Ideals in LA-rings

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Abstract. Our aim is to encourage research and maturity of associative algebraic structures by studying a class of non-associative and non-commutative algebraic structures (LA-ring).

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1. Introduction

In ternary operations, the commutative law is given by $abc = cba$. Kazim et al [6] have generalized this notion by introducing the paranthesis on the left side of this equation to get a new pseudo associative law, that is $(ab)c = (cb)a$. This law $(ab)c = (cb)a$ is the left invertive law. A groupoid S is a left almost semigroup (abbreviated as LA-semigroup), if it satisfies the left invertive law. An LA-semigroup is a midway structure between a commutative semigroup and a groupoid. Ideals in LA-semigroups have been investigated by Protic et al [7].

In [2] (resp. [1]), a groupoid S is to be medial (resp. paramedial) if $(ab)(cd) = (ac)(bd)$ (resp. $(ab)(cd) = (db)(ca)$). In [6], an LA-semigroup is medial, but in general an LA-semigroup needs not to be paramedial. Every LA-semigroup with left identity is paramedial by Protic et al [7] and also satisfies $a(bc) = b(ac)$, $(ab)(cd) = (dc)(ba)$.

Kamran [3], extended the notion of LA-semigroup to the left almost group (LA-group). An LA-semigroup G is a left almost group, if there exists a left identity $e \in G$ such that $ea = a$ for all $a \in G$ and for every $a \in G$ there exists $b \in G$ such that $ba = e$.

Shah et al [8], discussed the left almost ring (LA-ring) of finitely nonzero functions which is a generalization of commutative semigroup ring. By a left almost ring, we mean a non-empty set R with at least two elements such that $(R, +)$ is an LA-group, $(R, \cdot)$ is an LA-semigroup, both left and right distributive laws hold. For example, from a commutative ring $(R, +, \cdot)$, we can always obtain an LA-ring $(R, \oplus, \cdot)$ by defining for all $a, b \in R$, $a \oplus b = b - a$ and $a \cdot b$ is same as in the ring. Although the structure is non-associative and non-commutative, nevertheless, it possesses many interesting properties which we usually find in associative and commutative algebraic structures.

In this paper, we will define the left (resp. right, interior, quasi-, bi-, generalized bi-) ideals of an LA-ring R. We will establish a study by discussing the different properties of such ideals. We will also characterize regular (resp. intra-regular, both regular and intra-regular) LA-rings in terms of left (resp. right, quasi-, bi-, generalized bi-) ideals.

2. Ideals in LA-rings

We initiate the concept of LA-subrings and left (resp. right, interior, quasi-, bi-, generalized bi-) ideals of an LA-ring R.

A non-empty subset A of an LA-ring R is called an LA-subring of R if $a - b$ and $ab \in A$ for all $a, b \in A$. A is called a left (resp. right) ideal of R, if $(A, +)$ is an LA-group and $RA \subseteq A$ (resp. $AR \subseteq A$). A is called an ideal of R. if it is both a left ideal and a right ideal of R.

Example 1. Let $R = \{0, 1, 2, 3, 4, 5, 6, 7\}$. Define $+$ and $\cdot$ in R as follows:

$+$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	2	0	3	1	6	4	7	5
2	1	3	0	2	5	7	4	6
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	1	2	3
5	6	4	7	5	2	0	3	1
6	5	7	4	6	1	3	0	2
7	7	6	5	4	3	2	1	0

and

$\cdot$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	4	4	0	0	4	4	0
2	0	4	4	0	0	4	4	0
3	0	0	0	0	0	0	0	0
4	0	3	3	0	0	3	3	0
5	0	7	7	0	0	7	7	0
6	0	7	7	0	0	7	7	0
7	0	3	3	0	0	3	3	0

Let $A = \{0, 4\}$ be a subset of R. Since $(A, +)$ is an LA-subgroup and

$$\begin{aligned} A^2 &= AA = \{0,4\}\{0,4\} = \{0\} \subseteq A. \\ AR &= \{0,4\}\{0,1,2,3,4,5,6,7\} = \{0,3\} \nsubseteq A. \end{aligned}$$

Thus A is an LA-subring of R, but not right ideal of R.

A non-empty subset A of an LA-ring R is an interior ideal of R, if $(A, +)$ is an LA-group and $(RA)R \subseteq A$. A non-empty subset A of R is a quasi-ideal of R, if $(A, +)$ is an LA-group and $AR \cap RA \subseteq A$. An LA-subring A of an LA-ring R is a bi-ideal of R if $(AR)A \subseteq A$. A non-empty subset A of an LA-ring R is a generalized bi-ideal of R if $(A, +)$ is an LA-group and $(AR)A \subseteq A$. An ideal I of an LA-ring R is an idempotent ideal of R if $A^2 = A$.

Now we give the central properties of such ideals of an LA-ring R, which will be very helpful for further sections.

Lemma 2.1. *Let R be an LA-ring with left identity e, then $RR = R$ and $eR = R = Re$.*

Proof. Since $RR \subseteq R$ and $x = ex \in RR$, i.e., $R \subseteq RR$, thus $RR = R$. Obviously, $eR = R$ and $Re = (RR)\,e = (eR)\,R = RR = R$. □

Lemma 2.2 ([4, Lemma 8])**.** *Let R be an LA-ring with left identity e and $a \in R$. Then Ra is the smallest left ideal of R containing a.*

Lemma 2.3 ([4, Lemma 9])**.** *Let R be an LA-ring with left identity e and $a \in R$. Then Ra is a left ideal of R.*

Proposition 2.1 ([4, Proposition 5])**.** *Let R be an LA-ring with left identity e and $a \in R$. Then $aR \cup Ra$ is the smallest right ideal of R containing a.*

Lemma 2.4. *Let R be an LA-ring with left identity e. Then every right ideal of R is an ideal of R.*

Proof. Let I be a right ideal of R. Let $a \in I$ and $r \in R$. Now $ra = (er)\,a = (ar)\,e \in IR \subseteq I$. Thus I is an ideal of R. □

Lemma 2.5. *Every two-sided ideal of R is an interior ideal of R, but the converse is not true.*

Proof. Straight forward. □

Example 2. Let $A = \{0, 1, 2, 3\}$ be a subset of R, defined as in example 1. Since $(A, +)$ is an LA-subgroup and

$$\begin{aligned}(RA)R &= (\{0,1,2,3,4,5,6,7\}\{0,1,2,3\})\{0,1,2,3,4,5,6,7\} \\ &= \{0,3,4,7\}\{0,1,2,3,4,5,6,7\} = \{0,3\} \subseteq A. \\ RA &= \{0,1,2,3,4,5,6,7\}\{0,1,2,3\} = \{0,3,4,7\} \nsubseteq A. \\ AR &= \{0,1,2,3\}\{0,1,2,3,4,5,6,7\} = \{0,4\} \nsubseteq A.\end{aligned}$$

Thus A is an interior ideal of R, but not an ideal of R.

Proposition 2.2. *Let R be an LA-ring with left identity e. Then any non-empty subset I of R is an ideal of R if and only if I is an interior ideal of R.*

Proof. Let I be an interior ideal of R. Let $a \in I$ and $r \in R$. Now $ar = (ea)r \in (RI)R \subseteq I$, i.e., I is a right ideal of R. Thus I is an ideal of R by the Lemma 2.4. Converse is true by the Lemma 2.5. □

Lemma 2.6. *Every left (resp. right, two-sided) ideal of R is a bi-ideal of R.*

Proof. Straight forward. □

Example 3. Let $A = \{0, 4\}$ be a subset of R defined as in example 1. Since $(A, +)$ is an LA-subgroup and

$$\begin{aligned}A^2 &= AA = \{0,4\}\{0,4\} = \{0\} \subseteq A. \\ (AR)A &= (\{0,4\}\{0,1,2,3,4,5,6,7\})\{0,4\} \\ &= \{0,3\}\{0,4\} = \{0\} \subseteq A. \\ AR &= \{0,4\}\{0,1,2,3,4,5,6,7\} = \{0,3\} \nsubseteq A.\end{aligned}$$

Thus A is a bi-ideal of R, but not right ideal of R.

Lemma 2.7. *Every bi-ideal of R is a generalized bi-ideal of R.*

Proof. Obvious. □

Lemma 2.8. *Every left (resp. right, two-sided) ideal of R is a quasi-ideal of R.*

Proof. Let I be a left ideal of R and $IR \cap RI \subseteq RI \subseteq I$, i.e., I is a quasi-ideal of R. □

Proposition 2.3. *Let I be a right ideal and L be a left ideal of an LA-ring R, respectively. Then $I \cap L$ is a quasi-ideal of R.*

Proof. Since $(I \cap L)R \cap R(I \cap L) \subseteq IR \cap RL \subseteq I \cap L$, i.e., $I \cap L$ is a quasi-ideal of R. □

Lemma 2.9. *Let R be an LA-ring with left identity e such that $(xe)R = xR$ for all $x \in R$. Then every quasi-ideal of R is a bi-ideal of R.*

Proof. Let Q be a quasi-ideal of R. Now $(QR)Q \subseteq RQ$ and $(QR)Q \subseteq (QR)R = (QR)(eR) = (Qe)(RR) = (Qe)R = QR$, thus $(QR)Q \subseteq QR \cap RQ \subseteq Q$. Hence Q is a bi-ideal of R. □

3. Regular LA-rings

An LA-ring R is regular, if for every $a \in R$, there exists an element $x \in R$ such that $a = (ax)a$. We characterize regular LA-rings by the properties of (left, right, quasi-, bi-, generalized bi-) ideals.

Lemma 3.1. *Every right ideal of a regular LA-ring R is an ideal of R.*

Proof. Suppose that I is a right ideal of R. Let $a \in I$ and $r \in R$. This implies that there exists an element $x \in R$, such that $r = (rx)r$. Now $ra = ((rx)r)a = (ar)(rx) \in IR \subseteq I$. Hence I is an ideal of R. □

Lemma 3.2. *Every ideal of a regular LA-ring R is an idempotent.*

Proof. Let I be an ideal of R. Since $I^2 \subseteq I$ and $a \in I$. This means that there exists an element $x \in R$ such that $a = (ax)a$. Now $a = (ax)a \in (IR)I \subseteq II = I^2$, i.e., $I \subseteq I^2$. Thus $I^2 = I$. □

Remark 1. Every right ideal of a regular LA-ring R is an idempotent.

Proposition 3.1. *Let R be a regular LA-ring. Then any non-empty subset I of R is an ideal of R if and only if I is an interior ideal of R.*

Proof. Assume that I is an interior ideal of R. Let $a \in I$ and $r \in R$. Then there exists an element $x \in R$, such that $a = (ax)a$. Now $ar = ((ax)a)r = (ra)(ax) \in (RI)R \subseteq I$, i.e., $IR \subseteq I$. Thus I is an ideal of R by the Lemma 3.1. Converse is true by the Lemma 2.5. □

Proposition 3.2. *Let R be a regular LA-ring with left identity e. Then $IR \cap RI = I$ for every right ideal I of R.*

Proof. Let I be a right ideal of R. This implies that $IR \cap RI \subseteq I$, because every right ideal of R is a quasi-ideal of R. Let $a \in I$, this means that there exists an element $x \in R$ such that $a = (ax)a$. Now $a = (ax)a \in (IR)I \subseteq II \subseteq IR$, i.e., $I \subseteq IR$ and $a = (ax)a = (ax)(ea) = (ae)(xa) \in I(RI) = R(II) = RI$, i.e., $I \subseteq RI$. Thus $I \subseteq IR \cap RI$, hence $IR \cap RI = I$. □

Lemma 3.3. *Let R be a regular LA-ring. Then $DL = D \cap L$ for every right ideal D and for every left ideal L of R.*

Proof. Since $DL \subseteq D \cap L$ is obvious. Let $a \in D \cap L$, then there exists an element $x \in R$ such that $a = (ax)a$. Now $a = (ax)a \in (DR)L \subseteq DL$, i.e., $D \cap L \subseteq DL$. Hence $DL = D \cap L$. □

Theorem 1. *Let R be an LA-ring with left identity e such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

(1) *R is a regular.*
(2) *$D \cap L = DL$ for every right ideal D and for every left ideal L of R.*
(3) *$Q = (QR)Q$ for every quasi-ideal Q of R.*

Proof. Suppose that (1) holds. Let Q be a quasi-ideal of R and $a \in Q$, this implies that there exists an element $x \in R$ such that $a = (ax)a$. Now $a = (ax)a \in (QR)Q$, i.e., $Q \subseteq (QR)Q \subseteq Q$, because every quasi-ideal of R is a bi-ideal of R. Hence $Q = (QR)Q$, i.e., $(1) \Rightarrow (3)$. Assume that (3) holds, let D be a right ideal and L be a left ideal of R. Then D and L be quasi-ideals of R by the Lemma 2.8, so $D \cap L$ be also a quasi-ideal of R. Now $D \cap L = ((D \cap L)R)(D \cap L) \subseteq (DR)L \subseteq DL$. Since $DL \subseteq D \cap L$, so $DL = D \cap L$, i.e., $(3) \Rightarrow (2)$. Suppose that (2) is true, let $a \in R$, then Ra is a left ideal of R containing a by the Lemma 2.2 and $aR \cup Ra$ is a right ideal of R containing a by the Proposition 2.1. By our supposition

$$\begin{aligned}(aR \cup Ra) \cap Ra &= (aR \cup Ra)(Ra) = (aR)(Ra) \cup (Ra)(Ra). \\ \text{Now } (Ra)(Ra) &= ((Re)a)(Ra) = ((ae)R)(Ra) = (aR)(Ra).\end{aligned}$$

Thus

$$\begin{aligned}(aR \cup Ra) \cap Ra &= (aR)(Ra) \cup (Ra)(Ra) \\ &= (aR)(Ra) \cup (aR)(Ra) = (aR)(Ra).\end{aligned}$$

Since $a \in (aR \cup Ra) \cap Ra$, implies $a \in (aR)(Ra)$. Then $a = (ax)(ya) = ((ya)x)a = (((ey)a)x)a = (((ay)e)x)a = ((xe)(ay))a$
$= (a((xe)y))a \in (aR)a$ for any $x, y \in R$, i.e., $a \in (aR)a$. Hence a is a regular, so R is a regular, i.e., $(2) \Rightarrow (1)$. □

Theorem 2. *Let R be an LA-ring with left identity e such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

(1) *R is a regular.*
(2) *$Q = (QR)Q$ for every quasi-ideal Q of R.*
(3) *$B = (BR)B$ for every bi-ideal B of R.*
(4) *$G = (GR)G$ for every generalized bi-ideal G of R.*

Proof. $(1) \Rightarrow (4)$, is obvious. $(4) \Rightarrow (3)$, since every bi-ideal of R is a generalized bi-ideal of R by the Lemma 2.7. $(3) \Rightarrow (2)$, since every quasi-ideal of R is bi-ideal of R by the Lemma 2.9. $(2) \Rightarrow (1)$, by the Theorem 1. □

Theorem 3. *Let R be an LA-ring with left identity e such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

(1) *R is a regular.*

(2) *$Q \cap I = (QI)Q$ for every quasi-ideal Q and for every ideal I of R.*

(3) *$B \cap I = (BI)B$ for every bi-ideal B and every for ideal I of R.*

(4) *$G \cap I = (GI)G$ for every generalized bi-ideal G and for every ideal I of R.*

Proof. Suppose that (1) holds. Let G be a generalized bi-ideal and I be an ideal of R. Now $(GI)G \subseteq (RI)R \subseteq I$ and $(GI)G \subseteq (GR)G \subseteq G$, thus $(GI)G \subseteq G \cap I$. Let $a \in G \cap I$, this means that there exists an element $x \in R$ such that $a = (ax)a$. Now $a = (ax)a = (((ax)a)x)a = ((xa)(ax))a = (a((xa)x))a \in (GI)G$, thus $G \cap I \subseteq (GI)G$. Hence $G \cap I = (GI)G$, i.e., $(1) \Rightarrow (4)$. $(4) \Rightarrow (3)$, since every bi-ideal of R is a generalized bi-ideal of R by the Lemma 2.7. $(3) \Rightarrow (2)$, since every quasi-ideal of R is a bi-ideal of R by the Lemma 2.9. Assume that (2) is true. Now $Q \cap R = (QR)Q$, i.e., $Q = (QR)Q$, where Q is a quasi-ideal of R. Hence R is a regular by the Theorem 1, i.e., $(2) \Rightarrow (1)$. □

Theorem 4. *Let R be an LA-ring with left identity e such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

(1) *R is a regular.*

(2) *$D \cap Q \subseteq DQ$ for every quasi-ideal Q and for every right ideal D of R.*

(3) *$D \cap B \subseteq DB$ for every bi-ideal B and for every right ideal D of R.*

(4) *$D \cap G \subseteq DG$ for every generalized bi-ideal G and for every right ideal D of R.*

Proof. Since $(1) \Rightarrow (4)$, is obvious. $(4) \Rightarrow (3)$, since every bi-ideal of R is a generalized bi-ideal of R. $(3) \Rightarrow (2)$, since every quasi-ideal of R is a bi-ideal of R. Suppose that (2) is true. Now $D \cap Q \subseteq DQ$, where Q is a left ideal and D is right ideal of R, because every left ideal of R is a quasi-ideal of R. Since $DQ \subseteq D \cap Q$, thus $D \cap Q = DQ$. Hence R is a regular by the Theorem 1, i.e., $(2) \Rightarrow (1)$. □

Theorem 5. *Let R be an LA-ring with left identity e, such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

(1) *R is a regular.*

(2) *$Q \cap D \cap L \subseteq (QD)L$ for every quasi-ideal Q, every right ideal D and for every left ideal L of R.*

(3) *$B \cap D \cap L \subseteq (BD)L$ for every bi-ideal B, every right ideal D and for every left ideal L of R.*

(4) *$G \cap D \cap L \subseteq (GD)L$ for every generalized bi-ideal G, every right ideal D and for every left ideal L of R.*

Proof. Suppose that (1) holds. Let $x \in G \cap D \cap L$, where G is a generalized bi-ideal, D is a right ideal and L is a left ideal of R. Let $x \in R$, this implies that there exists an element $a \in R$ such that $x = (xa)x$. Now

$$\begin{aligned} x &= (xa)x. \\ xa &= ((xa)x)a = (ax)(xa) = x((ax)a). \\ (ax)a &= (a((xa)x))a = ((xa)(ax))a = (a(ax))(xa) \\ &= x((a(ax))a) = x(((ea)(ax))a) = x(((xa)(ae))a) \\ &= x((((ae)a)x)a) = x((nx)a) = x((nx)(ea)) = x((ae)(xn)) \\ &= x(x((ae)n)) = x(xm). \\ &\Rightarrow xa = x((ax)a) = x(x(xm)) = (ex)(x(xm)) = ((xm)x)(xe). \end{aligned}$$

Thus $x = (xa)x = (((xm)x)(xe))x \in (GD)L$, i.e., $G \cap D \cap L \subseteq (GD)L$. Hence $(1) \Rightarrow (4)$. It is clear that $(4) \Rightarrow (3)$ and $(3) \Rightarrow (2)$. Assume that (2) is true. Then $Q \cap R \cap L \subseteq (Q \circ R) \circ L$, where Q is a right ideal of R, i.e., $Q \cap L \subseteq Q \circ L$. Since $Q \circ L \subseteq Q \cap L$, so $Q \circ L = Q \cap L$. Therefore R is a regular by the Theorem 1, i.e., $(2) \Rightarrow (1)$. □

4. Intra-regular LA-rings

An LA-ring R is an intra-regular, if for every $a \in R$, there exist elements $x_i, y_i \in R$ such that $a = \sum_{i=1}^{n}(x_i a^2)y_i$. We characterize intra-regular LA-rings by the properties of (left, right, quasi-, bi-, generalized bi-) ideals.

Lemma 4.1. *Every left* (*right*) *ideal of an intra-regular LA-ring R is an ideal of R.*

Proof. Suppose that I is a right ideal of R. Let $i \in I$ and $a \in R$. This implies that there exist elements $x_i, y_i \in R$, such that $a = \sum_{i=1}^{n}(x_i a^2)y_i$. Now $ai = ((x_i a^2)y_i)i = (iy_i))(x_i a^2) \in IR \subseteq I$. Hence I is an ideal of R. □

Lemma 4.2. *Every ideal of an intra-regular LA-ring R with left identity e is an idempotent.*

Proof. Let I be an ideal of R. Since $I^2 \subseteq I$ and $a \in I$, this means that there exist elements $x_i, y_i \in R$ such that $a = \sum_{i=1}^{n}(x_i a^2)y_i$. Now

$$\begin{aligned} a &= (x_i a^2)y_i = (x_i(aa))y_i = (a(x_i a))y_i \\ &= (a(x_i a))(ey_i) = (ae)((x_i a)y_i) = (x_i a)((ae)y_i) \in II. \end{aligned}$$

Thus $I \subseteq I^2$, i.e., $I^2 = I$. □

Proposition 4.1. *Let R be an intra-regular LA-ring with left identity e. Then any non-empty subset I of R is an ideal of R if and only if I is an interior ideal of R.*

Proof. Assume that I is an interior ideal of R. Let $i \in I$ and $a \in R$. Then there exist elements $x_i, y_i \in R$ such that $a = \sum_{i=1}^{n}(x_i a^2)y_i$. Now

$$\begin{aligned} ia &= i((x_i a^2)y_i) = i((x_i(aa))y_i) \\ &= i((a(x_i a))y_i) = i((a(x_i a))(ey_i)) \\ &= i((ae)((x_i a)y_i)) = i((x_i a)((ae)y_i)) \\ &= (x_i a)(i((ae)y_i)) = (x_i i)(a((ae)y_i)) \in (RI)R \subseteq I. \end{aligned}$$

Thus I is a right ideal of R. Therefore I is an ideal of R by the Lemma 4.1. Converse is obvious. □

Lemma 4.3. *Let R be an intra-regular LA-ring. Then $L \cap D \subseteq LD$ for every left ideal L and every right ideal D of R.*

Proof. Let $a \in L \cap D$, where L is a left ideal and D is a right ideal of R. This implies that there exist elements $x_i, y_i \in R$ such that $a = \sum_{i=1}^{n}(x_i a^2)y_i$. Now

$$\begin{aligned} a &= (x_i a^2)y_i = (x_i(aa))y_i = (a(x_i a))y_i = (a(x_i a))(ey_i) \\ &= (ae)((x_i a)y_i) = (x_i a)((ae)y_i) \in LD. \end{aligned}$$

Thus $L \cap D \subseteq LD$. □

Theorem 6. *Let R be an LA-ring with left identity e such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

(1) *R is an intra-regular.*

(2) *$L \cap D \subseteq LD$ for every left ideal L and for every right ideal D of R.*

Proof. (1) $\Rightarrow$ (2) is true by the Lemma 4.3. Suppose that (2) holds and $a \in R$, then Ra is a left ideal of R containing a by the Lemma 2.2 and $aR \cup Ra$ is a right ideal of R containing a by the Proposition 2.1. By our supposition

$$\begin{aligned} Ra \cap (aR \cup Ra) &\subseteq (Ra)(aR \cup Ra) = (Ra)(aR) \cup (Ra)(Ra). \\ \text{Now } (Ra)(aR) &= (Ra)((ea)R) = (Ra)((Ra)e) = (Ra)((Ra)(ee)) \\ &= (Ra)((Re)(ae)) = (Ra)(R(ae)) = (Ra)(Ra). \end{aligned}$$

Thus

$$\begin{aligned} (aR \cup Ra) \cap Ra &\subseteq (Ra)(aR) \cup (Ra)(Ra) = (Ra)(Ra) \cup (Ra)(Ra) \\ &= (Ra)(Ra) = R^2a^2 = (RR)(a^2e) \\ &= (ea^2)(RR) = (Ra^2)(Re) = (Ra^2)R. \end{aligned}$$

Since $a \in (aR \cup Ra) \cap Ra$, implies $a \in (Ra^2)R$, thus a is an intra regular. Hence R is an intra-regular, i.e., (2) $\Rightarrow$ (1). □

Theorem 7. *Let R be an LA-ring with left identity e such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

(1) *R is an intra-regular.*

(2) *$Q \cap I = (QI)Q$ for every quasi-ideal Q and for every ideal I of R.*

(3) *$B \cap I = (BI)B$ for every bi-ideal B and for every ideal I of R.*

(3) *$G \cap I = (GI)G$ for every generalized bi-ideal G and for every ideal I of R.*

Proof. Suppose that (1) holds. Let $a \in G \cap I$, where G is a generalized bi-ideal and I is an ideal of R, this implies that there exist elements $x_i, y_i \in R$ such that $a = \sum_{i=1}^{n}(x_i a^2)y_i$. Now

$$\begin{aligned} a &= (x_i a^2)y_i = (x_i(aa))y_i = (a(x_i a))y_i = (y_i(x_i a))a. \\ y_i(x_i a) &= y_i(x_i((x_i a^2)y_i)) = y_i((x_i a^2)(x_i y_i)) = (x_i a^2)(y_i(x_i y_i)) \end{aligned}$$

$$\begin{aligned} &= (x_i a^2)(x_i y_i^2) = (x_i(aa))m_i, \text{ say } x_i y_i^2 = m_i \\ &= (a(x_i a))m_i = (m_i(x_i a))a. \\ m_i(x_i a) &= m_i(x_i((x_i a^2)y_i)) = m_i((x_i a^2)(x_i y_i)) = (x_i a^2)(m_i(x_i y_i)) \\ &= (x_i(aa))n_i, \text{ say } m_i(x_i y_i) = n_i \\ &= (a(x_i a))n_i = (n_i(x_i a))a \\ &= v_i a, \text{ say } n_i(x_i a) = v_i. \\ &\Rightarrow y_i(x_i a) = (m_i(x_i a))a = (v_i a)a = (v_i a)(ea) = (v_i e)(aa) = a((v_i e)a). \end{aligned}$$

Thus $a = (x_i a^2)y_i = (y_i(x_i a))a = (a((v_i e)a))a \in (GI)G$, i.e., $G \cap I \subseteq (GI)G$. Now $(GI)G \subseteq (RI)R \subseteq I$ and $(GI)G \subseteq (GR)G \subseteq G$, thus $(GI)G \subseteq G \cap I$. Hence $G \cap I = (GI)G$, i.e., $(1) \Rightarrow (4)$. $(4) \Rightarrow (3)$, every bi-ideal of R is a generalized bi-ideal of R by the Lemma 2.7. $(3) \Rightarrow (2)$, every quasi-ideal of R is a bi-ideal of R by the Lemma 2.9. Assume that (2) is true and let Q be a right ideal and I be a two-sided ideal of R. Now $I \cap Q = (QI)Q \subseteq (RI)Q \subseteq IQ$, since every right ideal of R is a quasi-ideal of R. Therefore R is an intra-regular by the Theorem 6, i.e., $(2) \Rightarrow (1)$. □

Theorem 8. *Let R be an LA-ring with left identity e such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

(1) *R is an intra-regular.*

(2) *$L \cap Q \subseteq LQ$ for every quasi-ideal Q and for every left ideal L of R.*

(3) *$L \cap B \subseteq LB$ for every bi-ideal B and for every left ideal L of R.*

(4) *$L \cap G \subseteq LG$ for every generalized bi-ideal G and for every left ideal L of R.*

Proof. Suppose that (1) holds. Let $a \in L \cap G$, where L is a left ideal and G is a generalized bi-ideal of R, this means that there exist elements $x_i, y_i \in R$ such that $a = \sum_{i=1}^{n}(x_i a^2)y_i$. Now $a = (x_i a^2)y_i = (x_i(aa))y_i = (a(x_i a))y_i =$

$(y_i(x_ia))a \in LG$, i.e., $a \in LG$. Thus $L \cap G \subseteq LG$, i.e., $(1) \Rightarrow (4)$. $(4) \Rightarrow (3)$, every bi-ideal of R is a generalized bi-ideal of R. $(3) \Rightarrow (2)$, every quasi-ideal of R is a bi-ideal of R. Assume that (2) is true and let Q be a right ideal and L be a left ideal of R. Now $L \cap Q \subseteq LQ$, where Q is a quasi-ideal of R. Hence R is an intra-regular by the Theorem 6, i.e., $(2) \Rightarrow (1)$. □

Theorem 9. *Let R be an LA-ring with left identity e such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

(1) *R is an intra-regular.*

(2) *$L \cap Q \cap D \subseteq (LQ)D$ for every quasi-ideal Q, every right ideal D and for every left ideal L of R.*

(3) *$L \cap B \cap D \subseteq (LB)D$ for every bi-ideal B, every right ideal D and for every left ideal L of R.*

(4) *$L \cap G \cap D \subseteq (LG)D$ for every generalized bi-ideal G, every right ideal D and for every left ideal L of R.*

Proof. Suppose that (1) holds. Let $a \in G \cap L \cap D$, where G is a generalized bi-ideal, L is a left ideal and D is a right ideal of R, this implies that there exist elements $x_i, y_i \in R$ such that $a = \sum_{i=1}^{n}(x_ia^2)y_i$. Now

$$\begin{aligned} a &= (x_ia^2)y_i = (x_i(aa))y_i = (a(x_ia))y_i = (y_i(x_ia))a. \\ y_i(x_ia) &= y_i(x_i((x_ia^2)y_i)) = y_i((x_ia^2)(x_iy_i)) = (x_ia^2)(y_i(x_iy_i)) \\ &= (x_ia^2)(x_iy_i^2) = (x_i(aa))m_i, \text{ say } x_iy_i^2 = m_i \\ &= (a(x_ia))m_i = (m_i(x_ia))a. \end{aligned}$$

Thus $a = (x_ia^2)y_i = (y_i(x_ia))a = ((m(x_ia))a)a \in (LG)R$, i.e., $a \in (LG)D$. Hence $G \cap L \cap D \subseteq (LG)D$, i.e., $(1) \Rightarrow (4)$. $(4) \Rightarrow (3)$, every bi-ideal of R is a generalized bi-ideal of R. $(3) \Rightarrow (2)$, every quasi-ideal of R is a bi-ideal of R. Assume that (2) is true. Now

$$\begin{aligned} L \cap R \cap D &\subseteq (LR)D = ((eL)R)D = ((RL)e)D \subseteq (Le)D \\ &= (e(Le))D \subseteq (R(Le))D \subseteq (RL)D \subseteq LD. \\ &\Rightarrow L \cap D \subseteq LD. \end{aligned}$$

Hence R is an intra-regular by the Theorem 6, i.e., $(2) \Rightarrow (1)$. □

5. Regular and intra-regular LA-rings

We characterize both regular and intra-regular LA-rings by the properties of (left, right, quasi-, bi-, generalized bi-) ideals.

Theorem 10. *Let R be an LA-ring with left identity e such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

(1) *R is a regular and an intra-regular.*

(2) *$B^2 = B$ for every bi-ideal B of R.*

(3) *$B_1 \cap B_2 = B_1B_2 \cap B_2B_1$ for all bi-ideals B_1, B_2 of R.*

Proof. Suppose that (1) holds. Let B be a bi-ideal of R and $B^2 \subseteq B$. Let $a \in B$, this implies that there exists an element $x \in R$ such that $a = (ax)a$, also there exist elements $y_i, z_i \in R$ such that $a = (y_i a^2)z_i$. Now

$$\begin{aligned}
a &= (ax)a = (ax)((y_i a^2)z_i) = (((y_i a^2)z_i)x)a. \\
((y_i a^2)z_i)x &= (xz_i)(y_i a^2) = m_i(y_i a^2), \text{ say } m_i = xz_i \\
&= m_i(y_i(aa)) = m_i(a(y_i a)) = a(m_i(y_i a)) \\
&= ((ax)a)(m_i(y_i a)) = ((ax)m_i)(a(y_i a)) \\
&= ((m_i x)a)(a(y_i a)) = (n_i a)(a(y_i a)), \text{ say } n_i = m_i x \\
&= ((en_i)a)(a(y_i a)) = ((an_i)e)(a(y_i a)) \\
&= ((an_i)a)(e(y_i a)) = ((an_i)a)(y_i a) = (s_i a)(y_i a), \text{ say } s_i = an_i \\
&= (aa)(y_i s_i) = (aa)t_i, \text{ say } t_i = y_i s_i \\
&= (((ax)a)a)t_i = ((aa)(ax))t_i = (t_i(ax))(aa) \\
&= (a(t_i x))(aa) = (aw_i)(aa), \text{ say } w_i = t_i x.
\end{aligned}$$

Thus $a = (((y_i a^2)z_i)x)a = ((aw_i)(aa))a \in ((BR)B)B \subseteq B^2$, i.e., $B \subseteq B^2$. Hence $B^2 = B$, i.e., (1) $\Rightarrow$ (3) . Assume that (2) is true. Let B_1, B_2 be bi-ideals of R, then $B_1 \cap B_2$ be also a bi-ideal of R. Now $B_1 \cap B_2 = (B_1 \cap B_2)(B_1 \cap B_2) \subseteq B_1 B_2$ and $B_1 \cap B_2 = (B_1 \cap B_2)(B_1 \cap B_2) \subseteq B_2 B_1$, thus $B_1 \cap B_2 \subseteq B_1 B_2 \cap B_2 B_1$. Now we show that $B_1 B_2$ is a bi-ideal of R. It is enough to show that $((B_1 B_2)R)(B_1 B_2) \subseteq B_1 B_2$. Now

$$\begin{aligned}
((B_1 B_2)R)(B_1 B_2) &= ((B_1 B_2)(RR))(B_1 B_2) \\
&= ((B_1 R)(B_2 R))(B_1 B_2) \\
&= ((B_1 R)B_1)((B_2 R)B_2) \subseteq B_1 B_2. \\
&\Rightarrow ((B_1 B_2)R)(B_1 B_2) \subseteq B_1 B_2.
\end{aligned}$$

Thus $B_1 B_2$ is a bi-ideal of R, similarly $B_2 B_1$ is also a bi-ideal of R. Then $B_1 B_2 \cap B_2 B_1$ is also a bi-ideal of R. Now

$$\begin{aligned}
B_1 B_2 \cap B_2 B_1 &= (B_1 B_2 \cap B_2 B_1)(B_1 B_2 \cap B_2 B_1) \\
&\subseteq (B_1 B_2)(B_2 B_1) \subseteq (B_1 R)(RB_1) \\
&= ((RB_1)R)B_1 = (((Re)B_1)R)B_1 \\
&= (((B_1 e)R)R)B_1 = ((B_1 R)R)B_1 \\
&= ((RR)B_1)B_1 = (RB_1)B_1 \\
&= ((Re)B_1)B_1 = ((B_1 e)R)B_1 \\
&= (B_1 R)B_1 \subseteq B_1. \\
&\Rightarrow B_1 B_2 \cap B_2 B_1 \subseteq B_1.
\end{aligned}$$

Similarly, we have $B_1 B_2 \cap B_2 B_1 \subseteq B_2$, thus $B_1 B_2 \cap B_2 B_1 \subseteq B_1 \cap B_2$. Therefore $B_1 \cap B_2 = B_1 B_2 \cap B_2 B_1$, i.e., (2) $\Rightarrow$ (3) . Suppose that (3) holds, let D be right ideal and I be an ideal of R. Then D and I be bi-ideals of R, because every right ideal and two-sided ideal of R is bi-ideal of R by the Lemma 2.6. Now $D \cap I = DI \cap ID$, this implies that $D \cap I \subseteq DI \cap ID$. Thus $D \cap I \subseteq DI$ and $D \cap I \subseteq ID$, where I is also a left ideal of R. Since $DI \subseteq D \cap I$, i.e., $DI = D \cap I$. Thus R is regular by the Theorem 1. Also, $D \cap I \subseteq ID$, Thus R is an intra-regular by the Theorem 6. Hence $(3) \Rightarrow (1)$. □

Theorem 11. *Let R be an LA-ring with left identity e such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

(1) *R is a regular and an intra-regular.*
(2) *Every quasi-ideal of R is an idempotent.*

Proof. Suppose that (1) holds. Let Q be a quasi-ideal of R and $Q^2 \subseteq Q$. Let $a \in Q$, this implies that there exists an element $x \in R$ such that $a = (ax)a$, also there exist elements $y_i, z_i \in R$ such that $a = (y_i a^2)z_i$. Now $a = (ax)a = (ax)((y_i a^2)z_i) = (((y_i a^2)z_i)x)a = ((aw_i)(aa))a$, where $((y_i a^2)z_i)x = (aw_i)(aa)$, by the Theorem 10. Thus $a = ((aw_i)(aa))a \in ((QR)Q)Q \subseteq QQ \subseteq Q^2$, i.e., $Q \subseteq Q^2$, because every quasi-ideal of R is a bi-ideal of R by the Lemma 2.9. Thus $Q^2 = Q$, i.e., $(1) \Rightarrow (2)$. Assume that (2) is true. Let $a \in R$, then Ra is a left ideal of R containing a, i.e., Ra is a quasi-ideal of R, because every left ideal of R is a quasi-ideal of R. Now $Ra = (Ra)^2 = (Ra)(Ra)$, i.e., $a \in (Ra)(Ra)$, thus R is an intra-regular by the Theorem 6. Now $Ra = (Ra)(Ra) = ((Re)a)(Ra) = ((ae)R)(Ra) = (aR)(Ra)$, i.e., $a \in (aR)(Ra)$, thus R is regular by the Theorem 1. Hence $(2) \Rightarrow (1)$. □

Theorem 12. *Let R be an LA-ring with left identity e such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

(1) *R is a regular and an intra-regular.*
(2) *Every quasi-ideal of R is an idempotent.*
(3) *Every bi-ideal of R is an idempotent.*

Proof. $(1) \Rightarrow (3)$, by the Theorem 10. $(3) \Rightarrow (2)$, every quasi-ideal of R is a bi-ideal of R, by the Lemma 2.9. $(2) \Rightarrow (1)$, by the Theorem 11. □

Theorem 13. *Let R be an LA-ring with left identity e such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.*

(1) *R is a regular and an intra-regular.*
(2) *$Q_1 \cap Q_2 \subseteq Q_1 Q_2$ for all quasi-ideals Q_1, Q_2 of R.*
(3) *$Q \cap B \subseteq QB$ for every quasi-ideal Q and for every bi-ideal B of R.*
(4) *$B \cap Q \subseteq BQ$ for every bi-ideal B and for every quasi-ideal Q of R.*
(5) *$B_1 \cap B_2 \subseteq B_1 B_2$ for all bi-ideals B_1, B_2 of R.*

Proof. Suppose that (1) holds. Let B_1, B_2 be bi-ideals of R, then $B_1 \cap B_2$ be also a bi-ideal of R. Since every bi-ideal of R is an idempotent by the Theorem 10, then $B_1 \cap B_2 = (B_1 \cap B_2)^2 = (B_1 \cap B_2)(B_1 \cap B_2) \subseteq B_1 B_2$, i.e., $(1) \Rightarrow (5)$. Since $(5) \Rightarrow (4) \Rightarrow (2)$ and $(5) \Rightarrow (3) \Rightarrow (2)$, because every quasi-ideal of R is a bi-ideal of R by the Lemma 2.9. Assume that (2) is true. Now $D \cap L \subseteq DL$, where D is a right ideal and L is a left ideal of R. Since $DL \subseteq D \cap L$, i.e., $D \cap L = DL$, thus R is regular. Again by (2) $L \cap D \subseteq LD$, thus R is an intra-regular. Hence (2) $\Rightarrow$ (1). □

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