Schur convexity of the dual form of complete symmetric function involving exponent parameter

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Abstract. In this paper, we generalize the dual form of complete symmetric function by introducing an exponent parameter, we also study the Schur-convexity, Schur-geometric convexity and Schur-harmonic convexity of this class of functions. As applications, we establish three new inequalities, which are associated with the arithmetic mean, geometric mean and harmonic mean.
Keywords: Schur-convexity, Schur-geometric convexity, Schur-harmonic convexity, complete symmetric function, dual form, inequality.

1. Introduction

The notion of majorization originates from characterizing the diversity of the components of an n-dimensional vector, it has led to large number of applications in different fields of pure and applied mathematics [1]. A famous monograph [2], published by Marshall and Olkin in 1979, covers many of the fundamental properties and research results in the majorization theory. Since then, the theory of majorization has received growing attention from mathematicians and has emerged as fascinating field.

In recent years, the Schur-convexity, Schur-geometric and Schur-harmonic convexity of various symmetric functions are hot topic in the field of majorization and inequalities. Among these investigations, the Schur convexities of the

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complete symmetric function and the dual form of complete symmetric function are of special interest and have attracted many researchers (see [3-17]). For being convenient to understand the subsequent contents, let us briefly recall the background of the relevant topic.

Throughout the paper, $\mathbb{R}$ denotes the set of real numbers, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ denotes n-tuple (n-dimensional real vectors), the set of vectors can be written as

$$\mathbb{R}^n = \{\boldsymbol{x} = (x_1, x_2, \cdots, x_n) : x_i \in \mathbb{R},\ i = 1, 2, \ldots, n\},$$

$$\mathbb{R}^n_+ = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_n) : x_i > 0,\ i = 1, 2, \ldots, n\},$$

$$\mathbb{R}^n_- = \{\boldsymbol{x} = (x_1, x_2, \ldots, x_n) : x_i < 0,\ i = 1, 2, \ldots, n\}.$$

In particular, the notations $\mathbb{R}$ and $\mathbb{R}_+$ denote $\mathbb{R}^1$ and $\mathbb{R}^1_+$, respectively.

In 1978, Menon [18] investigated certain inequalities related to the symmetric function, this kind of symmetric function is now known in the literature as complete symmetric function, which is defined by

$$c_n(\boldsymbol{x}, r) = \sum_{i_1 + i_2 + \cdots + i_n = r} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}, \tag{1}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$, $r \in \{1, 2, \ldots, n\}$, $i_1, i_2, \ldots, i_n$ are non-negative integers, $c_0(\boldsymbol{x}, r) = 1$.

Guan [19] discussed the Schur-convexity of the complete symmetric function $c_n(\boldsymbol{x}, r)$ and proved the following assertion:

Proposition 1. *For $r = 1, 2, \ldots, n$, $c_n(\boldsymbol{x}, r)$ is increasing and Schur-convex on $\mathbb{R}^n_+$.*

Inspired by the work of Guan in [19], Chu et al. [20] proved the following result:

Proposition 2. *For $r = 1, 2, \ldots, n$, $c_n(\boldsymbol{x}, r)$ is Schur-geometrically convex and Schur-harmonically convex on $\mathbb{R}^n_+$.*

In 2016, Shi et al. [21] further considered the Schur-convexity of $c_n(\boldsymbol{x}, r)$ on $\mathbb{R}^n_-$, they drew the conclusion asserted by Proposition 3 below.

Proposition 3. *If r is even integer (or odd integer, respectively), then $c_n(\boldsymbol{x}, r)$ is decreasing and Schur-convex (or increasing and Schur-concave, respectively) on $\mathbb{R}^n_-$.*

The dual form of the complete symmetric function $c_n(\boldsymbol{x}, r)$ is defined by $c^*_n(\boldsymbol{x}, r)$, i.e.,

$$c^*_n(\boldsymbol{x}, r) = \prod_{i_1 + i_2 + \cdots + i_n = r} \sum_{j=1}^{n} i_j x_j, \tag{2}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$, $r \in \{1, 2, \ldots, n\}$, $i_1, i_2, \ldots, i_n$ are non-negative integers, $c_0^*(\boldsymbol{x}, r) = 1$.

Zhang and Shi [22] proved the following results:

Proposition 4. *For $r = 1, 2, \ldots, n$, the following assertions are valid.*

(i) $c_n^*(\boldsymbol{x}, r)$ is increasing and Schur-concave on $\mathbb{R}_+^n$.

(ii) $c_n^*(\boldsymbol{x}, r)$ is Schur-geometrically convex and Schur-harmonically convex on $\mathbb{R}_+^n$.

By introducing an exponent parameter, Guan [23] studied the following generalized form of the complete symmetric function,

$$c_n\left(\boldsymbol{x}^{\frac{1}{r}}, r\right) = \sum_{i_1+i_2+\cdots+i_n=r} \left(\prod_{j=1}^{n} x_j^{i_j}\right)^{\frac{1}{r}}. \tag{3}$$

Guan [23] proved a result asserted by the Theorem A below.

Theorem A. *For $r = 1, 2, \ldots, n$, $c_n(\boldsymbol{x}^{\frac{1}{r}}, r)$ is an increasing Schur-concave function on $\mathbb{R}_+^n$.*

Chu and Sun [24] proved the assertion of Theorem B, i.e.,

Theorem B. *For $r = 1, 2, \ldots, n$, $c_n(\boldsymbol{x}^{\frac{1}{r}}, r)$ is Schur-harmonically concave on $\mathbb{R}_+^n$.*

Wang and Yang [25] obtained more extensive results described in Theorem C below.

Theorem C. *For $r = 1, 2, \ldots, n$, $c_n(\boldsymbol{x}^{\frac{1}{r}}, r)$ is Schur m-power concave on $\mathbb{R}_+^n$ when $m > 1$, and $c_n(\boldsymbol{x}^{\frac{1}{r}}, r)$ is Schur m-power convex on $\mathbb{R}_+^n$ when $m \leq \frac{1}{r}$.*

Motivated by the work of [23], [24] and [25], in this paper, we generalize the dual form of complete symmetric function by introducing an exponent parameter, the generalized version of $c_n^*(\boldsymbol{x}, r)$, written as $c_n^*(\boldsymbol{x}^{\frac{1}{r}}, r)$, is defined by

$$c_n^*\left(\boldsymbol{x}^{\frac{1}{r}}, r\right) = \prod_{i_1+i_2+\cdots+i_n=r} \sum_{j=1}^{n} i_j \left(x_j^{\frac{1}{r}}\right). \tag{4}$$

Also, we will study the Schur-convexity, Schur-geometric convexity and Schur-harmonic convexity of the function $c_n^*(\boldsymbol{x}, r)$. Our main results are stated in the Theorem 1 below.

Theorem 1. *For $r = 1, 2, \ldots, n$, the following assertions are valid.*

(i) The symmetric function $c_n^*(\boldsymbol{x}^{\frac{1}{r}}, r)$ is increasing and Schur-concave on $\mathbb{R}_+^n$;

(ii) The symmetric function $c_n^*(\boldsymbol{x}^{\frac{1}{r}}, r)$ is Schur-geometrically convex on $\mathbb{R}_+^n$;

(iii) The symmetric function $c_n^*(\boldsymbol{x}^{\frac{1}{r}}, r)$ is Schur-harmonically convex on $\mathbb{R}_+^n$.

2. Definitions and lemmas

In this section we quote some definitions and lemmas, which will be used in the proofs of the main results in subsequent sections.

Definition 1 ([2, 3]). Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$.

(*i*) $\boldsymbol{x} \geq \boldsymbol{y}$ means $x_i \geq y_i$ for all $i = 1, 2, \ldots, n$.

(*ii*) Let $\Omega \subset \mathbb{R}^n$, φ: $\Omega \to \mathbb{R}$ is said to be increasing if $\boldsymbol{x} \geq \boldsymbol{y}$ implies $\varphi(\boldsymbol{x}) \geq \varphi(\boldsymbol{y})$. φ is said to be decreasing if and only if $-\varphi$ is increasing.

Definition 2 ([2, 3]). Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$.

(*i*) $\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$ (in symbols $\boldsymbol{x} \prec \boldsymbol{y}$) if $\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}$ for $k = 1, 2, \ldots, n-1$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, where $x_{[1]} \geq x_{[2]} \geq \cdots \geq x_{[n]}$ and $y_{[1]} \geq y_{[2]} \geq \cdots \geq y_{[n]}$ are rearrangements of $\boldsymbol{x}$ and $\boldsymbol{y}$ in a descending order.

(*ii*) Let $\Omega \subset \mathbb{R}^n$, φ: $\Omega \to \mathbb{R}$ is said to be a Schur-convex function on Ω if $\boldsymbol{x} \prec \boldsymbol{y}$ on Ω implies $\varphi(\boldsymbol{x}) \leq \varphi(\boldsymbol{y})$. φ is said to be a Schur-concave function on Ω if and only if $-\varphi$ is Schur-convex function on Ω.

Definition 3 ([2, 3]). Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$.

(*i*) $\Omega \subset \mathbb{R}^n$ is said to be a convex set if $\boldsymbol{x}, \boldsymbol{y} \in \Omega, 0 \leq \alpha \leq 1$, implies $\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y} = (\alpha x_1 + (1-\alpha)y_1, \alpha x_2 + (1-\alpha)y_2, \ldots, \alpha x_n + (1-\alpha)y_n) \in \Omega$.

(*ii*) Let $\Omega \subset \mathbb{R}^n$ be convex set. A function φ: $\Omega \to \mathbb{R}$ is said to be a convex function on Ω if

$$\varphi(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \leq \alpha\varphi(\boldsymbol{x}) + (1-\alpha)\varphi(\boldsymbol{y})$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, and all $\alpha \in [0, 1]$. φ is said to be a concave function on Ω if and only if $-\varphi$ is convex function on Ω.

Definition 4 ([2, 3]). Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \Omega \subset \mathbb{R}^n$.

(*i*) A set $\Omega \subset \mathbb{R}^n$ is called a symmetric set, if $\boldsymbol{x} \in \Omega$ implies $\boldsymbol{x}P \in \Omega$ for every $n \times n$ permutation matrix P.

(*ii*) A function $\varphi : \Omega \to \mathbb{R}$ is called symmetric if for every permutation matrix P, $\varphi(\boldsymbol{x}P) = \varphi(\boldsymbol{x})$ for all $\boldsymbol{x} \in \Omega$.

Lemma 1 ([2, 3]). *Let $\Omega \subset \mathbb{R}^n$ be symmetric and have a nonempty interior convex set. Ω° is the interior of Ω. $\varphi : \Omega \to \mathbb{R}$ is continuous on Ω and differentiable in Ω°. Then φ is the Schur-convex (or Schur-concave, respectively) function if and only if φ is symmetric on Ω and*

$$(x_1 - x_2)\left(\frac{\partial\varphi}{\partial x_1} - \frac{\partial\varphi}{\partial x_2}\right) \geq 0 \quad (or \leq 0,\ respectively) \tag{5}$$

holds for any $\boldsymbol{x} \in \Omega^\circ$.

Definition 5 ([4])**.** Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+$ and $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n_+$.

(*i*) $\Omega \subset \mathbb{R}^n_+$ is called a geometrically convex set if $(x_1^\alpha y_1^\beta, x_2^\alpha y_2^\beta, \ldots, x_n^\alpha y_n^\beta) \in \Omega$ for all $\boldsymbol{x}$,$\boldsymbol{y} \in \Omega$ and α,$\beta \in [0, 1]$ such that $\alpha + \beta = 1$.

(*ii*) Let $\Omega \subset \mathbb{R}^n_+$. The function φ: $\Omega \to \mathbb{R}_+$ is said to be Schur-geometrically convex function on Ω if $(\log x_1, \log x_2, \ldots, \log x_n) \prec (\log y_1, \log y_2, \ldots, \log y_n)$ on Ω implies $\varphi(\boldsymbol{x}) \le \varphi(\boldsymbol{y})$. The function φ is said to be a Schur-geometrically concave function on Ω if and only if $-\varphi$ is Schur-geometrically convex function on Ω.

Lemma 2 ([4])**.** *Let $\Omega \subset \mathbb{R}^n_+$ be a symmetric and geometrically convex set with a nonempty interior Ω°. Let $\varphi : \Omega \to \mathbb{R}_+$ be continuous on Ω and differentiable in Ω°. If φ is symmetric on Ω and*

$$(\log x_1 - \log x_2)\left(x_1\frac{\partial\varphi}{\partial x_1} - x_2\frac{\partial\varphi}{\partial x_2}\right) \ge 0 \quad (or \le 0, respectively) \tag{6}$$

holds for any $\boldsymbol{x} \in \Omega^\circ$, then φ is a Schur-geometrically convex (or Schur-geometrically concave, respectively) function.

Definition 6 ([24])**.** Let $\Omega \subset \mathbb{R}^n_+$ or $\Omega \subset \mathbb{R}^n_-$.

(*i*) A set Ω is said to be harmonically convex if $\frac{\boldsymbol{xy}}{\lambda\boldsymbol{x}+(1-\lambda)\boldsymbol{y}} \in \Omega$, for every $\boldsymbol{x}, \boldsymbol{y} \in \Omega$ and $\lambda \in [0, 1]$, where $\frac{\boldsymbol{xy}}{\lambda\boldsymbol{x}+(1-\lambda)\boldsymbol{y}} = \left(\frac{x_1y_1}{\lambda x_1+(1-\lambda)y_1}, \cdots, \frac{x_ny_n}{\lambda x_n+(1-\lambda)y_n}\right)$.

(*ii*) A function $\varphi : \Omega \to \mathbb{R}_+$ is said to be Schur-harmonically convex on Ω if $\frac{1}{\boldsymbol{x}} \prec \frac{1}{\boldsymbol{y}}$ implies $\varphi(\boldsymbol{x}) \le \varphi(\boldsymbol{y})$, where $\frac{1}{\boldsymbol{x}} = \left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right)$. A function φ is said to be a Schur-harmonically concave function on Ω if and only if $-\varphi$ is a Schur-harmonically convex function.

Lemma 3 ([2, 3])**.** *Let $\mathbb{A}, \mathbb{B} \subset \mathbb{R}$, $\varphi : \mathbb{B}^n \to \mathbb{R}$, $f : \mathbb{A} \to \mathbb{B}$ and $\psi(x_1, x_2, \ldots, x_n) = \varphi(f(x_1), f(x_2), \ldots, f(x_n)) : \mathbb{A}^n \to \mathbb{R}$.*

(*i*) *If f is convex and φ is increasing and Schur-convex, then ψ is Schur-convex;*

(*ii*) *If f is concave, φ is increasing and Schur-concave, then ψ is Schur-concave.*

Lemma 4 ([2, 3])**.** *Let $\Omega \subset \mathbb{R}^n_+$. The function $\varphi : \Omega \to \mathbb{R}_+$ is differentiable.*

(*i*) *If φ is increasing and Schur-geometrically convex, then φ is Schur-harmonically convex.*

(*ii*) *If φ is decreasing and Schur-geometrically concave, then φ is Schur-harmonically concave.*

Lemma 5 ([2, 3])**.** *Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Then*

$$(A(\boldsymbol{x}), A(\boldsymbol{x}), \ldots, A(\boldsymbol{x})) \prec \boldsymbol{x} = (x_1, x_2, \ldots, x_n), \tag{7}$$

where $A(\boldsymbol{x}) = \frac{1}{n}\sum_{i=1}^n x_i$.

3. Proof of main results

Proof of Theorem 1.

(i) Define a function $g(t) = t^{\frac{1}{r}}$, $t \in (0, +\infty)$. Then

$$g'(t) = \frac{1}{r} t^{\frac{1}{r}-1}, \quad g''(t) = \frac{1}{r}\left(\frac{1}{r} - 1\right) t^{\frac{1}{r}-2}. \tag{8}$$

For $r = 1, 2, \ldots, n$, $g(t)$ is increasing and concave on $\mathbb{R}_+$. By using Proposition 4, Lemma 3 and Definition 1, it follows that $c_n^*(\boldsymbol{x}^{\frac{1}{r}}, r)$ is increasing and Schur-concave on $\mathbb{R}_+^n$.

(ii) For $r = 1$ and $r = 2$, it is easy to prove that the symmetric function $c_n^*(\boldsymbol{x}^{\frac{1}{r}}, r)$ is Schur-geometrically convex on $\mathbb{R}_+^n$.

Now consider the case of $r \geq 3$. By the symmetry of $c_n^*(\boldsymbol{x}^{\frac{1}{r}}, r)$, without loss of generality, we set $x_1 > x_2$. Note that

$$\begin{aligned} c_n^*\left(\boldsymbol{x}_j^{\frac{1}{r}}, r\right) = & \prod_{\substack{i_1+i_2+\cdots+i_n=r \\ i_1\neq 0, i_2=0}} \sum_{j=1}^n i_j x_j^{\frac{1}{r}} \times \prod_{\substack{i_1+i_2+\cdots+i_n=r \\ i_1=0, i_2\neq 0}} \sum_{j=1}^n i_j x_j^{\frac{1}{r}} \\ & \times \prod_{\substack{i_1+i_2+\cdots+i_n=r \\ i_1\neq 0, i_2\neq 0}} \sum_{j=1}^n i_j x_j^{\frac{1}{r}} \times \prod_{\substack{i_1+i_2+\cdots+i_n=r \\ i_1=0, i_2=0}} \sum_{j=1}^n i_j x_j^{\frac{1}{r}}. \end{aligned}$$

Then

$$\begin{aligned} \frac{\partial c_n^*\left(\boldsymbol{x}_j^{\frac{1}{r}}, r\right)}{\partial x_1} &= c_n^*\left(\boldsymbol{x}_j^{\frac{1}{r}}, r\right) \\ & \times \left(\sum_{\substack{i_1+i_2+\cdots+i_n=r \\ i_1\neq 0, i_2=0}} \frac{i_1 \frac{1}{r} x_1^{\frac{1}{r}-1}}{\sum_{j=1}^n i_j x_j^{\frac{1}{r}}} + \sum_{\substack{i_1+i_2+\cdots+i_n=r \\ i_1\neq 0, i_2\neq 0}} \frac{i_1 \frac{1}{r} x_1^{\frac{1}{r}-1}}{\sum_{j=1}^n i_j x_j^{\frac{1}{r}}} \right) \\ &= c_n^*\left(\boldsymbol{x}_j^{\frac{1}{r}}, r\right) \Big(\sum_{\substack{k+k_3+\cdots+k_n=r \\ k\neq 0}} \frac{k \frac{1}{r} x_1^{\frac{1}{r}-1}}{k x_1^{\frac{1}{r}} + \sum_{j=3}^n i_j x_j^{\frac{1}{r}}} \\ & + \sum_{\substack{k+m+i_3+\cdots+i_n=r \\ k\neq 0, m\neq 0}} \frac{k \frac{1}{r} x_1^{\frac{1}{r}-1}}{k x_1^{\frac{1}{r}} + m x_2^{\frac{1}{r}} + \sum_{j=3}^n i_j x_j^{\frac{1}{r}}} \Big). \end{aligned} \tag{9}$$

By the same arguments, we have

$$\frac{\partial c_n^*\left(\boldsymbol{x}_j^{\frac{1}{r}}, r\right)}{\partial x_2} = c_n^*\left(\boldsymbol{x}_j^{\frac{1}{r}}, r\right)\Big(\sum_{\substack{k+k_3+\cdots+k_n=r\\ k\neq 0}} \frac{k\frac{1}{r}x_2^{\frac{1}{r}-1}}{kx_2^{\frac{1}{r}} + \sum\limits_{j=3}^{n} i_j x_j^{\frac{1}{r}}}$$

$$+ \sum_{\substack{k+m+i_3+\cdots+i_n=r\\ k\neq 0, m\neq 0}} \frac{k\frac{1}{r}x_2^{\frac{1}{r}-1}}{kx_2^{\frac{1}{r}} + mx_1^{\frac{1}{r}} + \sum\limits_{j=3}^{n} i_j x_j^{\frac{1}{r}}}\Big). \tag{10}$$

$$x_1\frac{\partial c_n^*\left(\boldsymbol{x}_j^{\frac{1}{r}}, r\right)}{\partial x_1} - x_2\frac{\partial c_n^*\left(\boldsymbol{x}_j^{\frac{1}{r}}, r\right)}{\partial x_2} = c_n^*\left(\boldsymbol{x}_j^{\frac{1}{r}}, r\right)(A_1 + A_2),$$

where

$$A_1 = \sum_{\substack{k+k_3+\cdots+k_n=r\\ k\neq 0}} \left(\frac{k\frac{1}{r}x_1^{\frac{1}{r}}}{kx_1^{\frac{1}{r}} + \sum\limits_{j=3}^{n} i_j x_j^{\frac{1}{r}}} - \frac{k\frac{1}{r}x_2^{\frac{1}{r}}}{kx_2^{\frac{1}{r}} + \sum\limits_{j=3}^{n} i_j x_j^{\frac{1}{r}}}\right)$$

$$= k\sum_{\substack{k+k_3+\cdots+k_n=r\\ k\neq 0}} \frac{\frac{1}{r}(x_1^{\frac{1}{r}} - x_2^{\frac{1}{r}})\sum\limits_{j=3}^{n} i_j x_j^{\frac{1}{r}}}{(kx_1^{\frac{1}{r}} + \sum\limits_{j=3}^{n} i_j x_j^{\frac{1}{r}})(kx_2^{\frac{1}{r}} + \sum\limits_{j=3}^{n} i_j x_j^{\frac{1}{r}})} \tag{11}$$

and

$$A_2 = \sum_{\substack{k+m+i_3+\cdots+i_n=r\\ k\neq 0, m\neq 0}} \Big(\frac{k\frac{1}{r}x_1^{\frac{1}{r}}}{kx_1^{\frac{1}{r}} + mx_2^{\frac{1}{r}} + \sum\limits_{j=3}^{n} i_j x_j^{\frac{1}{r}}} - \frac{k\frac{1}{r}x_2^{\frac{1}{r}}}{kx_2^{\frac{1}{r}} + mx_1^{\frac{1}{r}} + \sum\limits_{j=3}^{n} i_j x_j^{\frac{1}{r}}}\Big)$$

$$= k\sum_{\substack{k+m+i_3+\cdots+i_n=r\\ k\neq 0, m\neq 0}} \frac{\delta}{(kx_1^{\frac{1}{r}} + mx_2^{\frac{1}{r}} + \sum\limits_{j=3}^{n} i_j x_j^{\frac{1}{r}})(kx_2^{\frac{1}{r}} + mx_1^{\frac{1}{r}} + \sum\limits_{j=3}^{n} i_j x_j^{\frac{1}{r}})} \tag{12}$$

where

$$\delta = \frac{m}{r}(x_1^{\frac{2}{r}} - x_2^{\frac{2}{r}}) + \frac{1}{r}(x_1^{\frac{1}{r}} - x_2^{\frac{1}{r}})\sum_{j=3}^{n} i_j x_j^{\frac{1}{r}}.$$

For $r = 1, 2, \ldots, n$ and $\boldsymbol{x} \in \mathbb{R}_+^n$, it is easy to see that $A_1 \geq 0$ and $A_2 \geq 0$, and then we have

$$x_1\frac{\partial c_n^*(\boldsymbol{x}_j^{\frac{1}{r}}, r)}{\partial x_1} - x_2\frac{\partial c_n^*(\boldsymbol{x}_j^{\frac{1}{r}}, r)}{\partial x_2} \geq 0,$$

by Lemma 2, it follows that $c_n^*(\boldsymbol{x}_j^{\frac{1}{r}}, r)$ is Schur-geometrically convex on $\mathbb{R}_+^n$.

(iii) Since $c_n^*(\boldsymbol{x}_j^{\frac{1}{r}}, r)$ is increasing and Schur-geometrically convex on $\mathbb{R}_+^n$, by Lemma 4, it follows that $c_n^*(\boldsymbol{x}_j^{\frac{1}{r}}, r)$ is Schur-harmonically convex on $\mathbb{R}_+^n$.

The proof of Theorem 1 is completed.

4. Applications

As an application of Theorem 1, we establish three new inequalities, which are associated with the arithmetic mean, geometric mean and harmonic mean.

Theorem 2. *For $\boldsymbol{x} \in \mathbb{R}_+^n$, $r = 1, 2, \ldots, n$, we have the following inequalities*

$$\prod_{i_1+i_2+\cdots+i_n=r} \sum_{j=1}^{n} i_j \left(x_j^{\frac{1}{r}} \right) \leq \left(r(A_n(\boldsymbol{x}))^{\frac{1}{r}} \right)^{\binom{n+r-1}{r}}, \tag{13}$$

$$\prod_{i_1+i_2+\cdots+i_n=r} \sum_{j=1}^{n} i_j \left(x_j^{\frac{1}{r}} \right) \geq \left(r(G_n(\boldsymbol{x}))^{\frac{1}{r}} \right)^{\binom{n+r-1}{r}}, \tag{14}$$

$$\prod_{i_1+i_2+\cdots+i_n=r} \sum_{j=1}^{n} i_j \left(x_j^{\frac{1}{r}} \right) \geq \left(r(H_n(\boldsymbol{x}))^{\frac{1}{r}} \right)^{\binom{n+r-1}{r}}, \tag{15}$$

where $A(\boldsymbol{x}) = \frac{1}{n}\sum_{i=1}^n x_i$, $G_n(\boldsymbol{x}) = \sqrt[n]{\prod_{i=1}^n x_i}$, $H_n(\boldsymbol{x}) = \frac{n}{\sum_{i=1}^n x_i^{-1}}$ *and* $\binom{n+r-1}{r} = \frac{(n+r-1)!}{r!((n+r-1)-r)!}$.

Proof. By using the assertion *(i)* of Theorem 1 and the majorization relation (7), we get $c_n^*(\boldsymbol{x}^{\frac{1}{r}}, r) \leq c_n^*(\boldsymbol{y}^{\frac{1}{r}}, r)$, $\boldsymbol{y} = (A(\boldsymbol{x}), A(\boldsymbol{x}), \ldots, A(\boldsymbol{x}))$, which is equivalent to the inequality (13).

A direct application of the majorization relation (7), it is easy to observe that

$$(\log G_n(\boldsymbol{x}), \log G_n(\boldsymbol{x}), \ldots, \log G_n(\boldsymbol{x})) \prec (\log x_1, \log x_2, \ldots, \log x_n).$$

Now, from the above majorization relation and the assertion *(ii)* of Theorem 1, we deduce the inequality (14).

From the majorization relation (7), it follows that

$$\left(\frac{1}{H_n(\boldsymbol{x})}, \frac{1}{H_n(\boldsymbol{x})}, \ldots, \frac{1}{H_n(\boldsymbol{x})} \right) \prec \left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n} \right).$$

Utilizing the above majorization relation and the assertion *(iii)* of Theorem 1 leads to the desired inequality (15).

This completes the proof of Theorem 2. □

Remark 1. Taking $r = 1$ in Theorem 2, from the inequalities (13) and (15), we obtain the inequalities

$$A(\boldsymbol{x}) \geq G_n(\boldsymbol{x}) \geq H_n(\boldsymbol{x}),$$

where $A(\boldsymbol{x})$, $G_n(\boldsymbol{x})$, $H_n(\boldsymbol{x})$ are the classical means known as arithmetic mean, geometric mean and harmonic mean respectively.

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