# Boundedness in a nonlinear gradient chemotaxis model with logistic source

**Xiaofei Yang**
*Department of Mathematics*
*Henan Institute of Science and Technology*
*Xinxiang 453003*
*P.R. China*

**Qingshan Zhang***
*Department of Mathematics*
*Henan Institute of Science and Technology*
*Xinxiang 453003*
*P.R. China*
*qingshan11@yeah.net*

**Wanyu Wu**
*Department of Mathematics*
*Henan Institute of Science and Technology*
*Xinxiang 453003*
*P.R. China*

**Abstract.** This paper deals with a nonlinear gradient chemotaxis system with logistic source motivated by the model of tumor lymphangiogenesis in a smooth bounded domain $\Omega \subset \mathbb{R}^n$. By using the iterative method and the test-function argument, we prove that the problem possesses a unique global solution which is uniformly bounded.
**Keywords:** nonlinear gradient chemotaxis model, global existence, boundedness, logistic source.

## 1. Introduction

In this paper we study the nonlinear gradient chemotaxis system with logistic source

$$
\begin{cases}
u_t = \Delta u - \chi \nabla \cdot (u\mathbf{F}(\nabla v)) + f(u), & x \in \Omega, \quad t > 0, \\
v_t = \Delta v - v + u, & x \in \Omega, \quad t > 0, \\
(\nabla u - \chi u \mathbf{F}(\nabla v)) \cdot \nu = \nabla v \cdot \nu = 0, & x \in \partial\Omega, \quad t > 0, \\
u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega,
\end{cases}
\tag{1.1}
$$

where $\Omega$ is a smooth bounded domain in $\mathbb{R}^n$, $\nu$ denotes the outward normal on $\partial\Omega$, $\chi > 0$, $\mathbf{F}$ is a vector-valued function and the initial data $u_0(x)$, $v_0(x)$ are given nonnegative functions. The unknown function $u$ denotes cell density and $v$ describes the concentration of the chemical signal.

*. Corresponding author

The problem (1.1) is a variation of the so-called minimal model of the Keller-Segel chemotaxis model which is obtained when $\mathbf{F}(\nabla v) = \nabla v$ and $f \equiv 0$. Such a model in which the chemotactic component to motion depends nonlinearly on signal gradient provides a more realistic depiction of individual cell migration [2]. Various versions of the model for chemotaxis have attracted much attention in recent years; see the survey articles [2, 3, 9] and the references therein for more information.

Throughout this paper we assume that the vector-valued function $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ belongs to $C^{1+\delta}(\mathbb{R}^n)$ for some $\delta > 0$ and that

$$|\mathbf{F}(\mathbf{a})| \leq A, \quad \text{for all} \quad \mathbf{a} \in \mathbb{R}^n. \tag{1.2}$$

The logistic function $f : \mathbb{R} \to \mathbb{R}$ is smooth satisfying $f(0) = 0$ and

$$f(s) \leq \kappa s - \mu s^2, \quad \text{for all} \quad s \geq 0 \tag{1.3}$$

with $\kappa > 0$ and $\mu > 0$. We note that there exist some important models in which the assumption (1.2) is satisfied. For instance,

$$\mathbf{F}(\nabla v) = \frac{1}{c}\Big(\tanh\big(\frac{cv_{x_1}}{1+c}\big), \cdots, \tanh\big(\frac{cv_{x_n}}{1+c}\big)\Big)$$

for describing migration of the flagella bacteria Escherichia coli [2]. Another specific choice of the function

$$\mathbf{F}(\nabla v) = \frac{\nabla v}{\sqrt{1 + c|\nabla v|^2}}$$

represents chemotactic factor in model of tumor lymphangiogenesis [1].

For the case $f(s) = 0$, the global existence of solutions has been shown by Hillen and Painter [2, Lemma 3] if $\mathbf{F}$ is uniformly bounded. The main purpose of this paper is to prove the global existence and the boundedness of solutions of (1.1) with nontrivial logistic source for all dimensions $n \geq 1$. The proof is based on a test-function argument and an iterative technique on $L^p$ norms[4, 5].

## 2. Preliminary lemmas and statements of the main result

In this paper we need the following well-known facts concerning the Laplacian on $\Omega$ supplemented with homogeneous Neumann boundary conditions (for instance, see [4, 6]). Firstly, the operator $-\Delta + 1$ is sectorial in $L^p(\Omega)$ and therefore possesses closed fractional powers $(-\Delta + 1)^\theta$, $\theta \in (0, 1)$, with dense domain $D((-\Delta + 1)^\theta)$. If $m \in \{0, 1\}$, $p \in [1, \infty]$ and $q \in (1, \infty)$ with $m - \frac{n}{p} < 2\theta - \frac{n}{q}$, then we have

$$\|\omega\|_{W^{m,p}(\Omega)} \leq c\|(-\Delta + 1)^\theta \omega\|_{L^q(\Omega)}, \tag{2.1}$$

for all $\omega \in D((-\Delta+1)^{\theta})$, where $c$ is a positive constant. Moreover, for $p < \infty$ the associated heat semigroup $(e^{t\Delta})_{t\geq 0}$ maps $L^p(\Omega)$ into $D((-\Delta+1)^{\theta})$ in any of the space $L^q(\Omega)$ for $q \geq p$, and there exist $c > 0$ and $\mu > 0$ such that

$$\|(-\Delta+1)^{\theta} e^{t(\Delta-1)}\omega\|_{L^q(\Omega)} \leq ct^{-\theta-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} e^{-\mu t}\|\omega\|_{L^p(\Omega)} \tag{2.2}$$

for all $\omega \in L^p(\Omega)$.

In the following proof, we shall use the ODE comparison principle repeatedly.

**Lemma 2.1.** *Let the function $y \in C^1[0,\infty)$ be positive and satisfy*

$$\begin{cases} y' + \gamma y^p \leq \delta y^q, & t > 0, \\ y(0) = y_0 \end{cases}$$

*with $\gamma > 0$, $p > q \geq 0$ and $\delta \geq 0$. Then, for all $t \geq 0$*

$$y(t) \leq \max\left\{ y_0, \left(\frac{\delta}{\gamma}\right)^{\frac{1}{p-q}} \right\}.$$

**Proof.** If $y_0 \leq \left(\frac{\delta}{\gamma}\right)^{\frac{1}{p-q}}$, then $y(t) \leq \left(\frac{\delta}{\gamma}\right)^{\frac{1}{p-q}}$ for all $t \geq 0$. If $y_0 > \left(\frac{\delta}{\gamma}\right)^{\frac{1}{p-q}}$, then there exists $t_0 \in (0,\infty)$ such that

$$\begin{aligned} y(t) &\geq \left(\frac{\delta}{\gamma}\right)^{\frac{1}{p-q}}, \quad t \in [0,t_0], \\ y(t) &\leq \left(\frac{\delta}{\gamma}\right)^{\frac{1}{p-q}}, \quad t \in [t_0,\infty). \end{aligned} \tag{2.3}$$

From (2.3), we have $y'(t) \leq 0$ in $(0,t_0)$, which implies $y(t) \leq y_0$ in $(0,t_0)$. Hence we complete the proof. □

In the proof of boundedness of $u$, we also use the following Lemma (see [8, 5] for details).

**Lemma 2.2.** *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary. For each $\varepsilon > 0$, there exists $C > 0$ which depends only on $n$ and $\Omega$ with the property that*

$$\|\varphi\|^2_{L^2(\Omega)} \leq \varepsilon\|\nabla\varphi\|^2_{L^2(\Omega)} + C(1+\varepsilon^{-\frac{n}{2}})\|\varphi\|^2_{L^1(\Omega)}, \quad \varphi \in W^{1,2}(\Omega). \tag{2.4}$$

The following lemma asserts that the system (1.1) has a unique local-in-time existence classical solution.

**Lemma 2.3.** *Let $u_0$ and $v_0$ be nonnegative functions such that $u_0 \in C^0(\bar{\Omega})$ and $v_0 \in W^{1,r}(\Omega)$ for some $r > n$. Suppose that $\mathbf{F}$ satisfies (1.2) and $f$ is given by (1.3). Then there exist the maximal existence time $T_{\max} \in (0,\infty]$ and a unique* *local-in-time classical nonnegative solution* $(u(x,t), v(x,t))$ *to the problem* (1.1) *such that*

$$u \in C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})),$$
$$v \in C^0(\bar{\Omega} \times [0, T_{\max})) \cap L^{\infty}_{loc}([0, T_{\max}); W^{1,r}(\Omega)) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max})).$$

*If* $T_{\max} < \infty$*, then*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} \to \infty \quad as \quad t \nearrow T_{\max}.$$

*In addition, there exists a constant* $M > 0$ *such that*

$$\int_\Omega u(x,t)dx \le M, \quad \text{for all} \quad t \in (0, T_{\max}). \tag{2.5}$$

**Proof.** The proof of local existence of a classical solution to (1.1) is similar to that of [6, Lemma 2.1] and [7, Lemma 1.1] and so is omitted. We now prove (2.5). Integrating the first equation in (1.1) over $\Omega$, we obtain

$$\frac{d}{dt}\int_\Omega u = \int_\Omega \Delta u - \chi \int_\Omega \nabla \cdot (u\mathbf{F}(\nabla v)) + \int_\Omega f(u).$$

Here the Green's formulas ensures that

$$\int_\Omega \Delta u = \int_{\partial\Omega} \nabla u \cdot \nu$$

and

$$-\chi \int_\Omega \nabla \cdot (u\mathbf{F}(\nabla v)) = -\chi \int_{\partial\Omega} u\mathbf{F}(\nabla v) \cdot \nu.$$

We next recall the boundary condition in (1.1) to see that

$$\int_\Omega \Delta u - \chi \int_\Omega \nabla \cdot (u\mathbf{F}(\nabla v)) = \int_{\partial\Omega} (\nabla u - \chi u\mathbf{F}(\nabla v)) \cdot \nu = 0.$$

Hence, the assumption (1.3) implies that

$$\frac{d}{dt}\int_\Omega udx \le \int_\Omega (\kappa u - \mu u^2)dx. \tag{2.6}$$

From Hölder's inequality, we get the inequality

$$-\int_\Omega u^2 dx \le -\frac{1}{|\Omega|}(\int_\Omega udx)^2. \tag{2.7}$$

It follows from (2.6) and (2.7) that

$$\frac{d}{dt}\int_\Omega udx \le \left(\kappa - \frac{\mu}{|\Omega|}\int_\Omega udx\right)\int_\Omega udx.$$

Let $y(t) = \int_\Omega u(x,t)dx$, then we have

$$y' \le \left(\kappa - \frac{\mu}{|\Omega|}y\right) y.$$

Hence, we apply Lemma 2.1 to obtain

$$\int_\Omega u dx \le M := \max\{\int_\Omega u_0 dx,\ \frac{\kappa}{\mu}|\Omega|\}.$$

This completes the proof. □

The main result of this paper reads as follows:

**Theorem 2.1.** *Under the assumptions of Lemma 2.2, problem* (1.1) *possesses a unique global classical solution* $(u, v)$ *such that both* $u$ *and* $v$ *are nonnegative and bounded in* $\Omega \times (0, \infty)$.

## 3. Proof of Theorem 2.1

We first prove uniform boundedness of the first component of the solution $u(\cdot, t)$ in the space $L^k(\Omega)$ for all $2 < k < \infty$. In the sequel we shall denote by $C$ various positive constants which may vary from line to line. Also $u(\cdot, t)$ will be denoted sometimes by $u$. Multiplying the first equation in (1.1) by $u^{k-1}$ and integrating over $\Omega$, we obtain

$$\frac{1}{k}\frac{d}{dt}\int_\Omega u^k dx \le \int_\Omega \Delta u u^{k-1} dx - \chi \int_\Omega \nabla \cdot (u\mathbf{F}(\nabla v))u^{k-1}dx + \int_\Omega u^{k-1}(\kappa u - \mu u^2)dx,$$

where by the Green's formulas

$$\int_\Omega \Delta u u^{k-1}dx = -(k-1)\int_\Omega u^{k-2}|\nabla u|^2 dx + \int_{\partial\Omega} u^{k-1}\nabla u \cdot \nu dS$$

and

$$-\chi \int_\Omega \nabla\cdot(u\mathbf{F}(\nabla v))u^{k-1}dx = \chi(k-1)\int_\Omega u^{k-1}\nabla u\cdot\mathbf{F}(\nabla v)dx - \chi\int_{\partial\Omega} u^k\mathbf{F}(\nabla v)\cdot\nu dS.$$

The boundary condition in (1.1) implies that

$$\begin{aligned}
&\int_\Omega \Delta u u^{k-1}dx - \chi\int_\Omega \nabla\cdot(u\mathbf{F}(\nabla v))u^{k-1}dx \\
&= -(k-1)\int_\Omega u^{k-2}|\nabla u|^2dx + \chi(k-1)\int_\Omega u^{k-1}\nabla u\cdot\mathbf{F}(\nabla v)dx \\
&\quad + \int_{\partial\Omega} u^{k-1}(\nabla u - \chi u\mathbf{F}(\nabla v))\cdot\nu dS \\
&= -(k-1)\int_\Omega u^{k-2}|\nabla u|^2dx + \chi(k-1)\int_\Omega u^{k-1}\nabla u\cdot\mathbf{F}(\nabla v)dx.
\end{aligned}$$

Hence, we get the estimate

$$\frac{1}{k}\frac{d}{dt}\int_\Omega u^k dx + (k-1)\int_\Omega u^{k-2}|\nabla u|^2 dx$$
$$\le \chi(k-1)\int_\Omega u^{k-1}\nabla u\cdot \mathbf{F}(\nabla v)dx + \int_\Omega u^{k-1}(\kappa u - \mu u^2)dx.$$

Applying (1.2) and Cauchy's inequality, we get

$$\chi(k-1)\int_\Omega u^{k-1}\nabla u\cdot \mathbf{F}(\nabla v)dx \le \frac{k-1}{2}\int_\Omega u^{k-2}|\nabla u|^2 dx + \frac{k-1}{2}(A\chi)^2\int_\Omega u^k dx.$$

In view of Hölder's inequality

$$\int_\Omega u^k dx \le \Big(\int_\Omega u^{k+1}dx\Big)^{\frac{k}{k+1}}|\Omega|^{\frac{1}{k+1}},$$

we have

$$-\int_\Omega u^{k+1}dx \le -|\Omega|^{-\frac{1}{k}}\Big(\int_\Omega u^k dx\Big)^{\frac{k+1}{k}}.$$

It follows that

$$\frac{1}{k}\frac{d}{dt}\int_\Omega u^k dx \le -\mu|\Omega|^{-\frac{1}{k}}\Big(\int_\Omega u^k dx\Big)^{\frac{k+1}{k}} + \Big(\frac{k-1}{2}(A\chi)^2+\kappa\Big)\int_\Omega u^k dx.$$

Therefore, Lemma 2.1 yields

$$\int_\Omega u^k dx \le C, \quad t\in(0,T_{\max}), \tag{3.1}$$

for all $2 < k < \infty$.

In order to complete the proof of Theorem 2.1, by means of Lemma 2.2, it is sufficient to prove that for any fixed $\tau\in(0,T_{\max})$

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} \le C(\tau), \quad t\in(\tau,T_{\max})$$

holds with some constant $C(\tau)>0$. To do so, we let $\tau\in(0,T_{\max})$ such that $\tau<1$ and pick $k>\max\{2,\frac{n}{2}\}$, so that we can choose $\theta\in(\frac{n}{2k},1)$. From the variation-of-constant formula, we get

$$v(\cdot,t) = e^{t(\Delta-1)}v_0 + \int_0^t e^{(t-s)(\Delta-1)}u(\cdot,s)ds, \quad t\in(0,T_{\max}).$$

By using (2.1), (2.2), (1.2) and (3.1), we have

$$\begin{aligned}
\|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} &\le C\|(-\Delta+1)^\theta v(\cdot,t)\|_{L^k(\Omega)}\\
&\le Ct^{-\theta}e^{-\mu t}\|v_0\|_{L^k(\Omega)} + C\int_0^t (t-s)^{-\theta}e^{-\mu(t-s)}\|u(\cdot,s)\|_{L^k(\Omega)}ds\\
&\le Ct^{-\theta} + C\int_0^t (t-s)^{-\theta}e^{-\mu(t-s)}ds\\
&\le C\big(\tau^{-\theta}+\Gamma(1-\theta)\big), \quad t\in(\tau,T_{\max}),
\end{aligned} \tag{3.2}$$

where $\Gamma(\cdot)$ denotes the Gamma function.

We now turn to establish a uniform bound of $u$. For any $p \geq 2$, using the first equation in (1.1), (1.2) and (1.3), we compute

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^p dx &= p\int_\Omega u^{p-1}u_t dx\\ &= p\int_\Omega u^{p-1}\Big(\Delta u - \chi\nabla(u\mathbf{F}(\nabla v)) + f(u)\Big)dx\\ &\leq -p(p-1)\int_\Omega u^{p-2}|\nabla u|^2 dx + \chi Ap(p-1)\int_\Omega u^{p-1}|\nabla u|dx + p\kappa\int_\Omega u^p dx\\ &= -\frac{4(p-1)}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 dx + 2\chi A(p-1)\int_\Omega u^{\frac{p}{2}}|\nabla u^{\frac{p}{2}}|dx + p\kappa\int_\Omega u^p dx\\ &\leq -\frac{2(p-1)}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 dx + \Big(\frac{(\chi A)^2}{2}p(p-1) + p\kappa\Big)\int_\Omega u^p dx,\end{aligned}$$

where we have used Young's inequality. Therefore,

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u^p dx + p(p-1)\int_\Omega u^p dx\\ &\leq -\frac{2(p-1)}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 dx + C_1 p(p-1)\int_\Omega u^p dx,\end{aligned} \tag{3.3}$$

where $C_1 = \frac{(\chi A)^2}{2} + \frac{\kappa}{p-1} + 1$. Using (2.4) with $\varphi = u^{\frac{p}{2}}$ and $\varepsilon = \frac{2}{C_1 p^2}$, we obtain

$$\begin{aligned}C_1 p(p-1)\int_\Omega u^p dx &= C_1 p(p-1)\|u^{\frac{p}{2}}\|^2_{L^2(\Omega)}\\ &\leq \frac{2(p-1)}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 dx + C_2 p(p-1)(1+p)^n\Big(\int_\Omega u^{\frac{p}{2}}dx\Big)^2,\end{aligned}$$

where $C_2 = CC_1\max\{1, (\frac{C_1}{2})^{\frac{n}{2}}\}$. Substituting in (3.3) we get

$$\frac{d}{dt}\int_\Omega u^p dx + p(p-1)\int_\Omega u^p dx \leq C_2 p(p-1)(1+p)^n\Big(\int_\Omega u^{\frac{p}{2}}dx\Big)^2.$$

Taking supremum for $t \in [0, T_{\max}]$ on the right hand-side, we get

$$\frac{d}{dt}\int_\Omega u^p dx + p(p-1)\int_\Omega u^p dx \leq C_2 p(p-1)(1+p)^n \sup_{0\leq t\leq T_{\max}}\Big(\int_\Omega u^{\frac{p}{2}}dx\Big)^2.$$

Therefore, it follows from Lemma 2.1 that

$$\int_\Omega u^p dx \leq \max\Big\{\int_\Omega u_0^p dx, \quad C_2(1+p)^n \sup_{0\leq t\leq T_{\max}}\big(\int_\Omega u^{\frac{p}{2}}dx\big)^2\Big\}. \tag{3.4}$$

Denoting

$$F(p) := \max\left\{\|u_0\|_{L^\infty(\Omega)}, \quad \sup_{0\leq t\leq T_{\max}}\|u(t)\|_{L^p(\Omega)}\right\}$$

and hence (3.4) yields

$$F(p) \leq \big(C_3(1+p)^n\big)^{\frac{1}{p}} F(\frac{p}{2}),$$

where $C_3 = \max\{|\Omega|, C_2\}$. Taking $p = 2^j$ $(j \in \mathbb{N}^+)$, we obtain

$$\begin{aligned} F(2^j) &\leq C_3^{\frac{1}{2^j}} (1+2^j)^{\frac{n}{2^j}} F(2^{j-1}) \\ &\leq \cdots \\ &\leq C_3^{\sum_{i=1}^{j} 2^{-i}} \cdot \prod_{i=1}^{j} (1+2^i)^{\frac{n}{2^i}} F(1) \\ &\leq C_3 2^{n \sum_{i=1}^{j} \frac{i}{2^i}} \cdot 2^{n \sum_{i=1}^{j} \frac{1}{2^i}} F(1) \\ &\leq C_3 2^{n(2-\frac{j}{2^{j+1}})} F(1). \end{aligned}$$

Letting $j \to \infty$, we infer that

$$\max\left\{ \|u_0\|_{L^\infty(\Omega)}, \quad \sup_{0\leq t\leq T_{\max}} \|u(t)\|_{L^\infty(\Omega)} \right\} =: F(\infty) \leq C_3 2^{2n} F(1)$$

according to (2.5). This along with (3.2) yields

$$\|u(t)\|_{L^\infty(\Omega)} + \|v(t)\|_{W^{1,\infty}(\Omega)} < \infty.$$

This is a contradiction with Lemma 2.3.

**Acknowledgements**

The authors would like to express thanks to the editor and the referees for their valuable comments on the original manuscript. This work is also supported by the key scientific research projects for higher education of Henan Province under Grant (No. 17A110020). The third author is supported by BNYC Grant 2016-2-11.

## References

[1] A. Friedman, G. Lolas, *Analysis of a mathematical model of tumor lymphangiogenesis,* Math. Models Methods Appl. Sci., 15 (2005), 95-107.

[2] T. Hillen, K. J. Painter, *A user's guide to PDE models for chemotaxis,* J. Math. Biol., 58 (2009), 183-217.

[3] D. Horstmann, *From 1970 until present: the Keller-Segel model in chemotaxis and its consequences,* I. Jahresber. Deutsch. Math.-Verein., 105 (2003), 103-165.

[4] C. Mu, L. Wang, P. Zheng, Q. Zhang, *Global existence and boundedness of classical solutions to a parabolic-parabolic chemotaxis system,* Nonlinear Anal. Real World Appl., 14 (2013), 1634-1642.

[5] Y. Tao, *Boundedness in a chemotaxis model with oxygen consumption by bacteria,* J. Math. Anal. Appl., 381 (2011), 521-529.

[6] M. Winkler, *Absence of collapse in a parabolic chemotaxis system with signal-dependent sensitivity,* Math. Nachr., 283 (2010), 1664-1673.

[7] M. Winkler, *Boundedness in the higher-dimensional parabolic-parabolic chemotaxis system with logistic source,* Comm. Partial Differential Equations, 35 (2010), 1516-1537.

[8] M. Winkler, *Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system*, J. Math. Pures Appl., 100 (2013), 748-767.

[9] N. Bellomo, A. Bellouquid, Y. Tao, M. Winkler, *Toward a mathematical theory of Keller-Segel models of pattern formation in biological tissues,* Math. Models Methods Appl. Sci., 25 (2015), 1663-1763.

Accepted: 26.02.2019