Characterizations of new open and closed mappings in topological spaces

Md. Hanif Page*
Department of Mathematics
KLE Technological University
Hubballi-580031, Karnataka
India
hanif01@yahoo.com

V. T. Hosamath
Department of Mathematics
K.L.E. Institute of Technology
(*Affiliated to V.T.U., Belagavi*), *Hubballi-580030, Karnataka*
India
vthosamath@gmail.com

Abstract. In this paper we present θ-semigeneralized pre-open maps, θ-semigeneralized pre-closed maps, pre-θ-semigeneralized pre-open maps, pre-θ-semigeneralized pre-closed maps by employing θ-sgp-closed sets. We also introduce higher separation axioms θ-semigeneralized pre-regular spaces and θ-semigeneralized pre-normal spaces using θ-sgp-closed sets in topological spaces and we investigate some new characterizations of these mappings and higher separation axioms.

Keywords: θ-sgp-open map, θ-sgp-closed map, pre-θ-sgp-open map, pre-θ-sgp-closed map, θ-sgp-regular space, θ-sgp-normal space.

1. Introduction

Malghan [10] defined generalized closed functions and obtained the preservation theorems of normality and regularity. Recently Arya and Nour [2] defined generalized semi-open sets and used them to obtain some characterizations of s-normal spaces due to Maheshwari and Prasad [9]. Dorsett ([5], [6]) defined and studied the concepts of semi-normal spaces and semi-regular spaces respectively. Quite recently, Devi et al. [4] have introduced generalized semi-closed functions and shown that the continuous generalized semi-closed surjective image of a normal space is s-normal. Present authors [13] introduced the notion of θ-sgp-closed set. Object of this article is to continue the study of θ-sgp-closed sets by intercalating new classes of mappings called θ-sgp-open map, θ-sgp-closed map, pre-θ-sgp-open map, pre-θ-sgp-closed map, contra pre-θ-sgp-open map and contra pre-θ-sgp-closed map using θ-sgp-open and θ-sgp-closed sets and also defined

*. Corresponding author

new spaces called θ-sgp-regular and θ-sgp-normal utilizing θ-sgp-closed set. We also investigated properties and characterizations of above mappings and spaces.

2. Preliminaries

In this paper, no separation axioms are taken up on the notations (X, τ) and (Y, σ) (or simply X and Y) which stands for topological spaces except clearly stated. If $M \subset X$ then Cl(M) and Int(M) stands for closure of M and the interior of M in X respectively.

We recall some definitions used in this paper to make it self-contained.

Definition 2.1. A subset A of a space X is called

(i) a semi-open set [8] if $A \subset Cl(Int(A))$.

(ii) a semi-closed set [2] if $Int(Cl(A)) \subset A$.

(iii) a pre-open [11] if $A \subseteq (Int(Cl(A)))$

Definition 2.2. The pre-θ-closure denoted by $pCl_\theta(A)$, is the set of all pre-θ-cluster points of A. A subset A is called pre-θ-closed set [1] if $A = pCl_\theta(A)$. The complement of pre-θ-closed set is pre-θ-open set.

Definition 2.3. A subset A of a topological space X is called θ-semigeneralized pre-closed set [14] (briefly, θ-sgp-closed) if $pCl_\theta(A) \subset U$ whenever $A \subset U$ and U is semi-open in X. The complement of θ-semigeneralized pre-closed set is called θ-semigeneralized pre-open (briefly, θ-sgp-open).

Definition 2.4. A space X is said to be $T_{\theta sgp}$ [16] if every θ-sgp-closed set is semi-closed set.

Definition 2.5. A function $f : X \to Y$ is called θ-sgp-irresolute [16](resp., semi-totally semi-continuous [7], θ-sgp-continuous [16], irresolute [3]) if $f^{-1}(F)$is θ-sgp-closed(resp., both semi-open and semi-closed, θ-sgp-closed, semi-closed) in X for every θ-sgp-closed set (resp., semi-open set, semi-closed set, semi-closed set) F of Y.

Definition 2.6. For a subset A of X, θ-semigeneralized pre-closure [15] of A, denoted by θ-sgpCl(A) and is defined as θ-sgpCl(A) $= \cap\{G : A \subseteq G$, G is θ-sgp-closed in X $\}$.

Definition 2.7. [18]A space X is said to be:

(i) θ-sgp-T_0, if for each pair $x, y \in X$ and $x \neq y$, there is a θ-sgp-open set containing one point but not the other.

(ii) θ-sgp-T_1, if for each pair of unequal points $x, y \in X$, there are disjoint θ-sgp-open sets U and V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.

(iii) θ-sgp-T_2, if for each pair of unequal points $x, y \in X$, there are disjoint θ-sgp-open sets U and V such that $x \in U$, $y \in V$ and $U \cap V = \phi$.

3. θ-sgp-open and θ-sgp-closed mappings

Definition 3.1. Let $f : X \to Y$ be a map. Then f is called θ-sgp-open (resp., θ-sgp-closed) if f(M) is θ-sgp-open set (resp., θ-sgp-closed set) in Y for every semi-open set (resp., semi-closed set) M in X.

Theorem 3.2. *A map $f : X \to Y$ is θ-sgp-closed if and only if for each subset P of Y and for each semi-open set Q containing $f^{-1}(P)$ there is a semi-open set of R of Y such that $P \subseteq R$ and $f^{-1}(R) \subseteq Q$.*

Proof. Assume that f is θ-sgp-closed map. Let P be a subset of Y and Q be a semi-open set of X such that $P \subseteq f(Q)$, that is $f^{-1}(P) \subseteq Q$. Now Q^c is semi-closed set in Y. Then $f(Q^c)$ is θ-sgp-closed in X, since f is θ-sgp-closed map. So, $Y - f(Q^c)$ is θ-sgp-open in X. Thus $R = Y - f(Q^c)$ is a θ-sgp-open set containing P such that $f^{-1}(R) \subseteq Q$.

Conversely, suppose that F is a semi-closed set in X. Then $f^{-1}(Y - f(F)) \subseteq X - F$ and X-F is semi-open. By hypothesis, there is a θ-sgp-open set R of Y such that $Y - f(F) \subseteq R$ and $f^{-1}(R) \subseteq X - F$ and so $F \subseteq X - f^{-1}(R)$. Hence $Y - R \subseteq f(F) \subseteq f(X - f^{-1}(R)) \subseteq Y - R$ which implies f(F)=Y-R. Since Y-R is θ-sgp-closed, f(F) is θ-sgp-closed and thus f is θ-sgp-closed map.

Theorem 3.3. If $f : X \to Y$ is irresolute, pre-θ-closed and B is θ-sgp-closed subset of X, then f(B) is θ-sgp-closed in Y.

Proof. Let M be a semi-open set in Y such that $f(B) \subseteq M$. Since f is irresolute, $f^{-1}(M)$ is a semi-open set containing B, that is $B \subseteq f^{-1}(M)$. Hence $pCl_\theta(B) \subseteq f^{-1}(M)$, as B is θ-sgp-closed in X. Since f is pre-θ-closed, $f(pCl_\theta(B))$ is pre-θ-closed. So, $f(pCl_\theta(B))$ is a θ-sgp-closed set contained in the semi-open set M, that is $f(pCl_\theta(B)) \subseteq M$. Now $pCl_\theta(f(B)) \subseteq pCl_\theta (f(pCl_\theta(B))) = f(pCl_\theta(B)) \subseteq M$. Hence $pCl_\theta(f(B)) \subseteq M$. Therefore f(B) is θ-sgp-closed set in Y.

Remark 3.4. The composition of two θ-sgp-closed maps need not be θ-sgp-closed as seen from the following example.

Example 3.5. Let X=Y=Z=$\{p, q, r\}$, $\tau = \{\phi, \{p\}, \{q\}, \{p, q\}, X\}$, $\sigma = \{\phi, \{p\}, \{q, r\}, Y\}$ and $\eta = \{\phi, \{p, q\}, Z\}$. Let $f : X \to Y$be the identity map and define $g : Y \to Z$ by g(p) = g(q) = r and g(r) = p. Then both f and g are θ-sgp-closed maps but their composition $g \circ f : X \to Z$ is not θ-sgp-closed map. Since for semi-closed set $\{r\}$ in X, $(g \circ f)(\{r\}) = g(f(\{r\})) = \{p\}$ which is not θ-sgp-closed set in Z.

Corollary 3.6. Let $f : X \to Y$ be a θ-sgp-closed map and $g : Y \to Z$ be pre-θ-closed and irresolute, then their composition $g \circ f : X \to Z$ is θ-sgp-closed.

Proof. Let B be a semi-closed set of X. Then by hypothesis f(B) is θ-sgp-closed in Y. Since g is pre-θ-closed and irresolute, by theorem 3.3, $g(f(B)) = (g \circ f)(B)$ is θ-sgp-closed in Z. Hence $(g \circ f)$ is θ-sgp-closed.

Theorem 3.7. Let $f : X \to Y$and $g : Y \to Z$ be θ-sgp-closed maps and Y be a $T_{\theta sgp}$. Then their composition $(g \circ f)$ is θ-sgp-closed.

Proof. Let D be a semi-closed set of X. Then by hypothesis f(D) is a θ-sgp-closed in Y. Since Y is a $T_{\theta sgp}$, f(D) is semi-closed in Y. Also by assumption g(f(D)) is θ-sgp-closed in Z. Hence $(g \circ f)$ is θ-sgp-closed.

Theorem 3.8. Let $f : X \to Y$ and $g : Y \to Z$ be two maps such that their composition $g \circ f : X \to Z$ is a θ-sgp-open. Then the following statements are true.

(i) If f is semi-totally semi-continuous and surjective, then g is θ-sgp-open.

(ii) If g is θ-sgp-irresolute and injective, then f is θ-sgp-open.

Proof. (i) Let D be a semi-open set in Y. Then $f^{-1}(D)$ is both semi-open and semi-closed in X as f is semi-totally semi-continuous. Since $(g \circ f)$ is θ-sgp-open map and f is surjective, $(g \circ f)(f^{-1}(D)) = g(D)$ is θ-sgp-open set in Z. Therefore g is a θ-sgp-open in X.

(ii) Let H be a semi-open set of X. Since $(g \circ f)$ is θ-sgp-open, $(g \circ f)(H)$ is a θ-sgp-open set in Z. Since g is θ-sgp-irresolute, $g^{-1}((g \circ f(H))) = g^{-1}(g(f(H))) = f(H)$ is θ-sgp-open set in Y, since g is injective. Thus f is a θ-sgp-open in X.

Theorem 3.9. For any bijection $f : X \to Y$ the following statements are identical:

(i) f^{-1} is θ-sgp-continuous.

(ii) f is a θ-sgp-open map.

(iii) f is a θ-sgp-closed map.

Proof. $(i) \rightarrowtail (ii)$: Let Q be a semi-open set of X. By assumption $(f^{-1})^{-1}(Q) = f(Q)$ is θ-sgp-open set in Y and so f is θ-sgp-open map.

$(ii) \rightarrowtail (iii)$: Let P be a semi-closed set of X. Then P^c is semi-open in X. By assumption $f(P^c)$ is θ-sgp-open set in Y, that is $f(P^c) = (f(P))^c$ θ-sgp-open set in Y and therefore f(P) is θ-sgp-closed set in Y. Hence f is θ-sgp-closed map.

$(iii) \rightarrowtail (i)$: Let P be a semi-closed set of X. By assumption f(P) is θ-sgp-closed set in Y. But $(f^{-1})^{-1}(P) = f(P)$ and therefore f is θ-sgp-continuous.

Theorem 3.10. If $f : X \to Y$ is semi-open, θ-sgp-irresolute onto and M is a θ-sgp-closed set in Y, then $f^{-1}(M)$ is a θ-sgp-closed set in X.

Proof. Let J be any semi-open set of X containing $f^{-1}(M)$. Then $M \subset f(J)$ and f(J) is semi-open in Y. Since M is θ-sgp-closed set in Y, $pCl_\theta(M) \subset f(J)$ and hence $f^{-1}(M) \subset f^{-1}(pCl_\theta(M)) \subset J$. Since f is θ-sgp-irresolute, $f^{-1}(pCl_\theta(M))$ is θ-sgp-closed in X and hence we have $pCl_\theta(f^{-1}(M)) \subset f^{-1}(pCl_\theta(M)) \subset J$. This shows that $f^{-1}(M)$ is θ-sgp-closed set in X.

Theorem 3.11. If $f : X \to Y$ is onto, θ-sgp-open. If X is θ-sgp-T_1 and $T_{\theta sgp}$, then Y is θ-sgp-T_1.

Proof. Let m_1 and m_2 be two points of Y. Since f is onto, there exist distinct points n_1 and n_2 of X such that $f(n_1) = m_1$ and $f(n_2) = m_2$. Since X is a θ-sgp-T_1, there exist θ-sgp-open sets J and K such that $n_1 \in J$ and $n_2 \notin J$ and $n_2 \in K$ and $n_1 \notin K$. Again since X is $T_{\theta sgp}$, J and K are semi-open sets in X. As f is θ-sgp-open map, f(J) and f(K) are θ-sgp-open sets such that $m_1 = f(n_1) \in f(J)$, $m_2 = f(n_2) \notin f(J)$ and $m_1 = f(n_1) \notin f(K), m_2 = f(n_2) \in f(K)$. Hence Y is θ-sgp-T_1.

4. Pre-θ-sgp-closed and pre-θ-sgp-open mappings

Definition 4.1. A map $f : X \to Y$is called pre-θ-sgp-closed(resp., pre-θ-sgp-open) if f(M) is θ-sgp-closed(resp., θ-sgp-open) in Y for every θ-sgp-closed(resp., θ-sgp-open) set M of X.

Theorem 4.2. If $f : X \to Y$ is semi-totally semi-continuous, pre-θ-sgp-closed and N is a θ-sgp-closed subset of X, then f(N) is θ-sgp-closed set in Y.

Proof. Let M be a semi-open set such that $f(N) \subseteq M$. Since f is semi-totally semi-continuous $f^{-1}(M)$ is a semi-open set containing N, that is $N \subseteq f^{-1}(M)$.Hence $pCl_\theta(N) \subseteq f^{-1}(M)$ as N is θ-sgp-closed set in X. Since f is pre-θ-sgp-closed, $f(pCl_\theta(N))$ is θ-sgp-closed set. So $f(pCl_\theta(N))$ is θ-sgp-closed set contained in semi-open set M, that is $f(pCl_\theta(N)) \subseteq M$ Now $pCl_\theta(f(N)) \subseteq pCl_\theta(f(\theta - sgpCl(N))) \subseteq pCl_\theta(f(pCl_\theta(N))) = f(pCl_\theta(N)) \subseteq M$. Hence $pCl_\theta(f(N)) \subseteq M$. Therefore f(N) is a θ-sgp-closed set in Y.

Theorem 4.3. The property of a space being θ-sgp-T_0 is preserved under one-one, onto, pre-θ-sgp-open map and hence is a topological property.

Proof. Let X be a θ-sgp-T_0-space and Y be any other topological space. Let $f : X \to Y$ be one-one, onto, pre-θ-sgp-open map from X to Y. Let y_1, $y_2 \in Y$ with $y_1 \neq y_2$ and since f is one-one, onto, there exist distinct points $x_1, x_2 \in X$ such that $f(x_1) = y_1, f(x_2) = y_2$. Since X is θ-sgp-T_0, there exists a θ-sgp-open set J in X such that $x_1 \in J$ but $x_2 \notin J$. Since f is pre-θ-sgp-open map, f(J) is θ-sgp-open set containing $f(x_1)$ but not containing $f(x_2)$. Thus there exist a θ-sgp-open set f(J) in Y such that $y_1 \in f(J)$ but $y_2 \notin f(J)$ and hence Y is θ-sgp-T_0. Again as the property of being θ-sgp-T_0 is preserved under one-one, onto map, it is also preserved under homeomorphism and hence is a topological property.

Theorem 4.4. Let $f : X \to Y$ be a θ-sgp-closed map and $g : Y \to Z$ be a pre-θ-sgp-closed map and semi-totally semi-continuous, then their composition $g \circ f : X \to Z$ is θ-sgp-closed.

Proof. Let K be a semi-closed set in X. Then by hypothesis f(K) is θ-sgp-closed set Y. Since g is pre-θ-sgp-closed map and semi-totally semi-continuous, by Theorem 4.2 $g(f(K)) = (g \circ f)(K)$ is θ-sgp-closed set in Z. Hence $(g \circ f)$ is θ-sgp-closed.

Definition 4.5. A map $f : X \to Y$ is called contra pre-θ-sgp-open if for every θ-sgp-open set O of X, f(O) is θ-sgp-closed set in Y.

Definition 4.6. A map $f : X \to Y$ is called contra pre-θ-sgp-closed if for every θ-sgp-closed set F of X, f(F) is θ-sgp-open set in Y.

The following examples show that contra pre-θ-sgp-closedness and contra pre-θ-sgp-openness are independent.

Example 4.7. Let X=Y= $\{p, q, r\}$, $\tau = \{X, \phi, \{p\}, \{q\}, \{p, q\}, \{p, r\}, \{q, r\}\}$ and $\sigma = \{Y, \phi \{p\}, \{q\}, \{p, q\}\}$. We have θ-sgp-open sets in X are $\{X, \{p\}, \{q, r\}\}$ and θ-sgp-open sets in Y are $\{Y, \{p\}, \{q\}, \{p, r\}, \{q, r\}\}$. Let $f : X \to Y$ is defined by f(p) = f(r) = r, f(q) = q and $g : X \to Y$ is by g(p) = g(q) = p, g(r) = q. Then f is contra pre-θ-sgp-open but not contra pre-θ-sgp-closed and g is contra pre-θ-sgp-closed but not contra pre-θ-sgp-open.

Remark 4.8. Contra pre-θ-sgp-closedness and contra pre-θ-sgp-openness are equivalent if the map is onto.

Theorem 4.9. For a map $f : X \to Y$ the following properties are equivalent:

(i) f is contra pre-θ-sgp-open.

(ii) For every subset D of Y and for every θ-sgp-closed subset E of X with $f^{-1}(D) \subseteq E$, there exists a θ-sgp-open subset G of Y with $D \subseteq G$ and $f^{-1}(G) \subseteq E$.

(iii)For every point $q \in Y$ and every θ-sgp-closed subset E of X with $f^{-1}(q) \subseteq E$, there exists a θ-sgp-open subset G of Y with $q \in G$ and $f^{-1}(G) \subseteq E$.

Proof. $(i) \rightarrowtail (ii)$: Let D be a subset of Y and let E be a θ-sgp-closed subset of X with $f^{-1}(D) \subseteq E$. For the case $D \neq \phi$, put $G = [f(E^c)]^c$. Then $f^{-1}(G) = [f^{-1}(E^c)]^c \subseteq E$ and G is a θ-sgp-open subset of Y. We claim that $D \subseteq G$. There are two cases to be considered.

Case 1: $f^{-1}(D) \neq \phi$. Since $f^{-1}(D) \subseteq E$, we have $f(E^c) \subseteq D^c$ and $D \subseteq G$.

Case 2: $f^{-1}(D) = \phi$. Since $f^{-1}(D) = \phi \subseteq E$, we have $f(E^c) \subseteq f(X)$. We have $D \cap f(X) = \phi$, because $f^{-1}(D) = \phi$ and $D \neq \phi$. Thus $D \subseteq [f(X)]^c \subseteq [f(E^c)]^c = G$. For the case $D = \phi$, put $G = \phi$. Then the set G is required θ-sgp-open of Y.

$(ii) \rightarrowtail (iii)$: It suffices to put D=q, in (ii).

$(iii) \rightarrowtail (i)$: Let H be a θ-sgp-open subset of X. Then $q \in [f(H)]^c$ and $E = H^c$. First we claim that $f^{-1}(q) \subset E$. For non empty set $f^{-1}(q)$, let $p \in f^{-1}(q)$; $f(p) = q \notin f(H)$. Suppose that $p \notin H$ and so $q = f(p) \in f(H)$. By contradiction, $p \in E$ for any $p \in f^{-1}(q)$, that is $f^{-1}(q) = \phi \subseteq E$. For the case where $f^{-1}(q) = \phi$, we have $f^{-1}(q) = \phi \subseteq E$. For both cases, we can use (iii) and get the following: there exists a θ-sgp-open set $G_q \subseteq Y$ such that $q \in G_q$ and $f^{-1}(G_q) \subseteq E = H^c$. Namely, $(\star)$ $f^{-1}(G_q) \cap H = \phi$ holds for each $q \in [f(H)]^c$. Finally we claim that $[f(H)]^c \cup \{G_q : q \in [f(H)]^c\}$. Obviously, we have that $[f(H)]^c \subseteq \bigcup \{G_q : q \in [f(H)]^c\}$

Conversely, let $p \in \bigcup\{G_q : q \in [f(H)]^c\}$. Then there exists a point $r \in [f(H)]^c$ such that $p \in G_r$. Suppose that $p \notin [f(H)]^c$. Then $p \in f(H)$ and there exists a point $b \in H$ such that f(b)=p. Thus we have that $f(b) \in G_r$ and so $b \in f^{-1}(G_r)$. We have a contradiction to $(\star)$ above, that is $b \in f^{-1}(G_r) \cap H$. Hence, we show that $\bigcup\{G_q : q \in [f(H)]^c\} \subseteq [f(H)]^c$ and so $[f(H)]^c = \bigcup\{G_q : q \in [f(H)]^c\}$. Hence f(H) is a θ-sgp-closed subset of Y.

5. θ-sgp-regular and θ-sgp-normal spaces

Definition 5.1. A topological space X is said to be θ-sgp-regular if for every θ-sgp-closed set P and each point x which is not in P, there exist disjoint semi-open sets M and N such that $x \in M, P \subseteq N$ and $M \cap N = \phi$.

Definition 5.2. A topological space X is said to be θ-sgp-normal if for every pair of disjoint θ-sgp-closed sets P_1 and P_2 in X, there exist disjoint semi-open sets M and N such that $P_1 \subset M, P_2 \subset N$ and $M \cap N = \phi$.

Theorem 5.3. Let X be a topological space. Then the following statements are identical: (i) X is θ-sgp-regular.

(ii) For each point $x \in X$ and for each θ-sgp-open nbd Z of x, there exists a semi-open set Q of x such that $pCl_\theta(Q) \subseteq Z$.

(iii) For each point $x \in X$ and for each θ-sgp-closed not containing x, there exists a semi-open set Q of x such that $pCl_\theta(Q) \cap F = \phi$.

Proof. Let Z be a θ-sgp-open nbd of x. Then there exists a θ-sgp-open set K such that $x \in K \subseteq Z$. Since K^c is θ-sgp-closed set and $x \notin K^c$, by hypothesis there exist semi-open sets P and Q such that $K^c \subseteq P$, $x \in Q$ and $P \cap Q = \phi$ and so $Q \subset P^c$. Now $pCl_\theta(Q) \subseteq pCl_\theta(P^c) = P^c$ and $K^c \subseteq P$ implies $P^c \subseteq K \subseteq Z$. Therefore $pCl_\theta(Q) \subseteq Z$. Hence (i) implies (ii).

Let F be any θ-sgp-closed set and $x \notin F$. Then $x \notin F^c$ and F^c is θ-sgp-open and so F^c is a θ-sgp-open nbd of x. By hypothesis, there exists a semi-open set Q of x such that $x \in Q$ and $pCl_\theta(Q) \subseteq F^c$, which implies $F \subseteq (pCl_\theta(Q))^c$. Then $(pCl_\theta(Q))^c$ is a semi-open set containing F and $Q \cap (pCl_\theta(Q))^c = \phi$. Therefore X is θ-sgp-regular. Hence (ii) implies (i).

Let $x \in X$ and F be a θ-sgp-closed set such that $x \notin F$. Then F^c is a θ-sgp-open nbd of x and by hypothesis there exists a semi-open set Q of x such that $pCl_\theta(Q) \subseteq F^c$ and therefore $pCl_\theta(Q) \cap F = \phi$. Hence (ii) implies (iii).

Let $x \in X$ and Z be a θ-sgp-open nbd of x. Then there exists a θ-sgp-open set K such that $x \in K \subseteq Z$. Since K^c is θ-sgp-closed set and $x \notin K^c$, by hypothesis there exist semi-open set Q of x such that $pCl_\theta(Q) \cap K^c = \phi$. Therefore $pCl_\theta(Q) \subseteq K \subseteq Z$. Hence (iii) implies (ii).

Theorem 5.4. A space X is a θ-sgp-regular if and only if given any $x \in X$ and any semi-open set B of X there is θ-sgp-open set C such that $x \in C \subset \theta - sgpCl(C) \subseteq B$.

Proof. Let B be a semi-open set, $x \in B$. So B^c is semi-closed set such that $x \notin B^c$. Since X is a θ-sgp-regular then there exist θ-sgp-open sets C_1 and C_2 such that $C_1 \cap C_2 = \phi$, $B^c \subset C_2, x \in C_1$. Since $C_1 \cap C_2 = \phi$, we have $\theta - sgpCl(C_1) \subset \theta - sgpCl(C_2^c) = C_2^c$. Since $B^c \subset C_2$, we have $C_2^c \subset B$. Hence we have $x \in C_1 \subset \theta - sgpCl(C_1) \subset C_2^c \subseteq B$.

Conversely, let F be a semi-closed set in X and $x \in X - F$. So F^c is a semi-open set such that $x \in F^c$. Hence there exist a θ-sgp-open set B such that $x \in B \subset \theta - sgpCl(B) \subseteq F^c$. Let C = X- θ-sgpCl(B). So C is a θ-sgp-open set which contains F and $B \cap C = \phi$. Hence X is θ-sgp-regular.

Theorem 5.5. Let X be a topological space. If X is a θ-sgp-regular and T_1 then X is a θ-sgp-T_2.

Proof. Suppose $x, y \in X$ such that $x \neq y$. Since X is T_1 then there is an open set M such that $x \in M$, $y \notin M$. Since X is θ-sgp-regular and M is semi-open set which contains x, then there is θ-sgp-open set N such that $x \in N \subset \theta - sgpCl(N) \subseteq M$. Since $y \notin M$, hence $y \notin \theta$-sgpCl(N). Therefore $y \in X - (\theta$-sgpCl(N)). Hence there are θ-sgp-open sets N and X-(θ-sgpCl(N)) such that (X-(θ-sgpCl(N)))$\cap N = \phi$. Hence X is a θ-sgp-T_2.

Theorem 5.6. Every subspace of θ-sgp-regular space is θ-sgp-regular.

Proof. Let X be a θ-sgp-regular space and Y be a subspace of X. Let $x \in Y$ and F be a θ-sgp-closed set of Y such that $x \notin F$. Then there is a semi-closed set and so θ-sgp-closed set A of X with $F = Y \cap A$ and $x \notin A$. Since X is a θ-sgp-regular, there exist semi-open sets K and L such that $x \in K$, $A \subseteq L$ and $K \cap L = \phi$. Note that $Y \cap K$ and $Y \cap L$ are semi-open sets in Y. Also $x \in K$ and $x \in Y$ which implies $x \in Y \cap K$ and $A \subseteq L$. This implies $Y \cap A \subseteq Y \cap K$, $F \subseteq Y \cap L$. Also $(Y \cap K) \cap (Y \cap L) = \phi$. Hence Y is a θ-sgp-regular.

Theorem 5.7. If $f : X \to Y$ is onto, θ-sgp-irresolute and semi-open from θ-sgp-regular space X into a topological space Y, then Y is θ-sgp-regular.

Proof. Let $y \in Y$ and F be a θ-sgp-closed set of Y with $y \notin F$. Since f is θ-sgp-irresolute, $f^{-1}(F)$ is θ-sgp-closed set in X. As f is onto, let f(x)=y and $x = f^{-1}(y)$ and $x \notin f^{-1}(F)$. Again since X is θ-sgp-regular space, there exist semi-open sets U and V such that $x \in U$ and $f^{-1}(F) \subseteq G$ and $U \cap V = \phi$. Since f is semi-open and onto, we have $y \in f(U), F \subseteq f(V)$ and $f(U) \cap f(V) = f(U \cap V) = f(\phi) = \phi$. Hence Y is θ-sgp-regular.

Theorem 5.8. If $f : X \to Y$ is one-to-one, θ-sgp-closed map from a topological space X into θ-sgp-regular space Y. If X is $T_{\theta sgp}$, then X is θ-sgp-regular.

Proof. Let $x \in X$ and F be a θ-sgp-closed set in X such that $x \notin F$. Since X is $T_{\theta sgp}$-space, F is semi-closed in X. Then f(F) is θ-sgp-closed set with $f(x) \notin f(F)$ in Y as f is θ-sgp-closed. Again Y is θ-sgp-regular, there exist disjoint semi-open sets U and V such that $f(x) \in U$ and $f(F) \subseteq V$. Therefore $x \in f^{-1}(U)$ and $F \subset f^{-1}(V)$. Hence X is θ-sgp-regular.

Theorem 5.9. A topological space X is said to be θ-sgp-normal if and only if for every semi-closed set F and for every semi-open set G contain F there exists θ-sgp-open set U such that $F \subset U \subset \theta - sgpCl(U) \subset G$.

Proof. Let F be a semi-closed set in X and G be semi-open set in X such that $F \subset G$, G^c is a semi-closed set and $G^c \cap F = \phi$. Since X is θ-sgp-normal then there exist θ-sgp-open sets U and V of X such that $U \cap V = \phi$, $G^c \subset V$ and $F \subset U$, $U \subset V^c$. We have θ-sgpCl(U) $\subset \theta - sgpCl(V^c) = V^c$. Hence $F \subset U \subset \theta - sgpCl(U) \subset V^c \subset G$.

Theorem 5.10. A θ-sgp-closed subspace of a θ-sgp-normal space is θ-sgp-normal.

Proof. Let X be a θ-sgp-normal space. Let Y be a θ-sgp-closed subspace of X. Let A and B be a pair of disjoint θ-sgp-closed sets in Y. Then A and B are disjoint θ-sgp-closed sets in X. Since X is θ-sgp-normal, there exist disjoint semi-open sets G and H in X such that $A \subseteq G$ and $B \subseteq H$. Since G and H are semi-open in X, then $Y \cap G$ and $Y \cap H$ are semi-open in Y. Also we have $A \subseteq G$ and $B \subseteq H$ which implies $Y \cap A \subseteq Y \cap G$, $Y \cap B \subseteq H$. So $A \subseteq Y \cap G$, $B \subseteq Y \cap H$ and $(Y \cap G) \cap (Y \cap H) = Y \cap (G \cap H) = \phi$. Thus for each pair of disjoint θ-sgp-closed sets A and B in Y, there exist disjoint semi-open sets $Y \cap G$ and $Y \cap H$ such that $A \subseteq Y \cap G$ and $B \subseteq Y \cap H$. Hence Y is θ-sgp-normal.

Theorem 5.11. Let X be a topological space. The following statements are equivalent: (i) X is θ-sgp-normal.

(ii) For each θ-sgp-closed set A and each θ-sgp-open set U such that $A \subseteq U$, there exists a semi-open set V such that $A \subseteq V \subseteq pCl_\theta(V) \subseteq U$.

(iii) For each pair A, B of distinct θ-sgp-closed sets there exists a semi-open set V such that $A \subseteq V$and $pCl_\theta(V) \cap B = \phi$.

(iv) For each pair A, B of distinct θ-sgp-closed sets there exists a semi-open sets U and V such that $A \subseteq U$, $B \subseteq V$ and $pCl_\theta(U) \cap pCl_\theta(V) = \phi$.

Proof. $(i) \rightarrowtail (ii)$: Let A be a θ-sgp-closed set and U be a θ-sgp-open set such that $A \subseteq U$. Then A and X-U are disjoint θ-sgp-closed sets in X. Since X is θ-sgp-normal, there exist disjoint semi-open sets V and W in X such that $A \subseteq V$ and $X - U \subseteq W$. Now $X - W \subseteq X - (X - U)$ that is $X - W \subseteq U$. Also $V \cap W = \phi$, which implies $V \subseteq X - W$ implies that $pCl_\theta(V) \subseteq pCl_\theta(X - W)$ that is $pCl_\theta(V) \subseteq X - W$. Therefore $pCl_\theta(V) \subseteq X - W \subseteq U$. So $pCl_\theta(V) \subseteq U$. Hence $A \subseteq V \subseteq pCl_\theta(V) \subseteq U$.

$(ii) \rightarrowtail (iii)$: Let A and B be a pair of disjoint θ-sgp-closed sets in X. Now $A \cap B = \phi$, so $A \subseteq X - B$, where A is θ-sgp-closed and X-B is θ-sgp-open. Then by (ii), there exists semi-open set V such that $A \subseteq V \subseteq pCl_\theta(V) \subseteq X - B$. Now $pCl_\theta(V) \subseteq X - B$ implies $pCl_\theta(V) \cap B = \phi$. Thus $A \subseteq V$ and $pCl_\theta(V) \cap B = \phi$.

$(iii) \rightarrowtail (iv)$: Let A and B be a pair of disjoint θ-sgp-closed sets in X. From (iii), there exists a semi-open set U such that $A \subseteq U$ and $pCl_\theta(U) \cap B = \phi$. Now

$pCl_\theta(V)$ is θ-sgp-closed set and B is θ-sgp-closed set. Therefore again from (iii), there exists a semi-open set V such that $B \subseteq V$ and $pCl_\theta(U) \cap pCl_\theta(V) = \phi$.

$(iv) \rightarrowtail (i)$: Let A and B be a pair of disjoint θ-sgp-closed sets in X. Then from (iv), there exists a semi-open sets U and V in X such that $A \subset U, B \subset V$ and $pCl_\theta(U) \cap pCl_\theta(V) = \phi$. So $A \subseteq U$, $B \subseteq V$ and $U \cap V = \phi$. Hence X is θ-sgp-normal.

6. Conclusion

The notions θ-sgp-open, θ-sgp-closed maps, θ-sgp-regular and θ-sgp-normal spaces have been introduced and investigated their properties. Compact and connected spaces may be discussed in future using θ-sgp-closed set.

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Accepted: 11.04.2019