Some common fixed point theorems for four maps in fuzzy metric-like spaces using α-ϕ and β-φ-fuzzy contractions

Satish Shukla
Department of Applied Mathematics
Shri Vaishnav Institute of Technology & Science
Gram Baroli, Sanwer Road, Indore, 453331, (M.P.)
India
satishmathematics@yahoo.co.in

Rajinder Sharma*
College of Applied Sciences, Sohar
Department of General Requirements (Mathematics Section)
Sultanate of Oman
rajind.math@gmail.com

Abstract. In this paper, some fixed point theorems for α-ϕ-fuzzy contractive and β-ψ-fuzzy contractive mappings in fuzzy metric-like spaces are proved. The results of this paper generalize, unify and improve the recent results of Shukla and Abbas [12] and Gopal and Vetro [7]. Some examples are provided which justify the importance of the results and notions introduced herein.

Keywords: common fixed point, fuzzy metric like space, α-ϕ-contractive maps, β-φ contractive maps.

1. Introduction

The fuzzy sets were first introduced by Zadeh [9] in 1965 which explain the vagueness in daily life problems. This concept is used to interpret and analyze several problems where system possesses fuzzy rather than stochastic nature. The concept of fuzzy metric was introduced by Kramosil and Michalek [8]. The fuzzy metric can be considered as the degree of nearness of points x and y of space. George and Veeramani [1] modified the notion of fuzzy metric spaces in such a way that the modified definition of fuzzy metric generates a Hausdorff and first countable topology. The Cauchy sequences in fuzzy metric spaces and the completeness of fuzzy metric spaces are defined in two ways. First, Grabiec [10] defined the notions of a Cauchy sequence in a fuzzy metric space and the completeness of a fuzzy metric space, which are often called the G-Cauchy sequences and the G-completeness respectively. The G-completeness defined by Grabiec [10] was a very strong notion so that even $\mathbb{R}$ fails to be G-complete (see, George and Veeramani [1]). Therefore, George and Veeramani

*. Corresponding author

[1] redefined the notions of Cauchy sequence and completeness which are called M-Cauchy sequence and and M-completeness respectively.

On the other hand, Harandi [2] introduced the notion of a metric-like space which generalizes the notion of partial metric spaces (see, [2] and the references therein). Shukla and Abbas [12] unified the notions of the metric-like spaces and the fuzzy metric spaces by introducing the notion of fuzzy metric-like spaces. In fuzzy metric-like spaces, the degree of nearness of points x and y of space, when $x = y$ is not perfect, and so, the self fuzzy distance may not be unity. Shukla and Abbas [12] defined the notions of Cauchy sequences and completeness of fuzzy metric-like spaces which were the extensions of the G-Cauchy sequence and G-completeness respectively, in fuzzy metric-like spaces. Recently, Shukla et al. [11] introduced various notions of Cauchy sequences and completeness of a fuzzy metric-like spaces. They showed that among the various notions 1-M-completeness is most general notion and is an extension and generalization of the M-completeness defined by George and Veeramani [1].

Gopal and Vetro [7] introduced the notions of α-ϕ-fuzzy contractive mapping and β-ψ-fuzzy contractive mapping and proved some fixed point theorems which ensure the existence and uniqueness of the fixed point for the classes of these two types of mappings. They extended and unified the fixed point results of Gregori and Sapena [13], Samet et al. [4] and several others (see, Gopal and Vetro [7]). The purpose of this paper is to extend and generalize and improve the recent results of Shukla and Abbas [12] and Gopal and Vetro [7] by proving fixed point theorem for α-ϕ-fuzzy contractive mappings and β-ψ-fuzzy contractive mappings in 1-M-complete fuzzy metric-like spaces. Some examples are given which verifies the generalization of this paper.

2. Preliminaries

We start with some known concepts and results.

Definition 2.1 (Zadeh [9])**.** A fuzzy set A in a nonempty set X is a function with domain X and values in $[0, 1]$.

Definition 2.2 (Schweizer and Sklar [6])**.** A binary operation $*\colon [0,1]\times[0,1] \to [0,1]$ is a continuous t-norm if $\{[0,1], *\}$ is an abelian topological monoid with unit 1 such that $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, $a, b, c, d \in [0, 1]$.

Three typical examples of t-norms are $a *_m b = \min\{a, b\}$ (minimum t-norm), $a *_p b = ab$ (product t-norm), and $a *_l b = \max\{a+b-1, 0\}$ (Lukasiewicz t-norm).

Definition 2.3 (George and Veeramani [1])**.** The triplet $(X, M, *)$ is a fuzzy metric space (in the sense of George and Veeramani) if X is an arbitrary set, $*$ is a continuous t-norm, and M is a fuzzy set in $X^2 \times (0, \infty)$ satisfying the following conditions:

(FM1) $M(x, y, t) > 0$;

(FM2) $M(x, y, t) = 1$ if and only if $x = y$;

(FM3) $M(x, y, t) = M(y, x, t)$;

(FM4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;

(FM5) $M(x, y, \cdot) \colon (0, \infty) \to [0, 1]$ is a continuous mapping;

for all $x, y, z \in X$ and $s, t > 0$.

Here M (endowed with $*$) is called a fuzzy metric on X. It is known that, for all $x, y \in X$, $M(x, y, \cdot)$ is a nondecreasing function. Several examples of fuzzy metric spaces can be found in [1, 3, 5, 14].

Definition 2.4 (Shukla and Abbas [12])**.** The triplet $(X, \mathcal{F}, *)$ is a fuzzy metric-like space if X is an arbitrary set, $*$ is a continuous t-norm, and $\mathcal{F}$ is a fuzzy set in $X^2 \times (0, \infty)$ satisfying the following conditions:

(FML1) $\mathcal{F}(x, y, t) > 0$;

(FML2) if $\mathcal{F}(x, y, t) = 1$ then $x = y$;

(FML3) $\mathcal{F}(x, y, t) = \mathcal{F}(y, x, t)$;

(FML4) $\mathcal{F}(x, y, t) * \mathcal{F}(y, z, s) \leq \mathcal{F}(x, z, t + s)$;

(FML5) $\mathcal{F}(x, y, \cdot) \colon (0, \infty) \to [0, 1]$ is a continuous mapping;

for all $x, y, z \in X$ and $s, t > 0$. Here $\mathcal{F}$ (endowed with $*$) is called a fuzzy metric-like on X.

If we choose $\mathcal{F}(x, x, t) = 1$ for all $x \in X$ and $t > 0$, then a fuzzy metric-like space reduces into a fuzzy metric space. Therefore, the class of fuzzy metric spaces are a particular case of the class of fuzzy metric-like spaces (with unit self fuzzy distance). We observe that $\mathcal{F}(x, x, t)$ may be less than or equal to 1 for some or may be for all $x \in X$ which is not possible in fuzzy metric spaces. We illustrate this fact by the following example:

Example 2.5. Let $X = [0, 1]$. Then, the triplet $(X, \mathcal{F}, *_l)$ is a fuzzy metric-like space, where the fuzzy set $\mathcal{F}$ is defined by

$$\mathcal{F}(x, y, t) = \begin{cases} 1, & \text{if } x = y = 0; \\ \frac{\max\{x,y\}}{a}, & \text{otherwise} \end{cases} \quad \text{for all } t > 0$$

where $a \geq 2$. Notice that, if $0 < x \leq 1$, then we have $\mathcal{F}(x, x, t) = \frac{x}{a} < 1$.

Remark 2.6. If $x \neq y$, then we must have $\mathcal{F}(x, y, t) < 1$, for all $t > 0$.

For various examples of fuzzy metric-like spaces, the reader is referred to Shukla and Abbas [12] and Shukla et al. [11].

Definition 2.7 (Shukla and Abbas [12] and Shukla et al. [11]). Let $(X, \mathcal{F}, *)$ be a fuzzy metric-like space and $\{x_n\}$ be a sequence in X. Then:

(*i*) $\{x_n\}$ is said to be convergent to $x \in X$ and x is called a limit of $\{x_n\}$ if,

$$\lim_{n\to\infty} \mathcal{F}(x_n, x, t) = \mathcal{F}(x, x, t) \quad \text{for all } t > 0.$$

(*ii*) $\{x_n\}$ is said to be G-Cauchy if, for all $t > 0$ and each $p \geq 1$, the limit $\lim_{n\to\infty} \mathcal{F}(x_{n+p}, x_n, t)$ exists.

(*iii*) $(X, \mathcal{F}, *)$ is said to be G-complete if every G-Cauchy sequence $\{x_n\}$ in X converges to some $x \in X$ such that $\lim_{n\to\infty} \mathcal{F}(x_n, x, t) = \mathcal{F}(x, x, t) = \lim_{n\to\infty} \mathcal{F}(x_{n+p}, x_n, t)$ for all $t > 0$ and each $p \geq 1$.

Definition 2.8 (Shukla et al. [11]). Let $(X, \mathcal{F}, *)$ be a fuzzy metric-like space and let $\{x_n\}$ be a sequence in X. The sequence $\{x_n\}$ is called a 1-G-Cauchy sequence if $\lim_{n\to\infty} \mathcal{F}(x_{n+p}, x_n, t) = 1$ for all $t > 0$ and each $p \geq 1$. The space $(X, \mathcal{F}, *)$ is called 1-G-complete if every 1-G-Cauchy sequence in X converges to some $x \in X$ such that $\mathcal{F}(x, x, t) = 1$ for all $t > 0$.

It is obvious that every 1-G-Cauchy sequence is a G-Cauchy sequence and every G-complete fuzzy metric-like space is 1-G-complete, while the converse of these facts do not hold (see, Shukla et al. [11]).

Definition 2.9 (Shukla et al. [11]). Let $(X, \mathcal{F}, *)$ be a fuzzy metric-like space and let $\{x_n\}$ be a sequence in X. The sequence $\{x_n\}$ is called a 1-M-Cauchy sequence if $\lim_{n,m\to\infty} \mathcal{F}(x_n, x_m, t) = 1$ for all $t > 0$. The space $(X, \mathcal{F}, *)$ is called 1-M-complete if every 1-M-Cauchy sequence in X converges to some $x \in X$ such that $\mathcal{F}(x, x, t) = 1$ for all $t > 0$.

Every complete fuzzy metric space in the sense of Grabiec [10] is 1-G-complete as a fuzzy metric-like space, and every complete fuzzy metric space in the sense of George and Veeramani [1] is 1-M-complete as a fuzzy metric-like space. Also, 1-G-completeness implies 1-M-completeness and the notion of 1-M-completeness is more general than that of 1-G-completeness, for detail see, Shukla et al. [11].

We now state the main results of this paper.

3. Main results

Suppose, Φ denotes the family of all right continuous functions $\phi\colon [0, \infty) \to [0, \infty)$ satisfying $\phi(r) < r$ for all $r > 0$. Then it is obvious that $\phi(0) = 0$.

Proposition 3.1. Let $\phi \in \Phi$ and $\{a_n\}$ be a sequence of positive numbers such that $a_{n+1} \leq \phi(a_n)$ for all $n \in \mathbb{N}$. Then, the sequence $\{a_n\}$ is convergent and $\lim_{n\to\infty} a_n = 0$.

Proof. Suppose, $\phi \in \Phi$ and $\{a_n\}$ be a sequence of positive numbers such that $a_{n+1} \leq \phi(a_n)$ for all $n \in \mathbb{N}$. Since $a_n > 0$ for all $n \in \mathbb{N}$, we have

$$a_{n+1} \leq \phi(a_n) < a_n \text{ for all } n \in \mathbb{N}.$$

It shows that $\{a_n\}$ is decreasing sequence of positive numbers, and so, it converges to some $a \in [0, \infty)$. Suppose $a > 0$. Again, since $a_n \leq \phi(a_{n-1})$ for all $n \in \mathbb{N}$ by the right continuity of ϕ we have

$$a = \lim_{n\to\infty} a_n \leq \phi(a) < a.$$

This contradiction shows that $a = 0$, i.e., $\lim_{n\to\infty} a_n = 0$. □

The following definition is an analogue of [7] in fuzzy metric-like spaces.

Definition 3.2. Let $(X, \digamma, *)$ be a fuzzy metric-like space. A mappings $T\colon X \to X$ is said to be an α-ϕ-fuzzy contractive mapping if there exist two functions $\alpha : X^2 \times (0, \infty) \to [0, \infty)$ and $\phi \in \Phi$ such that

$$\alpha(x, y, t)\left(\frac{1}{\digamma(Tx, Ty, t)} - 1\right) \leq \phi\left(\frac{1}{\digamma(x, y, t)} - 1\right) \tag{1}$$

for all $x, y \in X$ and $t > 0$.

Remark 3.3. If $\alpha(x, y, t) = 1$ for all $x, y \in X, t > 0$, and $\phi(r) = kr$ for all $r > 0$ and for some $k \in (0, 1)$, then Definition 3.2 reduces into the definition of fuzzy contractive mapping given by Shukla and Abbas [12]. It shows that every fuzzy contractive mapping is an α-ϕ-fuzzy contractive mapping; but the converse is not true. The following example verifies this fact.

Example 3.4. Let $X = [0, \infty)$ and define $\digamma\colon X^2 \times (0, \infty) \to [0, 1]$ by

$$\digamma(x, y, t) = \frac{1}{1 + \max\{x, y\}} \quad \text{for all } x, y \in X, t > 0.$$

Then $(X, \digamma, *_m)$ is a fuzzy metric-like space. Define the mappings $T\colon X \to X$, $\phi\colon [0, \infty) \to [0, \infty)$ and $\alpha\colon X^2 \times (0, \infty) \to [0, \infty)$ by:

$$Tx = \begin{cases} x - 1, & \text{if } x \in [1, \infty) \cap \mathbb{Q}^c; \\ x + 1, & \text{otherwise} \end{cases}, \quad \phi(r) = \begin{cases} \frac{r}{a}, & \text{if } r \leq \frac{1}{2}; \\ r - \frac{1}{b}, & \text{if } r > \frac{1}{2} \end{cases} \quad \text{and}$$

$$\alpha(x, y, t) = \begin{cases} 1, & \text{if } x \in \mathbb{Q},\ y \in \mathbb{Q}^c \text{ and } x + 1 \leq y - 1; \\ 0, & \text{otherwise} \end{cases}$$

where $a > 1, b \geq 2$ are such that $b = \frac{2a}{a-1}$. Then T is an α-ϕ-fuzzy contractive mapping, but not a fuzzy contractive mapping in the sense of Shukla and Abbas [12].

Notice that, although T is an α-ϕ-fuzzy contractive mapping, but it is a fixed point free mapping, i.e., has no fixed point. We denote by Fix(T) the set of all fixed points of T, i.e., $\mathrm{Fix}(T) = \{x \in X : Tx = x\}$.

Definition 3.5. Let X be a nonempty set. A mappings $T : X \to X$ is said to be an α-admissible mapping if there exists a function $\alpha : X^2 \times [0, \infty) \to [0, \infty)$ such that for all $x, y \in X$ and $t > 0$ the following condition is satisfied:

$$\alpha(x, y, t) \geq 1 \implies \alpha(Tx, Ty, t) \geq 1. \tag{2}$$

Example 3.6. Let X be a nonempty set and $T : X \to X$ be defined by $Tx = x$ for all $x \in X$. Then T is α-admissible for every function $\alpha : X^2 \times [0, \infty) \to [0, \infty)$.

Example 3.7. Let X be a nonempty set and $\alpha : X^2 \times [0, \infty) \to [0, \infty)$ be a function such that $\alpha(c, c, t) \geq 1$ for all $t > 0$ for some $c \in X$. Then, the mapping $T : X \to X$ defined by $Tx = c$ for all $x \in X$ is an α-admissible mapping.

Example 3.8. Let $X = [0, \infty)$, $T : X \to X$ be defined by $Tx = x + 1$ for all $x \in X$ and $\alpha : X^2 \times [0, \infty) \to [0, \infty)$ be defined by $\alpha(x, y, t) = \frac{xy}{t}$ for all $x, y \in X, t > 0$. Then T is an α-admissible mapping.

We next state a fixed point result for α-ϕ-fuzzy contractive mappings in 1-G-complete fuzzy metric-like spaces.

Theorem 3.9. *Let $(X, \digamma, *)$ be a 1-G-complete fuzzy metric-like space and $T : X \to X$ be an α-ϕ-fuzzy contractive mapping. Suppose, the following conditions are satisfied:*

(*i*) *T is α-admissible;*

(*ii*) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, t) \geq 1$ for all $t > 0$;*

(*iii*) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, t) \geq 1$ for all $n \in \mathbb{N}$ and $\{x_n\}$ converges to x, then $\alpha(x_n, x, t) \geq 1$ for all $n \in \mathbb{N}$.*

(*iv*) *for all $x, y \in X$ and $t > 0$ there exists $z \in X$ such that $\alpha(x, z, t) \geq 1$ and $\alpha(y, z, t) \geq 1$.*

Then, T has a unique fixed point $u \in X$ and $\digamma(u, u, t) = 1$ for all $t > 0$.

Proof. Let $x_0 \in X$ be the point such that $\alpha(x_0, Tx_0, t) \geq 1$, for all $t > 0$. Let us define a sequence $\{x_n\}$ in X such that $x_n = Tx_{n-1}$, for all $n \in \mathbb{N}$. If $x_n = x_{n-1}$ for some $n \in \mathbb{N}$, then x_n is a fixed point of T. Assume that $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$, and so, we must have $\digamma(x_n, x_{n-1}, t) < 1$ for all $n \in \mathbb{N}$ and $t > 0$. Since T is α-admissible, we have

$$\alpha(x_0, x_1, t) = \alpha(x_0, Tx_0, t) \geq 1 \implies \alpha(Tx_0, Tx_1, t) = \alpha(x_1, x_2, t) \geq 1.$$

Proceeding in the same way further we get by induction:

(3) $$\alpha(x_{n-1}, x_n, t) \geq 1 \text{ for all } n \in \mathbb{N}, t > 0.$$

By (3) we have

$$\frac{1}{F(x_n, x_{n+1}, t)} - 1 = \frac{1}{F(Tx_{n-1}, Tx_n, t)} - 1 \leq \alpha(x_{n-1}, x_n, t)\left[\frac{1}{F(x_{n-1}, x_n, t)} - 1\right].$$

Using (1) with $x = x_{n-1}$ and $y = x_n$ in the above inequality and using the fact that $\phi(r) < r$, we have

$$\frac{1}{F(x_n, x_{n+1}, t)} - 1 \leq \phi\left(\frac{1}{F(Tx_{n-1}, Tx_n, t)} - 1\right) < \frac{1}{F(x_{n-1}, x_n, t)} - 1.$$

Suppose, $F_n(t) = \frac{1}{F(x_n, x_{n+1}, t)} - 1$ for $t > 0$ and $n \in \mathbb{N}$. We shall show that $\lim_{n\to\infty} F_n(t) = 0$ for all $t > 0$. Then, since $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$ we have $F_n(t) > 0$ for all $t > 0$ and $n \in \mathbb{N}$. Therefore, by the above inequality we have

$$F_n(t) \leq \phi\left(F_{n-1}(t)\right) \text{ for all } n \in \mathbb{N}.$$

The above inequality with Proposition 3.1 yields

$$\lim_{n\to\infty} F_n(t) = 0, \text{ i.e., } \lim_{n\to\infty}\left[\frac{1}{F(x_n, x_{n+1}, t)} - 1\right] = 0.$$

Therefore

$$\lim_{n\to\infty} F(x_n, x_{n+1}, t) = 1 \text{ for all } t > 0.$$

Thus, if $n \in \mathbb{N}$ and $p \geq 1$, then we have

$$F(x_n, x_{n+p}, t) \geq F\left(x_n, x_{n+1}, \frac{t}{p}\right) * F\left(x_{n+1}, x_{n+2}, \frac{t}{p}\right) * \cdots * F\left(x_{n+p-1}, x_{n+p}, \frac{t}{p}\right).$$

Letting $n \to \infty$ in the above inequality we obtain

$$\lim_{n\to\infty} F(x_n, x_{n+p}, t) = 1 \text{ for all } t > 0.$$

Hence, $\{x_n\}$ is a G-Cauchy sequence in $(X, F, *)$. Since $(X, F, *)$ is G-complete, there exists $u \in X$ such that

(4) $$\lim_{n\to\infty} F(x_n, u, t) = \lim_{n\to\infty} F(x_{n+p}, x_n, t) = F(u, u, t) = 1 \text{ for all } t > 0, p \geq 1.$$

Let us show that u is the fixed point of T. For all $t > 0, n \in \mathbb{N}$, we obtain from (1) and (iii) that $\alpha(x_n, u, t) \geq 1$. Without loss of generality we can assume that $F(x_n, u, t) < 1$ for all $n \in \mathbb{N}$ and $t > 0$, and then, by (1) we have

$$\begin{aligned}\frac{1}{F(x_{n+1}, Tu, t)} - 1 &\leq \alpha(x_n, u, t)\left(\frac{1}{F(Tx_n, Tu, t)} - 1\right)\\ &\leq \phi\left(\frac{1}{F(x_n, u, t)} - 1\right)\\ &< \frac{1}{F(x_n, u, t)} - 1.\end{aligned}$$

The above inequality shows that $F(x_n, u, t) < F(x_{n+1}, Tu, t)$. Therefore, by (4) we obtain:

$$\lim_{n\to\infty} F(x_{n+1}, Tu, t) = 1 \text{ for all } t > 0. \tag{5}$$

By (FML4) we have

$$F(u, Tu, t) \geq F(u, x_{n+1}, t/2) * F(x_{n+1}, Tu, t/2).$$

Using (4) and (5) in the above inequality we obtain $F(u, Tu, t) = 1$ for all $t > 0$, i.e., $Tu = u$. Hence, u is a fixed point of T and $F(u, u, t) = 1$ for all $t > 0$.

For uniqueness, let v be the another fixed point of T and $u \neq v$. Then $F(u, v, t) < 1$ for all $t > 0$. It follows from (iv) that there exists $z \in X$ such that

$$\alpha(u, z, t) \geq 1 \text{ and } \alpha(v, z, t) \geq 1 \text{ for all } t > 0.$$

We shall show that

$$\lim_{n\to\infty} F(u, T^n z, t) = \lim_{n\to\infty} F(v, T^n z, t) = 1 \text{ for all } t > 0.$$

Since $\alpha(u, z, t) \geq 1$ and T is α-admissible, we get $\alpha(Tu, Tz, t) = \alpha(u, Tz, t) \geq 1$. Similarly, we obtain

$$\alpha(u, T^n z, t) \geq 1 \text{ and } \alpha(v, T^n z, t) \geq 1, \text{ for all } n \in \mathbb{N}, t > 0.$$

Therefore, using (1) we obtain

$$\begin{aligned} \frac{1}{F(u, T^n z, t)} - 1 &\leq \alpha(u, T^{n-1} z, t) \left(\frac{1}{F(u, T^n z, t)} - 1 \right) \\ &= \alpha(u, T^{n-1} z, t) \left(\frac{1}{F(Tu, TT^{n-1} z, t)} - 1 \right) \\ &\leq \phi \left(\frac{1}{F(u, T^{n-1} z, t)} - 1 \right). \end{aligned}$$

Suppose, $z_n(t) = \frac{1}{F(u,T^n z,t)} - 1$ for all $n \in \mathbb{N}$, $t > 0$. The it follows from the above inequality that

$$z_n(t) \leq \phi(z_{n-1}(t)) \text{ for all } n \in \mathbb{N}.$$

If $z_{n_0}(t) = 0$ for any $n_0 \in N$, then it follows from the above inequality that $z_n(t) = 0$ for all $n > n_0$, and so, we obtain $\lim_{n\to\infty} F(u, T^n z, t) = 1$. If it is not the case, we have $F(u, T^n z, t) < 1$ for all $n \in \mathbb{N}$, and the above inequality with Proposition 3.1 yields

$$\lim_{n\to\infty} z_n(t) = 0, \text{ i.e., } \lim_{n\to\infty} \left[\frac{1}{F(u, T^n z, t)} - 1 \right] = 0.$$

It shows that $\lim_{n\to\infty} \mathcal{F}(u, T^n z, t) = 1$.

Following similar arguments and replacing u by v in the above we have

$$\lim_{n\to\infty} \mathcal{F}(u, T^n z, t) = \lim_{n\to\infty} \mathcal{F}(v, T^n z, t) = 1 \text{ for all } t > 0. \tag{6}$$

Now, by (FML4) we have

$$\mathcal{F}(u, v, t) \geq \mathcal{F}(u, T^n z, t/2) * \mathcal{F}(T^n z, v, t/2).$$

Letting $n \to \infty$ and using (6) in the above inequality we obtain $\mathcal{F}(u, v, t) = 1$. This contradiction shows that $u = v$ and the uniqueness follows. □

The following corollary was the main result of Shukla and Abbas [12].

Corollary 3.10. *Let $(X, \mathcal{F}, *)$ be a G-complete fuzzy metric-like space and $T\colon X \to X$ be a fuzzy contractive mapping with fuzzy contractive constant k, then T has a unique fixed point $u \in X$ and $\mathcal{F}(u, u, t) = 1$ for all $t > 0$.*

Proof. Let us define two functions $\alpha\colon X^2 \times (0, \infty) \to [0, \infty)$ and $\phi\colon [0, \infty \to [0, \infty)$ by $\alpha(x, y, t) = 1$ and $\phi(r) = kr$ for all $x, y \in X$, $t > 0$ and $r \geq 0$. Then, it is easy to see that T is α-ϕ-fuzzy contractive mapping. We can show easily that all the hypotheses of Theorems 3.9 are satisfied. Consequently, T has a unique fixed point $u \in X$ and $\mathcal{F}(u, u, t) = 1$ for all $t > 0$. □

Remark 3.11. Gopal and Vetro [7] proved the existence and uniqueness of fixed of an α-ϕ-fuzzy contractive mapping (see, Theorem 3.6 of [7]). We notice that, they assumed that the fuzzy metric M is triangular. In the following corollary, we prove that this assumption is superfluous and Theorem 3.6, and so, Theorem 3.9 of [7] can be proved without this assumption.

Remark 3.12. It is obvious that the conditions (i), (ii) and (iii) are sufficient for the existence of fixed point of the mapping T. The condition (iv) is needed only when proving the uniqueness of fixed point.

Corollary 3.13. *Let $(X, M, *)$ be a G-complete fuzzy metric space. Let $T\colon X \to X$ be an α-ϕ-fuzzy contractive mapping satisfying the following conditions:*

(i) T is α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, t) \geq 1$ for all $t > 0$;

(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}, t) \geq 1$ for all $n \in \mathbb{N}$ and $\{x_n\}$ converges to x, then $\alpha(x_n, x, t) \geq 1$ for all $n \in \mathbb{N}$.

(iv) for all $x, y \in X$ and $t > 0$, there exists $z \in X$ such that $\alpha(x, z, t) \geq 1$ and $\alpha(y, z, t) \geq 1$.

Then, T has a unique fixed point $u \in X$.

Proof. Because every G-complete fuzzy metric spaces is a G-complete fuzzy metric-like space with unit self-fuzzy distance, therefore the proof follows from Theorem 3.9. $\square$

Definition 3.14. Let $(X, \preceq)$ be a partially ordered set such that $(X, \digamma, *)$ is a fuzzy metric-like space. A mapping $T\colon X \to X$ is called an ordered ϕ-fuzzy contractive mapping if there exists $\phi \in \Phi$ such that the following implication holds:

$$x, y \in X, x \preceq y \implies \frac{1}{\digamma(Tx, Ty, t)} - 1 \leq \phi\left(\frac{1}{\digamma(x, y, t)} - 1\right), \text{ for all } t > 0.$$

The next theorem is the ordered version of the main result of Shukla and Abbas [12].

Theorem 3.15. *Let $(X, \preceq)$ be a partially ordered set such that $(X, \digamma, *)$ is a 1-G-complete fuzzy metric-like space and $T\colon X \to X$ be an ordered ϕ-fuzzy contractive mapping. Suppose, the following conditions hold:*

(*i*) *T is a non-decreasing mapping with respect to $\preceq$;*

(*ii*) *there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;*

(*iii*) *if $\{x_n\}$ is a sequence in X such that $x_n \preceq x_{n+1}$ for all $n \in \mathbb{N}$ and $\{x_n\} \to x$ as $n \to \infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$;*

(*iv*) *for all $x, y \in X$ there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$.*

Then, T has a unique fixed point $u \in X$ and $\digamma(u, u, t) = 1$ for all $t > 0$.

We next introduce the notion of continuity and 1-continuity in fuzzy metric-like spaces.

Definition 3.16. Let $(X, \digamma_X, *_X)$ and $(Y, \digamma_Y, *_Y)$ be two fuzzy metric-like spaces and $T\colon X \to Y$ be a mapping. Then, T is called:

(A) continuous at $x \in X$ if for every sequence $\{x_n\} \subset X$ converging to x the sequence $\{Tx_n\} \subset Y$ converges to Tx, i.e., T is continuous at x if for every sequence $\{x_n\} \subset X$ such that $\lim_{n\to\infty} \digamma_X(x_n, x, t) = \digamma_X(x, x, t)$ for all $t > 0$, we have $\lim_{n\to\infty} \digamma_Y(Tx_n, Tx, t) = \digamma_Y(Tx, Tx, t)$ for all $t > 0$. T is called continuous on X if it is continuous at each $x \in X$.

(B) 1-continuous at $x \in X$ if for every sequence $\{x_n\} \subset X$ such that $\lim_{n\to\infty} \digamma_X(x_n, x, t) = \digamma_X(x, x, t) = 1$ for all $t > 0$, we have $\lim_{n\to\infty} \digamma_Y(Tx_n, Tx, t) = \digamma_Y(Tx, Tx, t) = 1$ for all $t > 0$.

The notions of continuity and 1-continuity are independent of each other which can be seen in the following examples:

Example 3.17. Let $X = [0,1]$ and $\mathcal{F}\colon X^2 \times (0,\infty) \to [0,1]$ be defined by

$$\mathcal{F}(x,y,t) = \frac{t}{t+\max\{x,y\}} \text{ for all } x,y \in X, t>0.$$

Then $(X, \mathcal{F}, *_p)$ is a fuzzy metric-like space. Suppose $T\colon X \to X$ be defined by

$$Tx = \begin{cases} \frac{x}{2}, & \text{if } x \in [0,1); \\ 0, & \text{if } x = 1. \end{cases}$$

Then, T is a 1-continuous mapping. Indeed, if $\{x_n\}$ be any sequence in X such that $\lim_{n\to\infty} \mathcal{F}(x_n, x, t) = \mathcal{F}(x,x,t) = 1$ for all $t>0$, then we have:

$$\lim_{n\to\infty} \frac{t}{t+\max\{x_n, x\}} = \frac{t}{t+x} = 1 \text{ for all } t>0.$$

It shows that $x = 0$ and the sequence $\{x_n\}$ must converge to 0 (in usual metric sense of $[0,1]$). Also, $T0 = 0$ and so by the above inequality we must have

$$\begin{aligned} \lim_{n\to\infty} \mathcal{F}(Tx_n, T0, t) &= \lim_{n\to\infty} \frac{t}{t+\max\{Tx_n, T0\}} \\ &= \lim_{n\to\infty} \frac{t}{t+x_n/2} = 1 \\ &= \mathcal{F}(T0, T0, t) \text{ for all } t>0. \end{aligned}$$

Thus, T is 1-continuous. On the other hand, T is not a continuous mapping. To see this choose a sequence $\{x_n\}$ in X, where $x_n = 1 - \frac{1}{2n}$, then we have

$$\lim_{n\to\infty} \mathcal{F}(x_n, 1, t) = \frac{t}{t+1} = \mathcal{F}(1,1,t).$$

So, $\{x_n\}$ converges to 1 in $(X, \mathcal{F}, *_p)$. But

$$\lim_{n\to\infty} \mathcal{F}(Tx_n, T1, t) = \lim_{n\to\infty} \frac{t}{t+\frac{1}{2}-\frac{1}{4n}} \neq \mathcal{F}(T1, T1, t).$$

Therefore, $Tx_n = \frac{1}{2} - \frac{1}{4n}$ does not converge to $T1 = 0$ in $(X, \mathcal{F}, *_p)$.

Example 3.18. Let $X = [0,1]$ and $\mathcal{F}\colon X^2 \times (0,\infty) \to [0,1]$ be defined by

$$\mathcal{F}(x,y,t) = \begin{cases} 1, & \text{if } x = y = 1; \\ \frac{t}{t+\max\{x,y\}}, & \text{otherwise} \end{cases}$$

for all $x, y \in X, t > 0$. Then $(X, \mathcal{F}, *_p)$ is a fuzzy metric-like space. Suppose $T\colon X \to X$ be defined by $Tx = \frac{x}{2}$ for all $x \in X$. Then, T is a continuous mapping. Indeed, if $\{x_n\}$ be any sequence in X such that $\lim_{n\to\infty} \digamma(x_n, x, t) = \digamma(x, x, t)$ for all $t > 0$, then we have $Tx = \frac{x}{2}$. Therefore,

$$\begin{aligned}
\lim_{n\to\infty} \digamma(Tx_n, Tx, t) &= \lim_{n\to\infty} \frac{t}{t + \max\{Tx_n, Tx\}} \\
&= \lim_{n\to\infty} \frac{t}{t + \max\{x_n/2, x/2\}} \\
&= \digamma(T(x/2), T(x/2), t) \text{ for all } t > 0.
\end{aligned}$$

Thus, T is continuous. On the other hand, T is not a 1-continuous mapping. To see this choose a sequence $\{x_n\}$ in X, where $x_n = 1 - \frac{1}{2n}$, then we have

$$\lim_{n\to\infty} \digamma(x_n, 1, t) = \frac{t}{t+1} = \digamma(1, 1, t) = 1.$$

So, $\{x_n\}$ converges to 1 in $(X, \digamma, *_p)$ with $\digamma(1, 1, t) = 1$ for all $t > 0$. But

$$\lim_{n\to\infty} \digamma(Tx_n, T1, t) = \lim_{n\to\infty} \frac{t}{t + \frac{1}{2}} = \digamma(T1, T1, t) \neq 1.$$

Therefore, $Tx_n = \frac{1}{2} - \frac{1}{4n}$ does not converge to $T1$ in $(X, \digamma, *_p)$ such that $\digamma(T1, T1, t) = 1$ for all $t > 0$.

In the next theorem, we show that the condition (iii) of Theorem 3.9 can be replaced by 1-continuity of the mapping T and thus, we find a generalization of Theorem 3.5 of Gopal and Vetro [7].

Theorem 3.19. *Let $(X, \digamma, *)$ be a 1-G-complete fuzzy metric-like space and $T : X \to X$ be an α-ϕ-fuzzy contractive mapping. Suppose, the following conditions are satisfied:*

(i) T is α-admissible;

(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, t) \geq 1$ for all $t > 0$;

(iii) T is 1-continuous on X;

(iv) for all $x, y \in X$ and $t > 0$, there exists $z \in X$ such that $\alpha(x, z, t) \geq 1$ and $\alpha(y, z, t) \geq 1$.

Then, T has a unique fixed point $u \in X$ and $\digamma(u, u, t) = 1$ for all $t > 0$.

Proof. Following the lines of the proof of Theorem 3.9 we obtain: for the sequence $\{x_n\}$ defined by $x_n = Tx_{n-1}$, $n \in \mathbb{N}$ is convergent and converges to $u \in X$ such that

$$\lim_{n\to\infty} \digamma(x_n, u, t) = \lim_{n\to\infty} \digamma(x_{n+p}, x_n, t) = \digamma(u, u, t) = 1 \text{ for all } t > 0, p \geq 1. \tag{7}$$

Let us show that u is the fixed point of T. By 1-continuity of T and the equation (7) we have

$$(8)\quad \lim_{n\to\infty} \digamma(Tx_n, Tu, t) = \lim_{n\to\infty} \digamma(x_{n+1}, Tu, t) = \digamma(Tu, Tu, t) = 1 \text{ for all } t > 0.$$

By (FML4) we have

$$\digamma(u, Tu, t) \geq \digamma(u, x_{n+1}, t/2) * \digamma(x_{n+1}, Tu, t/2).$$

Using (7) and (8) in the above inequality we obtain $\digamma(u, Tu, t) = 1$ for all $t > 0$, i.e., $Tu = u$. Thus, u is a fixed point of T.

The proof for uniqueness of fixed point is similar to the proof of uniqueness in Theorem 3.9. □

Again, one can see that in the proof of the above theorems conditions (i), (ii) and (iii) are sufficient for the existence of fixed point, and the condition (iv) is required only when proving the uniqueness of fixed point. In other words, if condition (iv) is not satisfied, then the fixed point of T may not be unique as shown in the following example.

Example 3.20. Let $X = [0, 1]$ and $\mathbb{Q}_{[0,1]} = [0, 1] \cap \mathbb{Q}$. Define $\digamma\colon X^2 \times (0, \infty) \to [0, 1]$ by

$$\digamma(x, y, t) = \begin{cases} 1, & \text{if } x = y = 2; \\ \frac{1}{1+\max\{x,y\}}, & \text{otherwise} \end{cases}$$

for all $x, y \in X, t > 0$. Then $(X, \digamma, *_p)$ is a G-complete fuzzy metric-like space. Suppose, the mappings $T\colon X \to X$ and $\alpha\colon X^2 \times (0, \infty) \to [0, \infty)$ be defined by

$$Tx = \begin{cases} x^2, & \text{if } x \in \mathbb{Q}_{[0,1]}; \\ x, & \text{otherwise} \end{cases} \quad \text{and} \quad \alpha(x, y, t) = \begin{cases} 1, & \text{if } x, y \in \mathbb{Q}_{[0,1]}; \\ 0, & \text{otherwise.} \end{cases}$$

Define the function $\phi\colon [0, \infty) \to [0, \infty)$ by

$$\phi(r) = \begin{cases} r^2, & \text{if } 0 \leq r < 1; \\ \frac{r}{2}, & \text{otherwise.} \end{cases}$$

Then, ϕ is a right continuous function such that $\phi(r) < r$ for all $r > 0$. Further, T is an α-admissible, 1-continuous mapping and it is α-ϕ-fuzzy contractive mapping. Also, if $x_0 \in \mathbb{Q}_{[0,1]}$ then $\alpha(x_0, Tx_0, t) \geq 1$ for all $t > 0$. Thus, all the conditions (except the condition (iv)) of Theorem 3.19 are satisfied and T has a fixed point $u(= 0) \in X$ such that $\digamma(u, u, t) = 1$ for all $t > 0$. Notice that, only the condition (iv) of Theorem 3.19 is not satisfied because there exists no $z \in X$ such that $\alpha(0, z, t) \geq 1$, $\alpha(i, z, t) \geq 1$, where $i \in [0, 1]$ is irrational. So, fixed point of T is not unique. Indeed, Fix $(T) = \{u\colon u \in X \setminus \mathbb{Q}_{[0,1]}\} \cup \{0\}$.

One can see easily that the continuity of mapping T assumed by Gopal and Vetro [7] in fuzzy metric spaces is a particular case of 1-continuity (as well as of continuity) of T in fuzzy metric-like spaces. Therefore, we obtain an improvement of Theorem 3.6 of [7] as a corollary of the above theorem.

Corollary 3.21. *Let $(X, M, *)$ be a G-complete fuzzy metric space and $T : X \to X$ be an α-ϕ-fuzzy contractive mapping. Suppose, the following conditions are satisfied:*

(*i*) *T is α-admissible;*

(*ii*) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0, t) \geq 1$ for all $t > 0$;*

(*iii*) *T is continuous on X;*

(*iv*) *for all $x, y \in X$ and $t > 0$, there exists $z \in X$ such that $\alpha(x, z, t) \geq 1$ and $\alpha(y, z, t) \geq 1$.*

Then, T has a unique fixed point $u \in X$.

We next define the β-ψ-contractions in fuzzy metric-like spaces.

Let Ψ be the class of all functions $\psi\colon [0, 1] \to [0, 1]$ satisfying the following properties:

(i) ψ is non-decreasing and left continuous;

(ii) $r < \psi(r)$ for all $r \in (0, 1)$.

It can be seen easily that $\psi(1) = 1$ and $\lim_{n\to\infty} \psi^n(r) = 1$ for all $r \in (0, 1)$.

Proposition 3.22. Let $\psi \in \Psi$ and $\{a_n\}$ be a sequence such that $a_n \in (0, 1)$ for all $n \in \mathbb{N}$ and $a_n \geq \phi(a_{n-1})$ for all $n \in \mathbb{N}$. Then, the sequence $\{a_n\}$ is convergent and $\lim_{n\to\infty} a_n = 1$.

Proof. Suppose, $\psi \in \Psi$ and $\{a_n\}$ be a sequence such that $a_n \in (0, 1)$ for all $n \in \mathbb{N}$ and $a_n \geq \phi(a_{n-1})$ for all $n \in \mathbb{N}$. By given condition we have

$$a_n \geq \psi(a_{n-1}) > a_{n-1} \text{ for all } n \in \mathbb{N}.$$

It shows that $\{a_n\}$ is increasing bounded above sequence of positive numbers, and so, it converges to some $a \in (0, 1]$. Suppose $a < 1$. Again, since $a_n \geq \psi(a_{n-1})$ for all $n \in \mathbb{N}$ by the left continuity of ψ we have

$$a = \lim_{n\to\infty} a_n \geq \psi(a) > a.$$

This contradiction shows that $a = 1$, i.e., $\lim_{n\to\infty} a_n = 1$. □

Definition 3.23. Let $(X, \digamma, *)$ be a fuzzy metric-like space and $T\colon X \to X$ be a mapping. Then, the mapping T is called a β-ψ-contractions if there exist two functions $\psi \in \Psi$ and $\beta\colon X^2 \times (0, \infty) \to (0, \infty)$ such that

$$\beta(x, y, t)\digamma(Tx, Ty, t) \geq \psi(\digamma(x, y, t)) \tag{9}$$

for all $t > 0$ and for all $x, y \in X$ with $x \neq y$.

Definition 3.24. Let $(X, \digamma, *)$ be a fuzzy metric-like space and $T\colon X \to X$ be a mapping. Suppose $\beta\colon X^2 \times (0, \infty) \to (0, \infty)$ be function. Then, the mapping T is called β-admissible if the following condition is satisfied:

$$x, y \in X, \beta(x, y, t) \leq 1 \implies \beta(Tx, Ty, t) \leq 1.$$

for all $t > 0$ and for all $x, y \in X$.

The identity mapping on X is a trivial example of β-admissible mapping on X for any arbitrary function $\beta\colon X^2 \times (0, \infty) \to (0, \infty)$.

We now state some fixed point results for β-ψ-contractions.

Theorem 3.25. *Let $(X, \digamma, *)$ be a 1-M-complete fuzzy metric-like space such that $\inf_{x,y \in X} \digamma(x, y, t) > 0$ for all $t > 0$. Suppose that $T : X \to X$ be an β-ψ-fuzzy contractive mapping and the following conditions are satisfied:*

(i) *T is β-admissible;*

(ii) *there exists $x_0 \in X$ such that $\beta(x_0, Tx_0, t) \leq 1$ for all $t > 0$;*

(iii) *if $\{x_n\}$ is a sequence in X such that $\beta(x_n, x_{n+1}, t) \leq 1$ for all $n \in \mathbb{N}, t > 0$, then there exists $n_0 \in \mathbb{N}$ such that $\beta(x_m, x_n, t) \leq 1$ for all $m > n > n_0$ and $t > 0$;*

(iv) *if $\{x_n\}$ is a sequence in X such that $\beta(x_n, x, t) \leq 1$ for all $n \in \mathbb{N}$ and $\{x_n\}$ converges to x, then $\beta(x_n, x, t) \leq 1$ for all $n \in \mathbb{N}$.*

(v) *for all $x, y \in X$ and $t > 0$, there exists $z \in X$ such that $\beta(x, z, t) \leq 1$ and $\beta(y, z, t) \leq 1$.*

Then, T has a unique fixed point $u \in X$ and $\digamma(u, u, t) = 1$ for all $t > 0$.

Proof. Let $x_0 \in X$ be the point such that $\beta(x_0, Tx_0, t) \leq 1$, for all $t > 0$. Let us define a sequence $\{x_n\}$ in X such that $x_n = Tx_{n-1}$, for all $n \in \mathbb{N}$. If $x_n = x_{n-1}$ for some $n \in \mathbb{N}$, then x_n is a fixed point of T. Assume that $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$, and so, we must have $\digamma(x_n, x_{n-1}, t) < 1$ for all $n \in \mathbb{N}$ and $t > 0$. Since T is β-admissible, we have

$$\beta(x_0, x_1, t) = \beta(x_0, Tx_0, t) \leq 1 \implies \beta(Tx_0, Tx_1, t) = \beta(x_1, x_2, t) \leq 1.$$

Proceeding in the same way further we get by induction:

$$\beta(x_{n-1}, x_n, t) \leq 1 \text{ for all } n \in \mathbb{N}, t > 0. \tag{10}$$

Therefore, by the condition (iii) there exists $n_0 \in \mathbb{N}$ such that

$$\beta(x_n, x_m, t) \leq 1 \text{ for all } n, m \geq n_0, t > 0. \tag{11}$$

By (9) we have

$$\digamma(x_n, x_{n+1}, t) = \digamma(Tx_{n-1}, Tx_n, t) \geq \beta(x_{n-1}, x_n, t)\digamma(Tx_{n-1}, Tx_n, t).$$

Using (9) with $x_n = x_{n-1}$ and $y = x_n$ in the above inequality and using the fact that $\psi(r) > r$, we have $\digamma(x_n, x_{n+1}, t) \geq \beta(x_{n-1}, x_n, t)\digamma(Tx_{n-1}, Tx_n, t) \geq \psi(\digamma(x_{n-1}, x_n, t)) > \digamma(x_{n-1}, x_n, t)$. Suppose, $F_n(t) = \digamma(x_n, x_{n+1}, t)$ for $t > 0$ and $n \in \mathbb{N}$. We shall show that $\lim_{n\to\infty} F_n(t) = 1$ for all $t > 0$. Then, since $x_n \neq x_{n-1}$ for all $n \in \mathbb{N}$ we have $F_n(t) < 1$ for all $t > 0$ and $n \in \mathbb{N}$. Therefore, by the above inequality we have

$$F_n(t) \geq \phi(F_{n-1}(t)) \text{ for all } n \in \mathbb{N}.$$

The above inequality with Proposition 3.22 yields $\lim_{n\to\infty} F_n(t) = 1$, i.e.,

$$\lim_{n\to\infty} \digamma(x_n, x_{n+1}, t) = 1 \text{ for all } t > 0.$$

We shall show that $\{x_n\}$ is a Cauchy sequence. Suppose

$$a_n(t) = \inf_{m>n} \digamma(x_m, x_n, t) \text{ for all } t > 0.$$

Using (10) we obtain from (9) that

$$\digamma(x_{n+1}, x_{m+1}, t) \geq \beta(x_m, x_n, t)\digamma(Tx_m, Tx_n, t) \geq \psi(\digamma(x_m, x_n, t)) > \digamma(x_m, x_n, t)$$

for all $n, m > n_0$ and $t > 0$. It shows that $a_{n+1}(t) \geq a_n(t)$ for all $n > n_0$. Therefore, the sequence $\{a_n(t)\}$ is convergent. Suppose, $\lim_{n\to\infty} a_n(t) = a(t) < 1$. Then from the above inequality and the properties of ψ we have

$$a(t) = \lim_{n\to\infty} a_n(t) \geq \psi((a)) > a(t).$$

This contradiction shows that $a(t) = 1$ for all $t > 0$, i.e., $\lim_{n\to\infty} a_n(t) = 1$, and so, for every given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\digamma(x_m, x_n, t) \geq a_n(t) > 1 - \varepsilon \text{ for all } m > n > \max\{n_0, N\} \text{ and for all } t > 0$$

or $\lim_{n,m\to\infty} \digamma(x_m, x_n, t) = 1$. Therefore, $\{x_n\}$ is a 1-M-Cauchy sequence in X. By 1-M-completeness of X, there exists $u \in X$ such that

$$\lim_{n\to\infty} \digamma(x_n, u, t) = \digamma(u, u, t) = 1 \text{ for all } t > 0. \tag{12}$$

We shall show that u is the fixed point of T.

For all $t > 0, n \in \mathbb{N}$, we obtain from (iii) that $\beta(x_n, u, t) \leq 1$. Without loss of generality we can assume that $\mathcal{F}(x_n, u, t) < 1$ for all $n \in \mathbb{N}$ and $t > 0$, and then, by (9) we have

$$\begin{aligned} \mathcal{F}(x_{n+1}, Tu, t) &= \beta(x_n, u, t)\mathcal{F}(Tx_n, Tu, t) \\ &\geq \psi\left(\mathcal{F}(x_n, u, t)\right) \\ &> \mathcal{F}(x_n, u, t). \end{aligned}$$

Therefore, $\mathcal{F}(x_n, u, t) < \mathcal{F}(x_{n+1}, Tu, t)$, and so, by (12) we obtain:

$$\lim_{n\to\infty} \mathcal{F}(x_{n+1}, Tu, t) = 1 \text{ for all } t > 0. \tag{13}$$

By (FML4) we have

$$\mathcal{F}(u, Tu, t) \geq \mathcal{F}(u, x_{n+1}, t/2) * \mathcal{F}(x_{n+1}, Tu, t/2).$$

Using (12) and (13) in the above inequality we obtain $\mathcal{F}(u, Tu, t) = 1$ for all $t > 0$, i.e., $Tu = u$. Hence, u is a fixed point of T and $\mathcal{F}(u, u, t) = 1$ for all $t > 0$.

For uniqueness, let v be the another fixed point of T and $u \neq v$, then $\mathcal{F}(u, v, t) < 1$ for all $t > 0$. It follows from (v) that there exists $z \in X$ such that

$$\beta(u, z, t) \leq 1 \text{ and } \beta(v, z, t) \leq 1 \text{ for all } t > 0.$$

Since $\beta(u, z, t) \leq 1$ and T is β-admissible, we get $\beta(Tu, Tz, t) = \beta(u, Tz, t) \leq 1$. Similarly, we obtain

$$\beta(u, T^n z, t) \leq 1 \text{ and } \beta(v, T^n z, t) \leq 1, \text{ for all } n \in \mathbb{N}, t > 0.$$

Therefore, using (9) we obtain

$$\begin{aligned} \mathcal{F}(u, T^n z, t) &\geq \beta(u, T^{n-1} z, t)\mathcal{F}(u, T^n z, t) \\ &= \beta(u, T^{n-1} z, t)\mathcal{F}(Tu, TT^{n-1} z, t) \\ &\geq \psi\left(\mathcal{F}(u, T^{n-1} z, t)\right). \end{aligned}$$

Suppose, $z_n(t) = \mathcal{F}(u, T^n z, t)$ for all $n \in \mathbb{N}$, $t > 0$. Then it follows from the above inequality that

$$z_n(t) \geq \psi\left(z_{n-1}(t)\right) \text{ for all } n \in \mathbb{N}.$$

If $z_{n_0} = 1$ for some $n_0 \in \mathbb{N}$, then it follows from the above inequality that $z_n(t) = 1$ for all $n > n_0$, and so, $\lim_{n\to\infty} \mathcal{F}(u, T^n z, t) = 1$ for all $t > 0$. If this is not the case, we have $\mathcal{F}(u, T^z, t) = 1$ for all $t > 0$ and the above inequality with Proposition 3.22 yields

$$\lim_{n\to\infty} z_n(t) = 1, \text{ i.e., } \lim_{n\to\infty} \mathcal{F}(u, T^n z, t) = 1.$$

Following similar arguments and replacing u by v in the above we have

$$\lim_{n\to\infty} \mathcal{F}(u, T^n z, t) = \lim_{n\to\infty} \mathcal{F}(v, T^n z, t) = 1 \text{ for all } t > 0. \tag{14}$$

Now, by (FML4) we have

$$\mathcal{F}(u, v, t) \geq \mathcal{F}(u, T^n z, t/2) * \mathcal{F}(T^n z, v, t/2).$$

Letting $n \to \infty$ and using (14) in the above inequality we obtain $\mathcal{F}(u, v, t) = 1$. This contradiction shows that $u = v$ and the uniqueness follows. □

The following corollary is an improvement of Theorem 4.4 and Theorem 4.6 of Gopal and Vetro [7] (see, Remark 3.27 below).

Corollary 3.26. *Let $(X, M, *)$ be an M-complete fuzzy metric-like space such that $\inf_{x,y\in X} M(x, y, t) > 0$ for all $t > 0$. Suppose that $T : X \to X$ be an β-ψ-fuzzy contractive mapping and the following conditions are satisfied:*

- *(i) T is β-admissible;*
- *(ii) there exists $x_0 \in X$ such that $\beta(x_0, Tx_0, t) \leq 1$ for all $t > 0$;*
- *(iii) if $\{x_n\}$ is a sequence in X such that $\beta(x_n, x_{n+1}, t) \leq 1$ for all $n \in \mathbb{N}, t > 0$, then there exists $n_0 \in \mathbb{N}$ such that $\beta(x_m, x_n, t) \leq 1$ for all $m > n > n_0$ and $t > 0$;*
- *(iv) if $\{x_n\}$ is a sequence in X such that $\beta(x_n, x_{n+1}, t) \leq 1$ for all $n \in \mathbb{N}$ and $\{x_n\}$ converges to x, then $\beta(x_n, x, t) \leq 1$ for all $n \in \mathbb{N}$.*
- *(v) for all $x, y \in X$ and $t > 0$, there exists $z \in X$ such that $\beta(x, z, t) \leq 1$ and $\beta(y, z, t) \leq 1$.*

Then, T has a unique fixed point $u \in X$.

Remark 3.27. In Theorem 3.25 we have proved the existence and uniqueness of the fixed point of a β-ψ-fuzzy contractive mapping in 1-M-fuzzy metric-like spaces. Note that, Gopal and Vetro [7] proved a similar result in M-complete fuzzy metric spaces, but with an additional assumption that the fuzzy metric space is *non-Archimedean*. Here, we have removed this superfluous assumption, as well as, we have extended and generalize their result in the setting of 1-M-fuzzy metric-like spaces.

Again, if we assume the 1-continuity of the mapping T, then we can omit the condition (iv) in Theorem 3.25. In this case, one can prove the following result by following the lines of proof of Theorem 3.25, and so, we omit the proof.

Theorem 3.28. *Let $(X, \mathcal{F}, *)$ be a 1-M-complete fuzzy metric-like space such that $\inf_{x,y\in X} \mathcal{F}(x, y, t) > 0$ for all $t > 0$. Suppose that $T : X \to X$ be an β-ψ-fuzzy contractive mapping and the following conditions are satisfied:*

(*i*) *T is β-admissible;*

(*ii*) *there exists $x_0 \in X$ such that $\beta(x_0, Tx_0, t) \leq 1$ for all $t > 0$;*

(*iii*) *if $\{x_n\}$ is a sequence in X such that $\beta(x_n, x_{n+1}, t) \leq 1$ for all $n \in \mathbb{N}, t > 0$, then there exists $n_0 \in \mathbb{N}$ such that $\beta(x_m, x_n, t) \leq 1$ for all $n, m > n_0$ and $t > 0$;*

(*iv*) *T is 1-continuous on X;*

(*v*) *for all $x, y \in X$ and $t > 0$, there exists $z \in X$ such that $\beta(x, z, t) \leq 1$ and $\beta(y, z, t) \leq 1$.*

Then, T has a unique fixed point $u \in X$ and $\digamma(u, u, t) = 1$ for all $t > 0$.

Again, in the above theorem the condition (v) is required only when we prove uniqueness of fixed point.

Example 3.29. Let $X = [0, 1]$ and define $\digamma\colon X^2 \times (0, \infty) \to [0, 1]$ by $\digamma(x, y, t) = e^{-\max\{x,y\}}$ for all $x, y \in X$ and $t > 0$. Then, $(X, \digamma, *_p)$ is a 1-M-complete fuzzy metric-like space. Define the mappings $T\colon X \to X$, $\beta\colon X^2 \times (0, \infty) \to (0, \infty)$ and $\psi\colon [0, 1] \to [0, 1]$ by:

$$Tx = \begin{cases} \frac{x}{2}, & \text{if } x \in [0, 1); \\ 1, & \text{if } x = 1; \end{cases} \qquad \beta(x, y, t) = \begin{cases} 1, & \text{if } x, y \in [0, 1) \text{ with } y \leq x; \\ 2, & \text{otherwise} \end{cases}$$

and $\psi(r) = \sqrt{r}$ for all $r \in [0, 1]$.

Then, ψ is a left continuous function such that $\psi(r) > r$ for all $r \in [0, 1]$. Further, T is an β-admissible, 1-continuous mapping and it is β-ψ-fuzzy contractive mapping. Also, if $x_0 \in [0, 1)$ then $\beta(x_0, Tx_0, t) \leq 1$ for all $t > 0$. Thus, all the conditions (except the condition (iv)) of Theorem 3.28 are satisfied and T has a fixed point $u(= 0) \in X$ such that $\digamma(u, u, t) = 1$ for all $t > 0$. Notice that, only the condition (iv) of Theorem 3.28 is not satisfied because there exists no $z \in X$ such that $\beta(0, z, t) \leq 1$, $\beta(1, z, t) \leq 1$. So, fixed point of T is not unique. Indeed, Fix $(T) = \{0, 1\}$.

References

[1] A. George and P. Veeramani, *On some results of analysis for fuzzy metric spaces,* Fuzzy Sets and Systems, 64 (1994), 395-399.

[2] A. A. Harandi, *Metric-like spaces, partial metric spaces and fixed points,* Fixed Point Theory Appl., Article ID 204 (2012), 1-10.

[3] A. Roldán, J. Martínez-Moreno and C. Roldán, *On interrelationship between fuzzy metric structures*, Iran. J. Fuzzy Syst., 10 (2013), 133-150.

[4] B. Samet, C. Vetro and P. Vetro, *Fixed point theorems for α-ψ-contractive type mappings,* Nonlinear Anal., 75 (2012), 2154-2165.

[5] A. Sapena, *A contribution to the study of fuzzy metric spaces,* Applied General Topology, 2 (2001), 63-75.

[6] B. Schweizer and A. Sklar, *Statistical metric spaces,* Pacific J. Math., 10 (1960), 313-334.

[7] D. Gopal and C. Vetro, *Some fixed point theorems in fuzzy metric spaces,* Ir. J. fuzzy systems, 11 (2014), 95-107.

[8] I. Kramosil and J. Michalek, *Fuzzy metric and statistical metric spaces,* Kybernetica, 15 (1975), 326-334.

[9] L. A. Zadeh, *Fuzzy sets,* Information and Control, 8 (1965), 338-353.

[10] M. Grabiec, *Fixed points in fuzzy metric spaces*, Fuzzy Sets and Systems, 27 (1988), 385-389.

[11] S. Shukla, D. Gopal and A-F Roldán-López-de-Hierro, *Some fixed point theorems in* 1-*M-complete fuzzy metric-like spaces,* International Journal of General Systems, 45 (2016), 815-829.

[12] S. Shukla and M. Abbas, *Fixed point results in fuzzy metric like spaces*, Ir. J. Fuzzy Systems, 11 (2014), 81-92.

[13] V. Gregori and A. Sapena, *On fixed point theorems in fuzzy metric spaces*, Fuzzy Sets and Systems, 125 (2002), 245-253.

[14] V. Gregori, S. Morillas and A. Sapena, *Examples of fuzzy metrics and applications*, Fuzzy Sets and Systems, 170 (2011), 95-111.

Accepted: 12.01.2019