Ideals in LA-rings

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Abstract. Our aim is to encourage research and maturity of associative algebraic structures by studying a class of non-associative and non-commutative algebraic structures (LA-ring).

Keywords: (left, right, interior, quasi-, bi-, generalized bi-) ideals, regular (intra-regular) LA-rings.

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1. Introduction

In ternary operations, the commutative law is given by \( abc = cba \). Kazim et al \[6\] have generalized this notion by introducing the parenthesis on the left side of this equation to get a new pseudo associative law, that is \((ab)c = (cb)a\). This law \((ab)c = (cb)a\) is the left invertive law. A groupoid \( S \) is a left almost semigroup (abbreviated as LA-semigroup), if it satisfies the left invertive law. An LA-semigroup is a midway structure between a commutative semigroup and a groupoid. Ideals in LA-semigroups have been investigated by Protic et al \[7\].

In \[2\] (resp. \[1\]), a groupoid \( S \) is to be medial (resp. paramedial) if \((ab)(cd) = (ac)(bd)\) (resp. \((ab)(cd) = (db)(ca))\). In \[6\], an LA-semigroup is medial, but in general an LA-semigroup needs not to be paramedial. Every LA-semigroup with left identity is paramedial by Protic et al \[7\] and also satisfies \( a(bc) = b(ac), (ab)(cd) = (dc)(ba)\).

Kamran \[3\], extended the notion of LA-semigroup to the left almost group (LA-group). An LA-semigroup \( G \) is a left almost group, if there exists a left identity \( e \in G \) such that \( ea = a \) for all \( a \in G \) and for every \( a \in G \) there exists \( b \in G \) such that \( ba = e \).

Shah et al \[8\], discussed the left almost ring (LA-ring) of finitely nonzero functions which is a generalization of commutative semigroup ring. By a left almost ring, we mean a non-empty set \( R \) with at least two elements such that \((R, +)\) is an LA-group, \((R, \cdot)\) is an LA-semigroup, both left and right distributive laws hold. For example, from a commutative ring \((R, +, \cdot)\), we can always obtain an LA-ring \((R, \oplus, \cdot)\) by defining for all \( a, b \in R, a \oplus b = b - a \) and \( a \cdot b \) is same as in the ring. Although the structure is non-associative and non-commutative, nevertheless, it possesses many interesting properties which we usually find in associative and commutative algebraic structures.

In this paper, we will define the left (resp. right, interior, quasi-, bi-, generalized bi-) ideals of an LA-ring \( R \). We will establish a study by discussing the different properties of such ideals. We will also characterize regular (resp. intra-regular, both regular and intra-regular) LA-rings in terms of left (resp. right, quasi-, bi-, generalized bi-) ideals.

2. Ideals in LA-rings

We initiate the concept of LA-subrings and left (resp. right, interior, quasi-, bi-, generalized bi-) ideals of an LA-ring \( R \).

A non-empty subset \( A \) of an LA-ring \( R \) is called an LA-subring of \( R \) if \( a - b \) and \( ab \in A \) for all \( a, b \in A \). \( A \) is called a left (resp. right) ideal of \( R \), if \((A, +)\) is an LA-group and \( RA \subseteq A \) (resp. \( AR \subseteq A \)). \( A \) is called an ideal of \( R \), if it is both a left ideal and a right ideal of \( R \).
Example 1. Let $R = \{0, 1, 2, 3, 4, 5, 6, 7\}$. Define $+$ and $\cdot$ in $R$ as follows:

\[
\begin{array}{cccccccc}
+ & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
0 & 1 & 2 & 0 & 3 & 1 & 6 & 4 & 7 \\
1 & 2 & 1 & 3 & 0 & 2 & 5 & 7 & 4 \\
2 & 3 & 2 & 1 & 0 & 7 & 6 & 5 & 4 \\
3 & 0 & 3 & 3 & 0 & 0 & 0 & 0 & 0 \\
4 & 4 & 5 & 6 & 7 & 0 & 1 & 2 & 3 \\
5 & 6 & 4 & 7 & 5 & 2 & 0 & 3 & 1 \\
6 & 5 & 7 & 4 & 6 & 1 & 3 & 0 & 2 \\
7 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
\cdot & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 4 & 0 & 0 & 4 & 4 & 0 & 0 \\
2 & 0 & 4 & 0 & 0 & 4 & 4 & 0 & 0 \\
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
4 & 0 & 3 & 0 & 0 & 3 & 3 & 0 & 0 \\
5 & 0 & 7 & 0 & 0 & 7 & 7 & 0 & 0 \\
6 & 0 & 7 & 0 & 0 & 7 & 7 & 0 & 0 \\
7 & 0 & 3 & 0 & 0 & 3 & 3 & 0 & 0 \\
\end{array}
\]

Let $A = \{0, 4\}$ be a subset of $R$. Since $(A, +)$ is an LA-subgroup and $A_2 = AA = \{0, 4\}\{0, 4\} = \{0\} \subseteq A$.

Thus $A$ is an LA-subring of $R$, but not right ideal of $R$.

A non-empty subset $A$ of an LA-ring $R$ is an interior ideal of $R$, if $(A, +)$ is an LA-group and $(RA)R \subseteq A$. A non-empty subset $A$ of $R$ is a quasi-ideal of $R$, if $(A, +)$ is an LA-group and $AR \cap RA \subseteq A$. An LA-subring $A$ of an LA-ring $R$ is a bi-ideal of $R$ if $(AR)A \subseteq A$. A non-empty subset $A$ of an LA-ring $R$ is a generalized bi-ideal of $R$ if $(A, +)$ is an LA-group and $(AR)A \subseteq A$. An ideal $I$ of an LA-ring $R$ is an ideal of $R$ if $A^2 = A$.

Now we give the central properties of such ideals of an LA-ring $R$, which will be very helpful for further sections.

**Lemma 2.1.** Let $R$ be an LA-ring with left identity $e$, then $RR = R$ and $eR = R = Re$.

**Proof.** Since $RR \subseteq R$ and $x = ex \in RR$, i.e., $R \subseteq RR$, thus $RR = R$. Obviously, $eR = R$ and $Re = (RR)e = (eR)R = RR = R$. \(\square\)

**Lemma 2.2** ([4, Lemma 8]). Let $R$ be an LA-ring with left identity $e$ and $a \in R$. Then $Ra$ is the smallest left ideal of $R$ containing $a$.

**Lemma 2.3** ([4, Lemma 9]). Let $R$ be an LA-ring with left identity $e$ and $a \in R$. Then $Ra$ is a left ideal of $R$.

**Proposition 2.1** ([4, Proposition 5]). Let $R$ be an LA-ring with left identity $e$ and $a \in R$. Then $aR \cup Ra$ is the smallest right ideal of $R$ containing $a$.

**Lemma 2.4.** Let $R$ be an LA-ring with left identity $e$. Then every right ideal of $R$ is an ideal of $R$. 
Proof. Let $I$ be a right ideal of $R$. Let $a \in I$ and $r \in R$. Now $ra = (er)a = (ar)e \in IR \subseteq I$. Thus $I$ is an ideal of $R$. 

Lemma 2.5. Every two-sided ideal of $R$ is an interior ideal of $R$, but the converse is not true.

Proof. Straight forward.

Example 2. Let $A = \{0, 1, 2, 3\}$ be a subset of $R$, defined as in example 1. Since $(A, +)$ is an LA-subgroup and

$$
(RA)R = (\{0, 1, 2, 3, 4, 5, 6, 7\}\{0, 1, 2, 3\})\{0, 1, 2, 3, 4, 5, 6, 7\} = \{0, 3\} \subseteq A.
$$

$$
RA = \{0, 1, 2, 3, 4, 5, 6, 7\}\{0, 1, 2, 3\} = \{0, 3, 4, 7\} \not\subseteq A.
$$

$$
AR = \{0, 1, 2, 3\}\{0, 1, 2, 3, 4, 5, 6, 7\} = \{0, 4\} \not\subseteq A.
$$

Thus $A$ is an interior ideal of $R$, but not an ideal of $R$.

Proposition 2.2. Let $R$ be an LA-ring with left identity $e$. Then any non-empty subset $I$ of $R$ is an ideal of $R$ if and only if $I$ is an interior ideal of $R$.

Proof. Let $I$ be an interior ideal of $R$. Let $a \in I$ and $r \in R$. Now $ar = (ea)r \in (RI)R \subseteq I$, i.e., $I$ is a right ideal of $R$. Thus $I$ is an ideal of $R$ by the Lemma 2.4. Converse is true by the Lemma 2.5.

Lemma 2.6. Every left (resp. right, two-sided) ideal of $R$ is a bi-ideal of $R$.

Proof. Straight forward.

Example 3. Let $A = \{0, 4\}$ be a subset of $R$ defined as in example 1. Since $(A, +)$ is an LA-subgroup and

$$
A^2 = AA = \{0, 4\}\{0, 4\} = \{0\} \subseteq A.
$$

$$
(AR)A = (\{0, 4\}\{0, 1, 2, 3, 4, 5, 6, 7\})\{0, 4\}
= \{0, 3\}\{0, 4\} = \{0\} \subseteq A.
$$

$$
AR = \{0, 4\}\{0, 1, 2, 3, 4, 5, 6, 7\} = \{0, 3\} \not\subseteq A.
$$

Thus $A$ is a bi-ideal of $R$, but not right ideal of $R$.

Lemma 2.7. Every bi-ideal of $R$ is a generalized bi-ideal of $R$.

Proof. Obvious.

Lemma 2.8. Every left (resp. right, two-sided) ideal of $R$ is a quasi-ideal of $R$.

Proof. Let $I$ be a left ideal of $R$ and $IR \cap RI \subseteq RI \subseteq I$, i.e., $I$ is a quasi-ideal of $R$. 

Proposition 2.3. Let $I$ be a right ideal and $L$ be a left ideal of an LA-ring $R$, respectively. Then $I \cap L$ is a quasi-ideal of $R$.

Proof. Since $(I \cap L)R \cap R(I \cap L) \subseteq IR \cap RL \subseteq I \cap L$, i.e., $I \cap L$ is a quasi-ideal of $R$. \hfill $\Box$

Lemma 2.9. Let $R$ be an LA-ring with left identity $e$ such that $(xe)R = xR$ for all $x \in R$. Then every quasi-ideal of $R$ is a bi-ideal of $R$.

Proof. Let $Q$ be a quasi-ideal of $R$. Now $(QR)Q \subseteq RQ$ and $(QR)Q \subseteq (QR)R = (QR)(eR) = (Qe)(RR) = (Qe)R = QR$, thus $(QR)Q \subseteq QR \cap RQ \subseteq Q$. Hence $Q$ is a bi-ideal of $R$. \hfill $\Box$

3. Regular LA-rings

An LA-ring $R$ is regular, if for every $a \in R$, there exists an element $x \in R$ such that $a = (ax)a$. We characterize regular LA-rings by the properties of (left, right, quasi-, bi-, generalized bi-) ideals.

Lemma 3.1. Every right ideal of a regular LA-ring $R$ is an ideal of $R$.

Proof. Suppose that $I$ is a right ideal of $R$. Let $a \in I$ and $r \in R$. This implies that there exists an element $x \in R$, such that $r = (rx)r$. Now $ra = ((rx)r)a = (ar)(rx) \in IR \subseteq I$. Hence $I$ is an ideal of $R$. \hfill $\Box$

Lemma 3.2. Every ideal of a regular LA-ring $R$ is an idempotent.

Proof. Let $I$ be an ideal of $R$. Since $I^2 \subseteq I$ and $a \in I$. This means that there exists an element $x \in R$ such that $a = (ax)a$. Now $a = (ax)a \in (IR)I \subseteq II = I^2$, i.e., $I \subseteq I^2$. Thus $I^2 = I$. \hfill $\Box$

Remark 1. Every right ideal of a regular LA-ring $R$ is an idempotent.

Proposition 3.1. Let $R$ be a regular LA-ring. Then any non-empty subset $I$ of $R$ is an ideal of $R$ if and only if $I$ is an interior ideal of $R$.

Proof. Assume that $I$ is an interior ideal of $R$. Let $a \in I$ and $r \in R$. Then there exists an element $x \in R$, such that $a = (ax)a$. Now $ar = ((ax)a)r = (ra)(ax) \in (RI)R \subseteq I$, i.e., $IR \subseteq I$. Thus $I$ is an ideal of $R$ by the Lemma 3.1. Converse is true by the Lemma 2.5. \hfill $\Box$

Proposition 3.2. Let $R$ be a regular LA-ring with left identity $e$. Then $IR \cap RI = I$ for every right ideal $I$ of $R$.

Proof. Let $I$ be a right ideal of $R$. This implies that $IR \cap RI \subseteq I$, because every right ideal of $R$ is a quasi-ideal of $R$. Let $a \in I$, this means that there exists an element $x \in R$ such that $a = (ax)a$. Now $a = (ax)a \in (IR)I \subseteq II \subseteq IR$, i.e., $I \subseteq IR$ and $a = (ax)a = (ax)(ea) = (ae)(xa) \in I(RI) = R(II) = RI$, i.e., $I \subseteq RI$. Thus $I \subseteq IR \cap RI$, hence $IR \cap RI = I$. \hfill $\Box$
Lemma 3.3. Let $R$ be a regular LA-ring. Then $DL = D \cap L$ for every right ideal $D$ and for every left ideal $L$ of $R$.

**Proof.** Since $DL \subseteq D \cap L$ is obvious. Let $a \in D \cap L$, then there exists an element $x \in R$ such that $a = (ax)a$. Now $a = (ax)a \in (DR)L \subseteq DL$, i.e., $D \cap L \subseteq DL$. Hence $DL = D \cap L$. \hfill \Box

**Theorem 1.** Let $R$ be an LA-ring with left identity $e$ such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.

(1) $R$ is a regular.

(2) $D \cap L = DL$ for every right ideal $D$ and for every left ideal $L$ of $R$.

(3) $Q = (QR)Q$ for every quasi-ideal $Q$ of $R$.

**Proof.** Suppose that (1) holds. Let $Q$ be a quasi-ideal of $R$ and $a \in Q$, this implies that there exists an element $x \in R$ such that $a = (ax)a$. Now $a = (ax)a \in (QR)Q$, i.e., $Q \subseteq (QR)Q \subseteq Q$, because every quasi-ideal of $R$ is a bi-ideal of $R$. Hence $Q = (QR)Q$, i.e., $(1) \Rightarrow (3)$. Assume that (3) holds, let $D$ be a right ideal and $L$ be a left ideal of $R$. Then $D$ and $L$ be quasi-ideals of $R$ by the Lemma 2.8, so $D \cap L$ be also a quasi-ideal of $R$. Now $D \cap L = ((D \cap L)R)(D \cap L) \subseteq (DR)L \subseteq DL$. Since $DL \subseteq D \cap L$, so $DL = D \cap L$, i.e., $(3) \Rightarrow (2)$. Suppose that (2) is true, let $a \in R$, then $Ra$ is a left ideal of $R$ containing $a$ by the Lemma 2.2 and $aR \cup Ra$ is a right ideal of $R$ containing $a$ by the Proposition 2.1. By our supposition

$$(aR \cup Ra) \cap Ra = (aR \cup Ra)(Ra) = (aR)(Ra) \cup (Ra)(Ra).$$

Now $(Ra)(Ra) = (((Ra)a)(Ra) = ((Ra)(Ra) = (aR)(Ra)).$

Thus

$$(aR \cup Ra) \cap Ra = (aR)(Ra) \cup (Ra)(Ra)$$

$$= (aR)(Ra) \cup (aR)(Ra) = (aR)(Ra).$$

Since $a \in (aR \cup Ra) \cap Ra$, implies $a \in (aR)(Ra)$. Then $a = (ax)(ya) = (((ya)a)x)a = (((ya)a)x)a = ((ya)(ay))a$

$= (a((xe)y))a \in (aR)a$ for any $x, y \in R$, i.e., $a \in (aR)a$. Hence $a$ is a regular, so $R$ is a regular, i.e., $(2) \Rightarrow (1)$. \hfill \Box

**Theorem 2.** Let $R$ be an LA-ring with left identity $e$ such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.

(1) $R$ is a regular.

(2) $Q = (QR)Q$ for every quasi-ideal $Q$ of $R$.

(3) $B = (BR)B$ for every bi-ideal $B$ of $R$.

(4) $G = (GR)G$ for every generalized bi-ideal $G$ of $R$.

**Proof.** $(1) \Rightarrow (4)$, is obvious. $(4) \Rightarrow (3)$, since every bi-ideal of $R$ is a generalized bi-ideal of $R$ by the Lemma 2.7. $(3) \Rightarrow (2)$, since every quasi-ideal of $R$ is bi-ideal of $R$ by the Lemma 2.9. $(2) \Rightarrow (1)$, by the Theorem 1. \hfill \Box
Theorem 3. Let $R$ be an LA-ring with left identity $e$ such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.

1. $R$ is a regular.
2. $Q \cap I = (QI)Q$ for every quasi-ideal $Q$ and for every ideal $I$ of $R$.
3. $B \cap I = (B\cap I)B$ for every bi-ideal $B$ and for every ideal $I$ of $R$.
4. $G \cap I = (GI)G$ for every generalized bi-ideal $G$ and for every ideal $I$ of $R$.

Proof. Suppose that (1) holds. Let $G$ be a generalized bi-ideal and $I$ be an ideal of $R$. Now $(GI)G \subseteq (RI)R \subseteq I$ and $(GI)G \subseteq (GR)G \subseteq G$, thus $(GI)G \subseteq G \cap I$. Let $a \in G \cap I$, this means that there exists an element $x \in R$ such that $a = (ax)a$. Now $a = (ax)a = ((ax)a)x = (xa)(ax))a = (a((xa)x))a \in (GI)G$, thus $G \cap I \subseteq (GI)G$. Hence $G \cap I = (GI)G$, i.e., (1) $\Rightarrow$ (4). (4) $\Rightarrow$ (3), since every bi-ideal of $R$ is a generalized bi-ideal of $R$ by the Lemma 2.9. Assume that (2) is true. Now $Q \cap R = (QR)Q$, i.e., $Q = (QR)Q$, where $Q$ is a quasi-ideal of $R$. Hence $R$ is a regular by the Theorem 1, i.e., (2) $\Rightarrow$ (1).

Theorem 4. Let $R$ be an LA-ring with left identity $e$ such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.

1. $R$ is a regular.
2. $D \cap Q \subseteq DQ$ for every quasi-ideal $Q$ and for every right ideal $D$ of $R$.
3. $D \cap B \subseteq DB$ for every bi-ideal $B$ and for every right ideal $D$ of $R$.
4. $D \cap G \subseteq DG$ for every generalized bi-ideal $G$ and for every right ideal $D$ of $R$.

Proof. Since (1) $\Rightarrow$ (4), is obvious. (4) $\Rightarrow$ (3), since every bi-ideal of $R$ is a generalized bi-ideal of $R$. (3) $\Rightarrow$ (2), since every quasi-ideal of $R$ is a bi-ideal of $R$. Suppose that (2) is true. Now $D \cap Q \subseteq DQ$, where $Q$ is a left ideal and $D$ is right ideal of $R$, because every left ideal of $R$ is a quasi-ideal of $R$. Since $DQ \subseteq D \cap Q$, thus $D \cap Q = DQ$. Hence $R$ is a regular by the Theorem 1, i.e., (2) $\Rightarrow$ (1).

Theorem 5. Let $R$ be an LA-ring with left identity $e$, such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.

1. $R$ is a regular.
2. $Q \cap D \cap L \subseteq (QD)L$ for every quasi-ideal $Q$, every right ideal $D$ and for every left ideal $L$ of $R$.
3. $B \cap D \cap L \subseteq (BD)L$ for every bi-ideal $B$, every right ideal $D$ and for every left ideal $L$ of $R$.
4. $G \cap D \cap L \subseteq (GD)L$ for every generalized bi-ideal $G$, every right ideal $D$ and for every left ideal $L$ of $R$.

Proof. Suppose that (1) holds. Let $x \in G \cap D \cap L$, where $G$ is a generalized bi-ideal, $D$ is a right ideal and $L$ is a left ideal of $R$. Let $x \in R$, this implies that
there exists an element \( a \in R \) such that \( x = (xa)x \). Now

\[
x = (xa)x.
\]
\[
xa = ((xa)x)a = (ax)(xa) = x((ax)a).
\]
\[
(ax)a = (a((xa)x))a = ((xa)(ax))a = (a(ax))((xa)a)
\]
\[
= x((a(ax))a) = x((ea)(ax)a) = x(((xa)(ae))a)
\]
\[
= x(x((ae)(n)a)) = x(xm).
\]
\[
\Rightarrow xa = x((ax)a) = x(x(xm)) = (ex)(x(xm)) = ((xm)x)(xe).
\]

Thus \( x = (xa)x = (((xm)x)(xe))x \in (GD)L \), i.e., \( G \cap D \cap L \subseteq (GD)L \). Hence (1) \( \Rightarrow \) (4). It is clear that (4) \( \Rightarrow \) (3) and (3) \( \Rightarrow \) (2). Assume that (2) is true. Then \( Q \cap R \cap L \subseteq (Q \circ R) \circ L \), where \( Q \) is a right ideal of \( R \), i.e., \( Q \cdot L \subseteq Q \circ L \). Since \( Q \circ L \subseteq Q \cap L \), so \( Q \circ L = Q \cap L \). Therefore \( R \) is a regular by the Theorem 1, i.e., (2) \( \Rightarrow \) (1).

4. Intra-regular LA-rings

An LA-ring \( R \) is an intra-regular, if for every \( a \in R \), there exist elements \( x_i, y_i \in R \) such that \( a = \sum_{i=1}^{n}(x_i a^2)y_i \). We characterize intra-regular LA-rings by the properties of (left, right, quasi-, bi-, generalized bi-) ideals.

**Lemma 4.1.** Every left (right) ideal of an intra-regular LA-ring \( R \) is an ideal of \( R \).

**Proof.** Suppose that \( I \) is a right ideal of \( R \). Let \( i \in I \) and \( a \in R \). This implies that there exist elements \( x_i, y_i \in R \), such that \( a = \sum_{i=1}^{n}(x_i a^2)y_i \). Now \( ai = ((x_i a^2)y_i) = (y_i)((x_i a^2)I \subseteq I. \) Hence \( I \) is an ideal of \( R \).

**Lemma 4.2.** Every ideal of an intra-regular LA-ring \( R \) with left identity \( e \) is an idempotent.

**Proof.** Let \( I \) be an ideal of \( R \). Since \( I^2 \subseteq I \) and \( a \in I \), this means that there exist elements \( x_i, y_i \in R \) such that \( a = \sum_{i=1}^{n}(x_i a^2)y_i \). Now

\[
a = (x_i a^2)y_i = (x_i(aa))y_i = (a(x_i a))y_i
\]
\[
= (a(x_i a))(ey_i) = (ae)((x_i a)y_i) = (x_i a)((ae)y_i) \in II.
\]

Thus \( I \subseteq I^2 \), i.e., \( I^2 = I \).

**Proposition 4.1.** Let \( R \) be an intra-regular LA-ring with left identity \( e \). Then any non-empty subset \( I \) of \( R \) is an ideal of \( R \) if and only if \( I \) is an interior ideal of \( R \).
Proof. Assume that $I$ is an interior ideal of $R$. Let $i \in I$ and $a \in R$. Then there exist elements $x_i, y_i \in R$ such that 
\[ a = \sum_{i=1}^{n} (x_i a^2) y_i. \] 
Now 
\[ ia = i((x_i a^2) y_i) = i((x_i(aa)) y_i) = i((a(x_i a)) (ey_i)) = i((ae)((x_i a)(y_i)) = (x_i a)(i((ae)y_i)) = (x_i i)((ae)y_i) \in (RI)R \subseteq I. \]

Thus $I$ is a right ideal of $R$. Therefore $I$ is an ideal of $R$ by the Lemma 4.1. Converse is obvious.

Lemma 4.3. Let $R$ be an intra-regular LA-ring. Then $L \cap D \subseteq LD$ for every left ideal $L$ and every right ideal $D$ of $R$.

Proof. Let $a \in L \cap D$, where $L$ is a left ideal and $D$ is a right ideal of $R$. This implies that there exist elements $x_i, y_i \in R$ such that 
\[ a = \sum_{i=1}^{n} (x_i a^2) y_i. \] 
Now 
\[ a = (x_i a^2) y_i = (x_i(aa)) y_i = (a(x_i a)) y_i = (a(x_i a)) (ey_i) = (ae)((x_i a)(y_i)) = (x_i a)((ae)y_i) \in LD. \]

Thus $L \cap D \subseteq LD$.

Theorem 6. Let $R$ be an LA-ring with left identity $e$ such that $(xe) R = xR$ for all $x \in R$. Then the following conditions are equivalent.
1. $R$ is an intra-regular.
2. $L \cap D \subseteq LD$ for every left ideal $L$ and for every right ideal $D$ of $R$.

Proof. (1) $\Rightarrow$ (2) is true by the Lemma 4.3. Suppose that (2) holds and $a \in R$, then $Ra$ is a left ideal of $R$ containing $a$ by the Lemma 2.2 and $aR \cup Ra$ is a right ideal of $R$ containing $a$ by the Proposition 2.1. By our supposition 
\[ Ra \cap (aR \cup Ra) \subseteq (Ra)(aR \cup Ra) = (Ra)(aR) \cup (Ra)(Ra). \] 
Now 
\[ (Ra)(aR) = (Ra)((ea)R) = (Ra)((Ra)e) = (Ra)((Re)(ae)) = (Ra)(R(Re)) = (Ra)(Ra). \]

Thus 
\[ (aR \cup Ra) \cap Ra \subseteq (Ra)(aR) \cup (Ra)(Ra) = (Ra)(Ra) \cup (Ra)(Ra). \]
\[ = (Ra)(Ra) = R a^2 = (RR)(a^2 e) = (ea^2)(RRR)(Re) = (Ra^2)R. \]

Since $a \in (aR \cup Ra) \cap Ra$, implies $a \in (Ra^2)R$, thus $a$ is an intra regular. Hence $R$ is an intra-regular, i.e., (2) $\Rightarrow$ (1).
Theorem 7. Let \( R \) be an LA-ring with left identity \( e \) such that \((xe)R = xR\) for all \( x \in R \). Then the following conditions are equivalent.

1. \( R \) is an intra-regular.
2. \( Q \cap I = (QI)Q \) for every quasi-ideal \( Q \) and for every ideal \( I \) of \( R \).
3. \( B \cap I = (BI)B \) for every bi-ideal \( B \) and for every ideal \( I \) of \( R \).
4. \( G \cap I = (GI)G \) for every generalized bi-ideal \( G \) and for every ideal \( I \) of \( R \).

Proof. Suppose that (1) holds. Let \( a \in G \cap I \), where \( G \) is a generalized bi-ideal and \( I \) is an ideal of \( R \), this implies that there exist elements \( x_i, y_i \in R \) such that

\[
a = \sum_{i=1}^{n} (x_i a^2) y_i.
\]

Thus \( a = (x_i a^2) y_i = (y_i (x_i a)) y_i = (y_i (x_i a)) a \),

\[
y_i (x_i a) = y_i (x_i ((x_i a^2) y_i)) = y_i ((x_i a^2) (x_i y_i)) = (x_i a^2) (y_i (x_i y_i))
\]

\[
= ((x_i a^2) (x_i y_i^2)) = (x_i (aa)) m_i, \text{ say } x_i y_i^2 = m_i
\]

\[
m_i (x_i a) = m_i (x_i ((x_i a^2) y_i)) = m_i ((x_i a^2) (x_i y_i)) = (x_i a^2) (m_i (x_i y_i))
\]

\[
= (x_i (aa)) n_i, \text{ say } m_i (x_i y_i) = n_i
\]

\[
= (a(x_i) a) n_i = (n_i (x_i a)) a
\]

\[
= v_i a, \text{ say } n_i (x_i a) = v_i
\]

\[
\Rightarrow y_i (x_i a) = (m_i (x_i a)) a = (v_i a) a = (v_i e)(aa) = a((v_i e)a).
\]

Thus \( a = (x_i a^2) y_i = (y_i (x_i a)) a = (a((v_i e) a)) a \in (GI)G \), i.e., \( G \cap I \subseteq (GI)G \). Now \((GI)G \subseteq (RI)R \subseteq I \) and \((GI)G \subseteq (GR)G \subseteq G \), thus \((GI)G \subseteq G \cap I \). Hence \( G \cap I = (GI)G \), i.e., (1) \( \Rightarrow \) (4). (4) \( \Rightarrow \) (3), every bi-ideal of \( R \) is a generalized bi-ideal of \( R \) by the Lemma 2.7. (3) \( \Rightarrow \) (2), every quasi-ideal of \( R \) is a bi-ideal of \( R \) by the Lemma 2.9. Assume that (2) is true and let \( Q \) be a right ideal and \( I \) be a two-sided ideal of \( R \). Now \( I \cap Q = (QI)Q \subseteq (RI)Q \subseteq I Q \), since every right ideal of \( R \) is a quasi-ideal of \( R \). Therefore \( R \) is an intra-regular by the Theorem 6, i.e., (2) \( \Rightarrow \) (1). \( \square \)

Theorem 8. Let \( R \) be an LA-ring with left identity \( e \) such that \((xe)R = xR\) for all \( x \in R \). Then the following conditions are equivalent.

1. \( R \) is an intra-regular.
2. \( L \cap Q \subseteq L Q \) for every quasi-ideal \( Q \) and for every left ideal \( L \) of \( R \).
3. \( L \cap B \subseteq L B \) for every bi-ideal \( B \) and for every left ideal \( L \) of \( R \).
4. \( L \cap G \subseteq L G \) for every generalized bi-ideal \( G \) and for every left ideal \( L \) of \( R \).

Proof. Suppose that (1) holds. Let \( a \in L \cap G \), where \( L \) is a left ideal and \( G \) is a generalized bi-ideal of \( R \), this means that there exist elements \( x_i, y_i \in R \) such that

\[
a = \sum_{i=1}^{n} (x_i a^2) y_i.
\]

Now \( a = (x_i a^2) y_i = (x_i (aa)) y_i = (a(x_i a)) y_i = \).
(y_i(x,a))a \in LG, i.e., a \in LG. Thus \( L \cap G \subseteq LG \), i.e., (1) \( \Rightarrow (4) \). (4) \( \Rightarrow (3) \), every bi-ideal of \( R \) is a generalized bi-ideal of \( R \). (3) \( \Rightarrow (2) \), every quasi-ideal of \( R \) is a bi-ideal of \( R \). Assume that (2) is true and let \( Q \) be a right ideal and \( L \) be a left ideal of \( R \). Now \( L \cap Q \subseteq LQ \), where \( Q \) is a quasi-ideal of \( R \). Hence \( R \) is an intra-regular by the Theorem 6, i.e., (2) \( \Rightarrow (1) \).

**Theorem 9.** Let \( R \) be an LA-ring with left identity \( e \) such that \((xe)R = xR \) for all \( x \in R \). Then the following conditions are equivalent.

1. \( R \) is an intra-regular.
2. \( L \cap Q \cap D \subseteq (LQ)D \) for every quasi-ideal \( Q \), every right ideal \( D \) and for every left ideal \( L \) of \( R \).
3. \( L \cap B \cap D \subseteq (LB)D \) for every bi-ideal \( B \), every right ideal \( D \) and for every left ideal \( L \) of \( R \).
4. \( L \cap G \cap D \subseteq (LG)D \) for every generalized bi-ideal \( G \), every right ideal \( D \) and for every left ideal \( L \) of \( R \).

**Proof.** Suppose that (1) holds. Let \( a \in G \cap L \cap D \), where \( G \) is a generalized bi-ideal, \( L \) is a left ideal and \( D \) is a right ideal of \( R \), this implies that there exist elements \( x_i, y_i \in R \) such that \( a = \sum_{i=1}^{n}(x_i a^2)y_i \). Now

\[
\begin{align*}
    a &= (x_i a^2)y_i = (x_i(aa))y_i = (a(x_a))y_i = (y_i(x_i a))a. \\
    y_i(x_i a) &= y_i(x_i((x_i a^2)y_i)) = y_i((x_i a^2)(x_i y_i)) = (x_i a^2)(y_i(x_i y_i)) \\
    &= (x_i a^2)(x_i y_i^2) = (x_i(aa))m_i, \text{ say } x_i y_i^2 = m_i \\
    &= (a(x_i a))m_i = (m_i(x_i a))a.
\end{align*}
\]

Thus \( a = (x_i a^2)y_i = (y_i(x_i a))a = ((m(x,a))a)a \in (LG)R \), i.e., \( a \in (LG)D \). Hence \( G \cap L \cap D \subseteq (LG)D \), i.e., (1) \( \Rightarrow (4) \). (4) \( \Rightarrow (3) \), every bi-ideal of \( R \) is a generalized bi-ideal of \( R \). (3) \( \Rightarrow (2) \), every quasi-ideal of \( R \) is a bi-ideal of \( R \). Assume that (2) is true. Now

\[
L \cap R \cap D \subseteq (LR)D = ((eL)R)D = ((RL)e)D \subseteq (Le)D = (e(Le))D \subseteq (R(Le))D \subseteq (RL)D \subseteq LD.
\]

\[
\Rightarrow L \cap D \subseteq LD.
\]

Hence \( R \) is an intra-regular by the Theorem 6, i.e., (2) \( \Rightarrow (1) \).

5. Regular and intra-regular LA-rings

We characterize both regular and intra-regular LA-rings by the properties of (left, right, quasi-, bi-, generalized bi-) ideals.

**Theorem 10.** Let \( R \) be an LA-ring with left identity \( e \) such that \((xe)R = xR \) for all \( x \in R \). Then the following conditions are equivalent.

1. \( R \) is a regular and an intra-regular.
2. \( B^2 = B \) for every bi-ideal \( B \) of \( R \).
3. \( B_1 \cap B_2 = B_1 B_2 \cap B_2 B_1 \) for all bi-ideals \( B_1, B_2 \) of \( R \).
Proof. Suppose that (1) holds. Let $B$ be a bi-ideal of $R$ and $B^2 \subseteq B$. Let $a \in B$, this implies that there exists an element $x \in R$ such that $a = (ax)a$, also there exist elements $y, z \in R$ such that $a = (y^2)z$. Now

$$
\begin{align*}
 a &= (ax)a = (ax)((y^2)z) = (((y^2)z)x)a. \\
((y^2)z)x &= (xz)(y^2) = m_i(y^2), \text{ say } m_i = xz, \\
&= m_i(y(aa)) = m_i(a(ya)) = a(m_i(ya)) \\
&= ((ax)a)(m_i(ya)) = (ax)m_i(a(ya)) \\
&= (m_i(x)a)(a(ya)) = (n_i)(a(ya)), \text{ say } n_i = m_ix \\
&= (en_i)(a)(ya) = (en_i)(a)(ya) \\
&= (an_i)(a)(ya) = (an_i)(a)(ya) = (s_i)(a)(ya), \text{ say } s_i = an_i \\
&= (aa)(yis_i) = (aa)t_i, \text{ say } t_i = yis_i \\
&= (((ax)a)t_i = ((aa)(ax))t_i = (t_i(ax))(aa) \\
&= (a(t_i,x))(aa) = (aw_i)(aa), \text{ say } w_i = t_ix.
\end{align*}
$$

Thus $a = (((y^2)z)x)a = ((aw_i)(aa))a \in ((BR)B)B \subseteq B^2$, i.e., $B \subseteq B^2$. Hence $B^2 = B$, i.e., (1) $\Rightarrow$ (3). Assume that (2) is true. Let $B_1, B_2$ be bi-ideals of $R$, then $B_1 \cap B_2$ be also a bi-ideal of $R$. Now $B_1 \cap B_2 = (B_1 \cap B_2)(B_1 \cap B_2) \subseteq B_1B_2$ and $B_1 \cap B_2 = (B_1 \cap B_2)(B_1 \cap B_2) \subseteq B_2B_1$, thus $B_1 \cap B_2 \subseteq B_1B_2 \cap B_2B_1$. Now we show that $B_1B_2$ is a bi-ideal of $R$. It is enough to show that $((B_1B_2)R)(B_1B_2) \subseteq B_1B_2$. Now

$$
\begin{align*}
((B_1B_2)R)(B_1B_2) &= ((B_1B_2)(R)(B_1B_2)) \\
&= ((B_1)(B_2)R)(B_1B_2) \\
&= ((B_1)(B_2)R)(B_1B_2) \subseteq B_1B_2. \\
\Rightarrow ((B_1B_2)R)(B_1B_2) \subseteq B_1B_2.
\end{align*}
$$

Thus $B_1B_2$ is a bi-ideal of $R$, similarly $B_2B_1$ is also a bi-ideal of $R$. Then $B_1B_2 \cap B_2B_1$ is also a bi-ideal of $R$. Now

$$
\begin{align*}
B_1B_2 \cap B_2B_1 &= (B_1B_2 \cap B_2B_1)(B_1B_2 \cap B_2B_1) \\
&\subseteq (B_1B_2)(B_2B_1) \subseteq (B_1)(B_2) \subseteq B_1B_2 \cap B_2B_1 \\
&= ((RB_1)R)B_1 = (((RB_1)R)B_1)B_1 \\
&= (((RB_1)R)B_1)B_1 = ((B_1)(B_2)R)B_1 \\
&= ((RB_1)B_1)B_1 = (RB_1)B_1 \\
&= ((RB_1)B_1)B_1 = (RB_1)B_1 \\
&= (B_1)(B_2)R \subseteq B_1, \\
\Rightarrow B_1B_2 \cap B_2B_1 \subseteq B_1.
\end{align*}
$$

Similarly, we have $B_1B_2 \cap B_2B_1 \subseteq B_2$, thus $B_1B_2 \cap B_2B_1 \subseteq B_1 \cap B_2$. Therefore $B_1 \cap B_2 = B_1B_2 \cap B_2B_1$, i.e., (2) $\Rightarrow$ (3). Suppose that (3) holds,
Let $D$ be right ideal and $I$ be an ideal of $R$. Then $D$ and $I$ be bi-ideals of $R$, because every right ideal and two-sided ideal of $R$ is bi-ideal of $R$ by the Lemma 2.6. Now $D \cap I = DI \cap ID$, this implies that $D \cap I \subseteq DI \cap ID$. Thus $D \cap I \subseteq DI$ and $D \cap I \subseteq ID$, where $I$ is also a left ideal of $R$. Since $DI \subseteq D \cap I$, i.e., $DI = D \cap I$. Thus $R$ is regular by the Theorem 1. Also, $D \cap I \subseteq ID$, Thus $R$ is an intra-regular by the Theorem 6. Hence $(3) \Rightarrow (1)$. □

**Theorem 11.** Let $R$ be an LA-ring with left identity $e$ such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.

1. $R$ is a regular and an intra-regular.
2. Every quasi-ideal of $R$ is an idempotent.

**Proof.** Suppose that (1) holds. Let $Q$ be a quasi-ideal of $R$ and $Q^2 \subseteq Q$. Let $a \in Q$, this implies that there exists an element $x \in R$ such that $a = (ax)a$, also there exist elements $y_i, z_i \in R$ such that $a = (y_ia^2)z_i$. Now $a = (ax)a = (ax)((y_ia^2)z_i) = (((y_ia^2)z_i)x)a = ((aw_1)(aa))a$, where $((y_ia^2)z_i)x = (aw_1)(aa)$, by the Theorem 10. Thus $a = ((aw_1)(aa))a \in ((QR)Q)Q \subseteq QQ \subseteq Q^2$, i.e., $Q \subseteq Q^2$, because every quasi-ideal of $R$ is a bi-ideal of $R$ by the Lemma 2.9. Thus $Q^2 = Q$, i.e., $(1) \Rightarrow (2)$. Assume that (2) is true. Let $a \in R$, then $Ra$ is a left ideal of $R$ containing $a$, i.e., $Ra$ is a quasi-ideal of $R$, because every left ideal of $R$ is a quasi-ideal of $R$. Now $Ra = (Ra)^2 = (Ra)(Ra)$, i.e., $a \in (Ra)(Ra)$, thus $R$ is an intra-regular by the Theorem 6. Now $Ra = (Ra)(Ra) = ((Re)a)(Ra) = ((ae)R)(Ra) = (aR)(Ra)$, i.e., $a \in (aR)(Ra)$, thus $R$ is regular by the Theorem 1. Hence $(2) \Rightarrow (1)$. □

**Theorem 12.** Let $R$ be an LA-ring with left identity $e$ such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.

1. $R$ is a regular and an intra-regular.
2. Every quasi-ideal of $R$ is an idempotent.
3. Every bi-ideal of $R$ is an idempotent.

**Proof.** $(1) \Rightarrow (3)$, by the Theorem 10. $(3) \Rightarrow (2)$, every quasi-ideal of $R$ is a bi-ideal of $R$, by the Lemma 2.9. $(2) \Rightarrow (1)$, by the Theorem 11. □

**Theorem 13.** Let $R$ be an LA-ring with left identity $e$ such that $(xe)R = xR$ for all $x \in R$. Then the following conditions are equivalent.

1. $R$ is a regular and an intra-regular.
2. $Q_1 \cap Q_2 \subseteq Q_1Q_2$ for all quasi-ideals $Q_1, Q_2$ of $R$.
3. $Q \cap B \subseteq QB$ for every quasi-ideal $Q$ and for every bi-ideal $B$ of $R$.
4. $B \cap Q \subseteq BQ$ for every bi-ideal $B$ and for every quasi-ideal $Q$ of $R$.
5. $B_1 \cap B_2 \subseteq B_1B_2$ for all bi-ideals $B_1, B_2$ of $R$.

**Proof.** Suppose that (1) holds. Let $B_1, B_2$ be bi-ideals of $R$, then $B_1 \cap B_2$ be also a bi-ideal of $R$. Since every bi-ideal of $R$ is an idempotent by the Theorem 10, then $B_1 \cap B_2 = (B_1 \cap B_2)^2 = (B_1 \cap B_2)(B_1 \cap B_2) \subseteq B_1B_2$, i.e., $(1) \Rightarrow (5)$. Since $(5) \Rightarrow (4) \Rightarrow (2)$ and $(5) \Rightarrow (3) \Rightarrow (2)$, because every quasi-ideal of $R$ is
a bi-ideal of $R$ by the Lemma 2.9. Assume that (2) is true. Now $D \cap L \subseteq DL$, where $D$ is a right ideal and $L$ is a left ideal of $R$. Since $DL \subseteq D \cap L$, i.e., $D \cap L = DL$, thus $R$ is regular. Again by (2) $L \cap D \subseteq LD$, thus $R$ is an intra-regular. Hence (2) $\Rightarrow$ (1).

References


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