Edge-to-vertex and edge-to-edge D-distances

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Abstract. In a (simple and connected) graph G, the concept of distance is one of the important concept. Earlier we introduced the concept of D-distance between vertices of a graph.

In this article we study edge-to-vertex and edge-to-edge distances w.r.t. D-distance. We obtain relation between their eccentrics and also determine centers of some graphs. We prove the relation between the edge-to-vertex and edge-to-edge D-eccentricities w.r.t. D-distance.We also prove that for any graph G either $C_2^D(G) \subseteq C_3^D(G)$ or $C_3^D(G) \subseteq C_2^D(G)$.

Keywords: D-distance, radius, diameter, center, eccentricity.

1. Introduction

The distance between vertices of a graph plays an important role in many places. The adjacency problems can be converted into distance problems and vice versa. This concept has been extended to distance between vertex-to-edge distance edge-to-vertex distance and edge-to-edge distance.

The first two authors have introduced the concept of D-distance between vertices of a graph in [2] and extended it vertex-to-edge D-distance in [3].

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In the present work, we extend the concept of D-distance to edge-to-vertex and edge-to-edge D-distance and study some properties. We give a result connecting edge-to-vertex D-centers and edge-to-edge D-centers. Also we prove a result on edge-to-vertex center of a tree.

2. Edge to vertex and edge to edge distances

By a graph $G(V, E)$ or simply G, we mean a non-trivial, finite, undirected graph without multiple edges and loops. Further unless otherwise specified all graphs will be connected.

In this section we introduce some definitions by extending the vertex to vertex concepts to edge to vertex and edge to edge. We begin with D-distance in graphs introduced the authors in (see [2]).

Definition 1. *If u, v are vertices of a connected graph G. The D-length of a connected $u - v$ path s is defined as $l^D(s) = l(s) + deg(v) + deg(u) + \sum deg(w)$ where sum runs over all intermediate vertices w of s and $l(s)$ is the length of the path.*

Definition 2 (D-distance)**.** *The* D-distance $d^D(u, v)$ *between two vertices u, v of a connected graph G is defined as $d^D(u, v) = 0$ if $u = v$ and $d^D(u, v) = \min \left\{l^D(s)\right\}$ if u, v are distinct, where the minimum is taken over all $u-v$ paths s in G. In other words,*

$$d^D(u, v) = \begin{cases} \min_s \left\{l^D(s) + deg(u) + deg(v) + \sum deg(w)\right\}, & \text{if } u \neq v \\ 0, & \text{if } u = v \end{cases}$$

where the sum runs over all intermediate vertices w in s and minimum is taken over all $u - v$ paths s in G.

Now we define the edge to vertex D-distance.

Definition 3. *Let G be a connected graph taken an edgee $= xy \in E(G)$ and vertex v in G. Then the D-distance between* edge (e) and vertex (v) *is defined as $d^D(e, v) = \min \left\{d^D(x, v), d^D(y, v)\right\}$.*

Definition 4. *For any edge f of G in a connected graph G, the* edge-to-vertex D-eccentricity *of $e_2^D(f)$, is given by $e_2^D(f) = \max \left\{d^D(e, v) : v \in V\right\}$.*

Definition 5. *Any edge f for which $e_2^D(f)$ is minimum is called* edge-to-vertex central edge *of G. The set of all edge-to-vertex central edges of G is the* edge-to-vertex center *and is denoted as $C_2^D(G)$.*

Next, we extend the distance concept to edge to edge D-distance.

Definition 6. *Let G be a connected graph and $e = xy$, $f = uv$ be two edges of G. Then the* edge-to-edge D-distance *is defined as*

$$d^D(e, f) = \min \left\{d^D(x, u), d^D(x, v), d^D(y, u), d^D(y, v)\right\}.$$

Definition 7. *For any edge f in a connected graph G, the* edge-to-edge D-eccentricity *off $e_3^D(f)$, is defined as $e_3^D(f) = \max\left\{d^D(f,e) : e \in E(G)\right\}$.*

One more definition.

Definition 8. *Any edge f of G, for which e_3^D is minimum is called* central edge *of G. The set of all central edges of G is the edge to edge D-distance* center *of G and is denoted by $C_3^D(G)$. Any edge e for which $e_3^D(e) = d^D(f,e)$ is called an* eccentric edge *of f.*

Remark 9. Observe that $d^D(e,f) = 0 \Leftrightarrow e, f$ are neighbor edges i.e., they have one common vertex

Below we give an example.

Example 10. Consider the $(6,6)$ graph G, shown in the figure 1. In this graph, the edge-to-vertex and edge-to-edge D-distances, are as shown in the tables 1 and 2. From these tables, we write various eccentricities of edges which are presented in table 3.

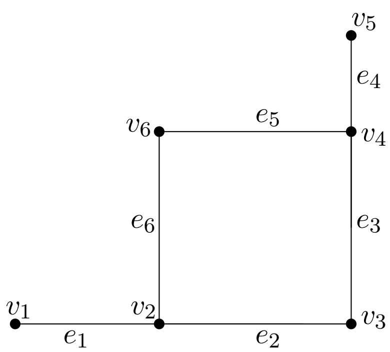

Figure 1: Graph G

The following table shows the edge-to-vertex D-distances of the graph G.

	v_1	v_2	v_3	v_4	v_5	v_6
e_1	0	0	6	10	12	6
e_2	5	0	0	6	8	6
e_3	8	6	0	0	5	6
e_4	12	10	6	0	0	6
e_5	8	6	6	0	5	0
e_6	5	0	6	6	8	0

Table 1: D-distance between edges and vertices of G.

The following table shows D-distances between edges of graph G.

	e_1	e_2	e_3	e_4	e_5	e_6
e_1	0	0	6	10	6	0
e_2	0	0	0	6	6	0
e_3	6	0	0	0	0	6
e_4	10	6	0	0	0	6
e_5	6	6	0	0	0	0
e_6	0	0	6	6	0	0

Table 2: D-distance between edges of G.

From the above two tables we write down the D-eccentricities of edges which are as follows:

	e_1	e_2	e_3	e_4	e_5	e_6
$e_2^D(e)$	12	8	8	12	8	8
$e_3^D(e)$	10	6	6	10	6	6

Table 3: D-eccentricities of G

Remark 11. In the above example observe that $C_2^D(G) = \{e_2, e_3, e_5, e_6\}$ and $C_3^D(G) = \{e_2, e_3, e_5, e_6\}$ and hence $C_2^D(G) = C_3^D(G)$.

3. Edge-to-vertex and edge-to-edge centers

We begin this with a result on the edge to vertex and edge to edge eccentricities.

Theorem 12. *For any edge e in a connected graph G, we have $e_2^D(e) - deg(e) - 1 \leq e_3^D(e) \leq e_2^D(e)$. Further, $e_3^D(e) = e_2^D(e)$ if and only if both the vertices of any eccentric edge of e are eccentric vertices of e.*

Proof. Let e be any edge of G and let f be an eccentric edge of e. Let $p = u = v_0, v_1, v_2, \cdots, v_n = v$ be a $u - v$ minimum D-path and let $e = v_{n-1}v_n$. Then

$$\begin{aligned} d^D(e, v) &= k + deg(v_0) + deg(v_1) + deg(v_2) + \cdots + deg(v_n) \\ &= e_2^D(e), \end{aligned}$$

where k is length of the path.

Let $f = v_{m-1}v_m$, since

$$\begin{aligned} d^D(e, f) &= k - 1 + deg(v_0) + deg(v_1) + deg(v_2) + \cdots + deg(v_{n-1}) \\ &= e_2^D(e) - deg(v) - 1, \end{aligned}$$

it follows that $e_3^D(e) \geq e_2^D(e) - deg(v) - 1$. Also since $e_3^D(e) = \min\left\{d^D(e, f) : e, f \in E\right\}$, it follows that that $e_3^D(e) \leq d^D(e, v) = e_2^D(e)$. Thus $e_2^D(e) - deg(v) - 1 \leq e_3^D(e) \leq e_2^D(e)$. □

Next, we prove a result on edge to vertex and edge to edge centers.

Theorem 13. *For any graph G, either $C_2^D(G) \subseteq C_3^D(G)$ or $C_3^D(G) \subseteq C_2^D(G)$.*

Proof. Suppose the result is false. Then thereexit two edges e and f such that $e \in C_3^D(G) - C_2^D(G)$ and $f \in C_2^D(G) - C_3^D(G)$ i.e., $e \in C_3^D(G)$, $e \notin C_2^D(G)$ and $f \in C_2^D(G)$, $f \notin C_3^D(G)$. Then

$$e_2^D(f) < e_2^D(e) \tag{3.1}$$

and

$$e_3^D(e) < e_3^D(f). \tag{3.2}$$

If $e_3^D(e) = e_2^D(e)$ we have $e_2^D(f) < e_2^D(e) = e_3^D(e) < e_3^D(f) \Rightarrow e_2^D(f) < e_3^D(f)$ which is contradiction to theorem(12). If $e_3^D(f) = e_2^D(f) - deg(f) - 1$ we have $e_3^D(e) - e_3^D(f) = e_2^D(f) - deg(f) - 1 < e_2^D(e) - deg(e) - 1 \Rightarrow e_3^D(e) < e_2^D(e) - deg(e) - 1$ which is again contradiction to theorem(12). Thus

$$e_3^D(e) = e_2^D(e) - deg(e) - 1 \tag{3.3}$$

and

$$e_3^D(f) = e_2^D(f). \tag{3.4}$$

From equation (3.2) we have $e_3^D(e) < e_3^D(f) = e_2^D(f) \Rightarrow e_3^D(e) < e_2^D(f) \Rightarrow e_2^D(e) - deg(e) - 1 < e_2^D(f) \Rightarrow e_2^D(e) < e_2^D(f)$ which is contradiction to equation (3.1). Hence the theorem. □

Below we construct a class of graphs for which both of these centers are equal.

Theorem 14. *Let $G = (K_{n1} \cup\ K_{n2} \cup\ K_{n3} \cup \ldots \cup\ K_{nr} \cup\ kK_1) + v$ be a block graph of order $p \geq 4$, $n_i \geq 2$ $(1 \leq i \leq r)$ and $n_1 + n_2 + n_3 + \cdots + n_r + k = p - 1$, then $C_2^D(G) = C_3^D(G)$.*

Proof. Let $r \geq 2$. For any edge $g = xy$ incident at vertex v then $e_2^D(g) = e_3^D(g) = deg(v) + deg(w)$, $\forall w \in K_{nr}$, $n_1 = n_2 = n_3 = \cdots = n_r$, $e_2^D(g) = e_3^D(g) = 1 + deg(v) + deg(w)$, $\forall w \in K_{nr}$, $n_1, n_2, n_3, \cdots, n_r$ are different and $e_2^D(f) = e_3^D(f) = 2 + deg(v) + deg(w_1) + deg(w_2)$. Thus $C_2^D(G) = C^D 3(G) =$ the set of all edges incident at v. □

Example 15. Consider the $(5, 5)$ graph kite (K), shown in the figure 2. For this, we see $C_3^D \subseteq C_2^D$.

Example 16. Consider the $(5, 6)$ graph Hut (H), shown in the figure 3. For this, we see $C_2^D \subseteq C_3^D$.

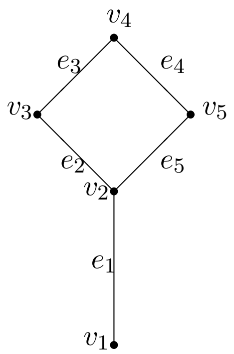

Figure 2: Kite (K)

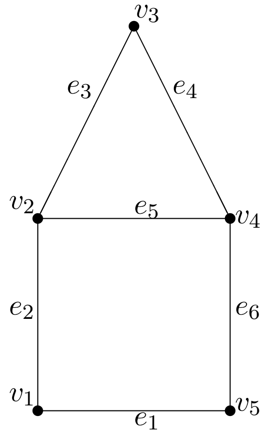

Figure 3: Hut (H)

Next, we present a result on edge to vertex center of a tree.

Theorem 17. *The edge-to-vertex center $C_2^D(T)$ of a non trivial tree T induces a star.*

Proof. If T is a tree and $T \neq K_2$, then $e_2^D(e) = n+1$ for any edge e. If $T = K_2$ then $e_2^D(e) = 0$ thus the subgraph induced by $C_2^D(T)$ is T it self. If T is not a star, we first prove that the subgraph induced by $C_2^D(T)$ is connected other wise, there exists two edges $e = v_1v_2$ and $f = v_3v_4$ in $C_2^D(T)$. Such that the path connecting e and f contains an edge, say $g = xy$ with g not in $C_2^D(T)$. Let $e_2^D(e) = e_2^D(f) = m$ and $e_2^D(g) = n$ then $n > m$. Let z be an eccentric vertex of g so that $e_2^D(g) = d^D(g, z) = n$. If this D-distance attained at x, then $e_2^D(f) > n > m$ which is contradiction. If the D-distance attained at y then $e_2^D(e) > n > m$ which is again a contradiction thus $C_2^D(T)$ induces a star other wise, the subgraph induced by $C_2^D(T)$ contains a path $p : v_1, v_2, v_3, v_4$ of length $deg(v_1) + deg(v_2) + deg(v_3) + deg(v_4) + 3$. Let $e = v_1v_2$, $f = v_2v_3$ and $g = v_3v_4$ since $e, f, g \in C_2^D(T)$. We have $e_2^D(e) = e_2^D(f) = e_2^D(g) = m$ (say). Let x be an eccentric vertex of f so that $e_2^D(f) = d^D(f, x) = m$. If the D-distance attained at v_2 it is easy to see that $d^D(g,x) = m + deg(x) + 1$. So that $e_2^D(g) > m + deg(x) + 1$ which a contradiction and if the D-distance $d^D(g,x)$ is attained at v_3 it is easy to see that $d^D(e,x) = m + deg(x) + 1$, so that $e_2^D(e) > m + deg(x) + 1$ which is again contradiction. Hence $C_2^D(T)$ induces a star. □

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Accepted: 22.02.2019