open subset $U \subset H$ such that for each $x \in U$, the orbit $Orb(T,x)$ is closed;*
2. *there is a non empty open subset $U \subset H$ such that for each $x \in U$, $\|T^n x\| \longrightarrow \infty$, as $n \longrightarrow \infty$;*
3. *for each $x \in X$, $\|T^n x\| \longrightarrow 0$;*
4. *the set $\overline{Orb(T)}^{SOT}$ is countable and strongly compact;*
5. $\sigma(T) \cap \{\lambda \in \mathbb{C} : |\lambda| = 1\} = \emptyset$.

Theorem 2.2 ([10, Theorem 7])**.** *Let X be a Banach space and $T \in B(X)$ satisfying $\sum_{n=1}^{\infty} \frac{1}{\|T^n\|} < \infty$, or when $X = H$ is a complex Hilbert space $\sum_{n=1}^{\infty} \frac{1}{\|T^n\|^2} < \infty$. Then T is orbit reflexive.*

Corollary 2.1 ([10, Corollary 8])**.** *Let X be a Banach space and $T \in B(X)$. If $r(T) \neq 1$, then T is orbit reflexive.*

Corollary 2.2. *Let H be a Hilbert space and $T \in B(H)$. If $r(T) \neq 1$, then T^* is orbit reflexive.*

Proof. Follows from, $r(T) = r(T^*)$. □

Corollary 2.3. *Let X be a Banach space and $T \in B(X)$. If T is self-adjoint and $r(T) \neq 1$, Then T^n is orbit reflexive for all $n \geq 1$.*

Proof. Follows from $r(T^n) = r(T)^n$ for all $n \geq 2$ and Corollary 2.1. □

Let $\sigma(T)$, $\sigma_p(T)$ and $\sigma_{ap}(T)$ the spectrum, the point spectrum and the approximate point spectrum of T, where

- $\sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bijective}\}$.
- $\sigma_p(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{is not injective }\}$.
- $\sigma_{ap}(T)$=$\{\lambda \in \mathbb{C} : \lambda I - T$ is not injective or $Rg(\lambda I - T)$ is not closed in $X\}$.

Theorem 2.3 ([8, Theorem 3.1]). *Let X be an infinite dimensional reflexive Banach space and $T, S \in B(X)$. If $\lambda \in \sigma_{ap}(T)\backslash\sigma_p(T)$ and $\mu \in \sigma_{ap}(S)\backslash\sigma_p(S)$, then for any two positive sequences $(\alpha_n)_{n\geq 1}$ and $(\beta_n)_{n\geq 1}$ with $\sum_{n\geq 1} \alpha_n < +\infty$ and $\sum_{n\geq 1} \beta_n < +\infty$, in every open ball in X with radius $\sum_{n\geq 1}(\alpha_n + \beta_n)$, there is $x \in X$ satisfying*

$$\|T^n x\| \geq \alpha_n \frac{|\lambda|^n}{2}, \|S^n x\| \geq \beta_n \frac{|\mu|^n}{2}.$$

Corollary 2.4. *Let $T, S \in B(X)$, if the sets $\sigma_{ap}(T)\backslash\sigma_p(T)$ and $\sigma_{ap}(S)\backslash\sigma_p(S)$ both have a non-empty intersection with the domain $\{\lambda \in \mathbb{C} : |\lambda| > 1\}$ then, T and S are orbit reflexive.*

Proof. From theorem 2.3 there is a non empty open subset $U \subset X$ such that for each $x \in U$,

$$\|T^n x\| \longrightarrow \infty, \|S^n x\| \longrightarrow \infty \text{ as } n \longrightarrow \infty$$

and from (2) of theorem 2.1 then, T and S are orbit reflexive. □

Let Ω be a non empty subset of the complex plane whose boundary consists of finite number of rectifiable Jordan curves, oriented in the positive sense and $Hol(\Omega)$ the set of all holomorphic functions on some open neighborhood of the closure of Let Ω.

To obtain the reflexivity of the orbit of the inverse operator we need the following theorem.

Theorem 2.4. *If $T \in B(X)$, $\sigma(T) \subset \Omega$ and $f \in \Omega$ is injective and non-constant function on each of the components of Ω and the set $\sigma_{ap}(T)\backslash\sigma_p(T)$ has a non-empty intersection with the both domains $\{\lambda \in \mathbb{C} : |\lambda| > 1\}$ and $\{\lambda \in \mathbb{C} : |f(\lambda)| > 1\}$ then, the both T and $f(T)$ are orbit reflexive.*

Proof. Follows from corollary 2.4. □

If $T \in B(X)$ is an invertible operator, then

$$\sigma(T) \subset \left\{\lambda \in \mathbb{C}/ \; [r(T^{-1})]^{-1} \leq |\lambda| \leq r(T)\right\}.$$

Since $f(\lambda) = \lambda^{-1}$ is holomorphic, non-constant and injective function on $\Omega = \{\lambda \in \mathbb{C} : m \leq |\lambda| \leq M\} \supset \sigma(T)$, where $0 < m < [r(T^{-1})]^{-1}$ and $M > r(T)$.

So, according to theorem 4.4 in [8] and corollary 2.4, we deduce that the both T and T^{-1} are orbit reflexive if the set $\sigma_{ap}(T)\backslash\sigma_p(T)$ has a non-empty intersection with both domains $\{\lambda \in \mathbb{C} : |\lambda| > 1\}$ and $\{\lambda \in \mathbb{C} : |\lambda| < 1\}$.

2.2 n-tuple of operators orbit reflexive

Definition 2.1 ([2])**.** *Let $\mathcal{T} = (T_1, T_2, \ldots, T_n)$ be a n-tuple of operators acting on an infinite dimensional Banach space X. We will let,*

$$\mathcal{F} = \left\{T_1^{k_1} T_2^{k_2}, \ldots, T_n^{k_n} : k_i \geq 0, i = 1, \ldots, n\right\}$$

be the semi-group generated by $\mathcal{T}$. For $x \in X$, the orbit of x under the tuple $\mathcal{T}$ is the set

$$Orb(\mathcal{T}, x) = \{Sx : S \in \mathcal{F}\}.$$

Definition 2.2 ([13, Definition 1.2])**.** *The orbit of x under the tuples $\mathcal{T}$ tending to infinity if:*

$$\left\|T_1^{k_1} T_2^{k_2} \ldots T_n^{k_n} x\right\| \to \infty \text{ as } k_i \to \infty \text{ with } k_i \geq 0, \forall i = 1, \ldots, n.$$

Lemma 2.1 ([4])**.** *Let $A, S_1, S_2, \ldots \in B(X)$.*

If the set of vectors $u \in X$ for which $Au \in \{S_1u, S_2u, \ldots\}$ is of second category, then $A \in \{S_1, S_2, \ldots\}$

Proposition 2.1. *Let $A, T_1, T_2, \ldots, T_n \in B(X)$ and $\mathcal{T} = (T_1, T_2, \ldots, T_n)$ be an n-tuple of commuting operators. If the set of vectors $x \in X$ for which $Ax \in Orb(\mathcal{T}, x)$ is of second category, then $A \in Orb(\mathcal{T})$.*

Proof. Follows from Lemma 2.1. □

Theorem 2.5. *Let $T_1, T_2, \ldots, T_n \in B(X)$ and $\mathcal{T} = (T_1, T_2, \ldots, T_n)$ be an n-tuple of commuting operators. $\mathcal{T}$ is orbit reflexive in any of the following cases:*

1. *there is a non empty open subset $U \subset X$such that for each $x \in U$, the orbit $Orb(\mathcal{T}, x)$ is closed;*
2. *there is a non empty open subset $U \subset X$ such that for each $x \in U$, $\left\|T_1^{k_1} T_2^{k_2} \ldots T_n^{k_n} x\right\| \to \infty$ as $k_i \to \infty$ with $k_i \geq 0$, for all $i = 1, \ldots, n$;*
3. *The set $\overline{Orb(\mathcal{T})}^{SOT}$ is countable and strongly compact.*

Proof. Throughout the proof, suppose that $Ax \in \overline{Orb(\mathcal{T}, x)}$ for every $x \in X$. We need to show that $A \in \overline{Orb(\mathcal{T})}^{SOT}$ in each of the cases.

1. Follows from Lemma 2.1,

2. Since $\left\|T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} x\right\| \to \infty$ as $k_i \to \infty$ then $Orb(\mathcal{T}, x)$ is closed, it follows from (1) that $\mathcal{T}$ is orbit reflexive.

3. Let $S \in \mathcal{F}$ and suppose that $Ax = \lim S^{n_k} x$ for some sequence $(n_k)_{k \geq 0}$ of positive integers.

Since $\overline{Orb(\mathcal{T})}^{SOT}$ is strongly compact, the sequence$(S^{n_k})_{k \geq 0}$has a strongly convergent subsequence. Therefore $Ax \in \left\{Bx : B \in \overline{Orb(\mathcal{T})}^{SOT}\right\}$ for each $x \in X$, and according to Lemma 2.1, we have $A \in \overline{Orb(\mathcal{T})}^{SOT}$. □

Example 1. Let S be the unilateral forward shift on $\ell^2(\mathbb{N})$:

$$Se_i = e_{i+1}; i \geq 1,$$

where $\{e_i : i \in \mathbb{N}^*\}$ is the standard orthonormal basis for $\ell^2(\mathbb{N})$.

Given a sequence of positive numbers $(a_i)_{i \geq 1}$ so that $a_i > 1$ for all $i \geq 1$ and $a_i \to 1$ as $i \to \infty$ and let

$$T_i = a_i S; i = 1, \dots, n.$$

So, of after [13, Example 1.1] there is a non empty open subset $U \subset X$ such that for each $x \in U$, the orbit of x under the tuple $(T_1, T_2, \dots, T_n)$ tend strongly to infinity, then follows from (2) of Theorem 2.5, the tuple $(T_1, T_2, \dots, T_n)$ it orbit reflexive.

Proposition 2.2. *Let X be an infinite dimensional reflexive Banach space and $\mathcal{T} = (T_1, T_2, \dots, T_n)$ be the n-tuple of operators in $B(X)$ bounded below for all $n \geq 1$. If there is $x \in X$ such that the orbit of x under T_i for all $i \in \{1, 2, \dots, n\}$ tend strongly to infinity then $\mathcal{T}$ is orbit reflexive.*

Proof. From [13, Corollary 1.1] the orbit of x under the tuple $\mathcal{T}$ tend strongly to infinity and from (2) of Theorem 2.5, $\mathcal{T}$ is orbit reflexive. □

2.3 Operator weakly orbit reflexive

By a weak orbit of an operator $T \in B(X)$ we mean a sequence of the form $(\langle T^n x, x^* \rangle)_{\infty}^{n=0}$, where $x \in X$ and $x^* \in X^*$, and we write:

$$W - Orb(T, x) = (\langle T^n x, x^* \rangle)_{\infty}^{n=0}.$$

Definition 2.3. *Let X be a Banach space, an operator $T \in B(X)$ is said to be weakly orbit reflexive if every operator $A \in B(X)$ belongs to $\overline{W - Orb(T)}^{WOT}$ (the closure in the weak operator topology), whenever $\langle Ax, x^* \rangle \in \overline{W - Orb(T, x)}^{W}$ (the closure in the weak topology) for each $x \in X$ and $x^* \in X^*$.*

Example 2. Since the norm topology is strictly stronger than the weak topology, then every operator orbit reflexive is weakly orbit reflexive. So in a Hilbert space a normal operators, contractions, algebraic operators, weighted shifts and compact operators are weak orbit-reflexive.

Each Banach space operator have spectral radius different from 1, is weak orbit-reflexive.

Theorem 2.6. *1. If T is weakly orbit reflexive and S is invertible operator then STS^{-1} is also weakly orbit reflexive.*

2. If T is weakly orbit reflexive then $T \oplus T$ is also weakly orbit reflexive.

Proof. 1. The image by an operator of a weakly convergent sequence is a sequence weakly convergent. Hence $\langle SAS^{-1}x, x^*\rangle \in \overline{W - Orb(STS^{-1}, x)}^W$ for all $x \in X$ and $x^* \in X^*$ is equivalent to $\langle Ax, x^*\rangle \in \overline{W - Orb(T, x)}^W$ for all $x \in X$ and $x^* \in X^*$, or T is weakly orbit reflexive, then $A \in \overline{W - Orb(T)}^{WOT}$, so $SAS^{-1} \in \overline{W - Orb(STS^{-1})}^{WOT}$.

2. Let $A \in B(X \oplus X)$ and suppose that $\langle Ax, x^*\rangle \in \overline{W - Orb(T \oplus T, x)}^W$ for all $x \in X \oplus X$ and $x^* \in X^* \oplus X^*$.

In particular, for any $a \in X$ we have:

$$\langle A(a, 0), x^*\rangle \in \overline{W - Orb(T \oplus T, (a, 0))}^W \subset \mathbb{C} \times \{0\}.$$

And similarly for the second component, so that both copies of X are A-invariant. Hence A can be written as $B \oplus C$ where $B, C \in B(X)$. Moreover, if $a \in X$ then: $\langle (Ba, Ca), x^*\rangle = \langle A(x, x), x^*\rangle \in \overline{W - Orb(T \oplus T, x)}^W \subset \{(\lambda, \lambda) : \lambda \in \mathbb{C}\}$ So, $B = C$.

Now, since T is weakly orbit reflexive and $Ba \in \overline{W - Orb(T, x)}^W$ for all $a \in X$, we have $B \in \overline{W - Orb(T)}^{WOT}$, and therefore:

$$A = B \oplus B \in \overline{W - Orb(T \oplus T)}^{WOT}.$$

□

Proposition 2.3. *Let X be a Banach space and $T \in B(X)$ satisfying*

$$\sum_{n=1}^{\infty} \frac{1}{\|T^n\|^{1/2}} < \infty,$$

or when $X = H$ is a complex Hilbert space $\sum_{n=1}^{\infty} \frac{1}{\|T^n\|} < \infty$. Then T is weakly orbit reflexive.

Proof. Let $A \in B(X)$ and suppose that $\langle Ax, x^*\rangle \in \overline{W - Orb(T, x)}^W$, for all $x \in X$ and $x^* \in X^*$.

Suppose, that $A \neq T^n$ for all $n \in \mathbb{N}$, otherwise there is nothing to prove. Since $\sum_{n=1}^{\infty} \frac{1}{\|T^n\|^{1/2}} < \infty$, then $\|T^n\|^{1/2} \to \infty$ and we have:

$$\|T^n - A\| \geq \|T^n\| - \|A\| \geq \frac{1}{2}\|T^n\| \text{ for all } n \text{ large enough.}$$

So, $\sum_{n=1}^{\infty} \frac{1}{\|T^n - A\|^{1/2}} < \infty$. Therefore, the operators $S_n := T_n - A$ satisfy the conditions in [1, Theorem 6]. So there exists a dense set of pairs $x \in X$; $x^* \in X$ with:

$$|\langle (T_n - A)x, x^* \rangle| > 0, \forall n \geq 1, |\langle (T_n - A)x, x^* \rangle| \to \infty.$$

Thus there is a constant $C > 0$ such that $\inf_n |\langle (T_n - A)x, x^* \rangle| > 0$ and we have a contradiction with the assumption that $\langle Ax, x^* \rangle \in \overline{W - Orb(T, x)}^{-W}$.

The Hilbert space case can be proved similarly. □

We finish this paper by posing the open question related to existence of operator weakly orbit reflexive but not orbit reflexive.

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