Computation of edge metric dimension of barycentric subdivision of Cayley graphs

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Abstract. Let $G = (V, E)$ be a connected graph, let $x \in V(G)$ be a vertex and $e = yz \in E(G)$ be an edge. The distance between the vertex x and the edge e is given by $d_G(x, e) = \min\{d_G(x, y), d_G(x, z)\}$. A vertex $t \in V(G)$ distinguishes two edges $e, f \in E(G)$ if $d_G(t, e) \neq d_G(t, f)$. A set $R \subseteq V(G)$ is an edge metric generator for G if every two edges of G are distinguished by some vertex of R. The minimum cardinality of R is called the edge metric dimension and is denoted by $edim(G)$. In this paper, we compute the edge metric dimension of barycentric subdivision of Cayley graphs $Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2)$.

Keywords: metric dimension, edge metric dimension, resolving set, barycentric subdivision, Cayley graph

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1. Introduction

The concept of metric dimension was introduced by Slater [1] and studied independently by Harary and Melter [2]. This problem has been investigated widely since then. The metric dimension has a lot of applications in different areas of science and technology. The concept of the edge metric dimension is a recent advancement in this line of research. Next we reveal some of the applications of metric dimension in various subjects. The metric dimension arises in many diverse areas, including navigation of robots [3], telecommunications [4], combinatorial optimization [5] and sonar and coast guard Loran [1] and applications to chemistry in [6, 7, 8]. Furthermore, this topic has some applications to problems of pattern recognition and image processing, some of which involve the use of hierarchical data structures [9]. Metric dimension of several interesting classes of graphs can be seen in [10, 11, 12, 13, 14, 15, 16, 17]. Let $G = (V, E)$ be a simple and connected graph. A vertex $x \in V(G)$ distinguishes two vertices $y, z \in V(G)$ if $d_G(y, x) \neq d_G(z, x)$, where $d_G(x, y)$ denotes the length of the shortest path between the vertices x and y in G. The minimum cardinality of any metric generator for G is the metric dimension of G, denoted by $dim(G)$. Let $R_1 = \{r_1, r_2, \ldots, r_s\}$ be an ordered set of vertices of G and let $x \in V(G)$, then the *representation* $r(x|R_1)$ of x with respect to R_1 is the s-tuple $(d_G(x, r_1), d_G(x, r_2), d_G(x, r_3), \ldots, d_G(x, r_s))$. Since the set R_1 has the minimum cardinality, therefore this is also known as the basis of G, and its cardinality is called the metric dimension or location number [18].

Similarly, let $x \in V(G)$ be a vertex and $e = yz \in E(G)$ be an edge. The distance between the vertex x and the edge e is given by $d_G(x, e) = \min\{d_G(x, y), d_G(x, z)\}$. A vertex $t \in V(G)$ distinguishes two edges $e, f \in E(G)$ if $d_G(t, e) \neq d_G(t, f)$. A set $R \subseteq V(G)$ is an edge metric generator for G if every two edges of G are distinguished by some vertex of R. The minimum cardinality of R is called the edge metric dimension and is denoted by $edim(G)$ [19]. Let $R = \{r_1, r_2, \ldots, r_t\}$ be an ordered set of vertices of G and let $e \in E(G)$, then the *representation* $r(e|R)$ of e with respect to R is the t-tuple $(d_G(e, r_1), d_G(e, r_2), d_G(e, r_3), \ldots, d_G(e, r_t))$. In addition, combined (mixed) form of these two parameters depicted above is of fascinate. A vertex $x \in V(G)$ distinguishes two elements (vertices or edges) $u, v \in V(G) \cup E(G)$ if $d_G(x, u) \neq d_G(x, v)$. A set $R^m \subseteq V(G)$ is a mixed metric generator for G if every two distinct elements (vertices or edges) of G are distinguished by some vertex of R^m. The smallest cardinality of R^m is the mixed metric dimension and is denoted by $mdim(G)$ [20].

Geometrically, an operation that splits an edge into two edges by inserting a new vertex into the interior of an edge is known as subdividing an edge. If we are performing a sequence of edge-subdivision operations, then it is called *Subdividing a graph G* and resulting graph is called a *subdivision of the graph G*. The subdivision of graph can be used to convert a general graph into a simple graph. If we subdividing each edge of the graph G, then this subdivision is called the *barycentric subdivision* of G. Gross and Yellen [21] proved the results that the barycentric subdivision of any graph is a simple and bipartite graph. A graph G is *planar* if it can be drawn in the plane without edge crossings. Subdivision of graphs play a very important role in characterization of planar graphs. A graph G is planar if and only if every subdivision of G is planar. Two graphs are said to be homeomorphic if they are subdivisions of the same graph G. The next theorem gives a nice characterization of planar graphs.

Theorem 1.1. [21] *A graph is planar if and only if it does not contain a subdivision of* K_5 *or* $K_{3,3}$.

In this paper, we study the edge metric dimension of barycentric subdivision of Cayley graphs $Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2)$. We prove that these subdivisions of Cayley graphs have constant edge metric dimension and only three vertices chosen appropriately suffice to resolve all the vertices of these subdivision of Cayley graphs $Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2)$.

2. Results and discussions

As expressed, there are a several graphs in which metric generator and edge metric generator are the same. In this sense, one could believe that most likely any edge metric generator R is likewise a standard metric generator. In any case, this is again further far from the truth, despite the fact that there are a few families of graphs in which such actualities happen. Kelenc *et al.* [19] explained some comparison between the edge metric generator and standard metric generator in details. We show a few results concerning the edge metric dimension of graphs. The first importance result about the complexity is as follows:

Theorem 2.1 ([19]). *Computing the edge metric dimension of graphs is NP-hard.*

The edge metric dimension of Cartesian product of two paths P_r and P_t with r and t vertices is determined in the following proposition.

Proposition 2.2 ([19]). *Let G be the grid graph $G = P_r \square P_t$, with $r \geq t \geq 2$. Then $edim(G) = dim(G) = 2$.*

Kelenc *et al.* [19] proved in the next proposition the edge metric dimension of wheel graphs and observe it is strictly larger than the value for the metric dimension, except in the case $W_{1,3}$. The wheel graph $W_{1,n}$ is the graph obtained from a cycle $C_n, n \geq 3$ by joining all vertices of C_n to an additional vertex. In [19], they determined the edge metric dimension of wheel graph $W_{1,n}$ in the following proposition:

Proposition 2.3 ([19]). *Let $W_{1,n}$ be a wheel graph. Then*

$$edim(W_{1,n}) = \begin{cases} n, & for n = 3, 4 \\ n-1, & for n \geq 5 \end{cases}$$

The fan graph $F_{1,n}$ is the graph obtained by joining each vertices of a path P_n, to an additional vertex. In the next proposition, the edge metric dimension of fan graph $F_{1,n}$ is determined.

Proposition 2.4 ([19]). *Let $F_{1,n}$ be a fan graph. Then*

$$edim(F_{1,n}) = \begin{cases} n, & for\, n = 1,2,3 \\ n-1, & for\, n \geq 4 \end{cases}$$

Kelenc *et al.* [19] also determined the edge metric dimension of path, cycle, complete graph, complete bipartite, cartesian product of cycles and bounds for some families of graphs.

3. The edge metric dimension of barycentric subdivision of Cayley graphs $Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2)$

Let G be a semigroup, and let H be a nonempty subset of G. The Cayley graph $Cay(G,H)$ of G relative to H is defined as the graph with vertex set G and edge set $E(H)$ consisting of those ordered pairs (a,b) such that $ha = b$ for some $h \in H$. Cayley graphs of groups are significant both in group theory and in constructions of interesting graphs with nice properties. The Cayley graph $Cay(G,H)$ of a group G is symmetric or undirected if and only if $H = H^{-1}$.

The Cayley graphs $Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2), n \geq 3$, is a 3-regular graph which is also known as the cartesian product $C_n \square P_2$ of a cycle of order n with a path of order 2. The Cayley graphs $Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2), n \geq 3$ consists of an inner n-cycle $a_1a_2a_3\ldots a_n$, an outer n-cycle $x_1x_2x_3\ldots x_n$ and n spokes $a_ix_i, 1 \leq i \leq n$. This implies that the order and size of $Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2)$ is $2n$ and $3n$, respectively. The metric dimension of Cayley graphs $Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2)$ has been determined in [14] while the metric dimension of Cayley graphs $Cay(\mathbb{Z}_n : H)$ for all $n \geq 7$ and $H = \{\pm 1, \pm 3\}$ has been determined in [22].

The barycentric subdivision graph $BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2))$ can be obtained by splitting edges a_ia_{i+1} by inserting a new vertices b_i, splitting edges a_ix_i by inserting a new vertices c_i and splitting edges x_ix_{i+1} by inserting a new vertices y_i. From this we observe that , $BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2))$ contains $5n$ vertices among of these $3n$ vertices of degree 2 and $2n$ vertices of degree 3 and $6n$ edges. In the next theorem, we prove that the edge metric dimension of the barycentric subdivision of $BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2))$ is constant and only three vertices appropriately chosen suffice to resolve all the vertices of the $BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2))$.

Theorem 3.1. *Let $BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2))$ be the barycentric subdivision of Cayley graphs $(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2))$. Then $edim(BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2))) = 3$ for every $n \geq 6$.*

Proof. We will prove the above equality by double inequalities.

Case 1. When n is even.

Let $R = \{a_1, a_{\frac{n}{2}+1,a_n}\} \subset V(BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2)))$. We show that R is a resolving set for $BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2))$ in this case. For this we give representations of any edge of $E(BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2)))$ with respect to R.

Representations for the edges of $BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2))$ are

$$r(a_ib_i|R) = \begin{cases} (2i-2, n-2i+1, 2i), & \text{for } 1 \le i \le \frac{n}{2}-1 \\ (n-2, 1, n-1), & \text{for } i = \frac{n}{2} \\ (2n-2i+1, 2i-n-2, 2n-2i-1), & \text{for } \frac{n}{2}+1 \le i \le n-1 \\ (1, n-2, 0), & \text{for } i = n \end{cases}$$

and

$$r(b_ia_{i+1}|R) = \begin{cases} (2i-1, n-2i, 2i+1), & \text{for } 1 \le i \le \frac{n}{2}-1 \\ (n-1, 0, n-2), & \text{for } i = \frac{n}{2} \\ (2n-2i, 2i-n-1, 2n-2i-2), & \text{for } \frac{n}{2}+1 \le i \le n-1 \\ (0, n-1, 1), & \text{for } i = n \end{cases}$$

Representations for the set of interior edges of $BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2))$ are

$$r(a_ic_i|R) = \begin{cases} (2i-2, n-2i+2, 2i), & \text{for } 1 \le i \le \frac{n}{2} \\ (2n-2i+2, 2i-n-2, 2n-2i), & \text{for } \frac{n}{2}+1 \le i \le n \end{cases}$$

and

$$r(c_ix_i|R) = \begin{cases} (2i-1, n-2i+3, 2i+1), & \text{for } 1 \le i \le \frac{n}{2} \\ (2n-2i+3, 2i-n-1, 2n-2i+1), & \text{for } \frac{n}{2}+1 \le i \le n \end{cases}$$

Representations for the edges on the outer cycle of $BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2))$ are

$$r(x_iy_i|R) = \begin{cases} (2i, n-2i+3, 2i+2), & \text{for } 1 \le i \le \frac{n}{2}-1 \\ (n, 3, n+1), & \text{for } i = \frac{n}{2} \\ (2n-2i+3, 2i-n, 2n-2i+1), & \text{for } \frac{n}{2}+1 \le i \le n-1 \\ (3, n, 2), & \text{for } i = n \end{cases}$$

and

$$r(y_ix_{i+1}|R) = \begin{cases} (2i+1, n-2i+2, 2i+3), & \text{for } 1 \le i \le \frac{n}{2}-1 \\ (n+1, 2, n), & \text{for } i = \frac{n}{2} \\ (2n-2i+2, 2i-n+1, 2n-2i), & \text{for } \frac{n}{2}+1 \le i \le n-1 \\ (2, n+1, 3), & \text{for } i = n \end{cases}$$

We note that there are no two edges having the same representations implying that $edim(BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2))) \le 3$.

On the other hand, we show that $edim(BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2))) \ge 3$. Suppose on contrary that $edim(BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2))) = 2$, then there are the following possibilities to be discussed.

(1) Both vertices are in the inner cycle. Here are the following subcases.

• Both vertices belong to the set $\{a_i : 1 \le i \le n\}$. Without loss of generality, we can suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is a_k $(2 \le k \le \frac{n}{2}+1)$. Then for $2 \le k \le \frac{n}{2}$, we have $r(a_1c_1|\{a_1, a_k\}) = r(a_1b_n|\{a_1, a_k\}) = (0, 2k-2)$, and for $k = \frac{n}{2}+1$, we have $r(a_1b_1|\{a_1, a_{\frac{n}{2}+1}\}) = r(a_1b_n|\{a_1, a_{\frac{n}{2}+1}\}) = (0, n-1)$, a contradiction.

• Both vertices belong to the set $\{b_i : 1 \le i \le n\}$. Without loss of generality, we can suppose that one resolving vertex is b_1. Suppose that the second resolving vertex is b_k $(2 \leq k \leq \frac{n}{2}+1)$. Then for $2 \leq k \leq \frac{n}{2}$, we have $r(a_1c_1|\{b_1, b_k\}) = r(a_1b_n|\{(b_1, b_k\}) = (1, 2k-1)$, and for $k = \frac{n}{2}+1$, we have $r(a_1b_1|\{b_1, b_{\frac{n}{2}+1}\}) = r(a_2b_1|\{b_1, b_{\frac{n}{2}+1}\}) = (0, n-1)$, a contradiction.

• One vertex belong to the set $\{a_i : 1 \leq i \leq n\}$ and the second vertex belong to the set $\{b_i : 1 \leq i \leq n\}$. Without loss of generality, we can suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is b_k $(1 \leq k \leq \frac{n}{2}+1)$. Then for $1 \leq k \leq \frac{n}{2}$, we have $r(a_1b_n|\{a_1, b_k\}) = r(a_1c_1|\{a_1, b_k\}) = (0, 2k-1)$, and for $k = \frac{n}{2}+1$, we have $r(a_1b_1|\{a_1, b_{\frac{n}{2}+1}\}) = r(a_1c_1|\{a_1, b_{\frac{n}{2}+1}\}) = (0, n-1)$, a contradiction.

(2) Both vertices are in the interior vertices. Without loss of generality, we can suppose that one resolving vertex is c_1. Suppose that the second resolving vertex is c_k $(2 \leq k \leq \frac{n}{2}+1)$. Then for $2 \leq k \leq \frac{n}{2}+1$, we have $r(x_1c_1|\{c_1, c_k\}) = r(a_1c_1|\{c_1, c_k\}) = (0, 2k-1)$, a contradiction.

(3) Both vertices are in the outer cycle. Due to the symmetry of the graph, this case is analogous to case **(1)**.

(4) One vertex is in the inner cycle and the other one is in the set of interior vertices. Here are the two subcases.

• One vertex is in the set $\{a_i : 1 \leq i \leq n\}$ and the other one is in the set of interior vertices $\{c_i : 1 \leq i \leq n\}$. Without loss of generality, we can suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is c_k $(1 \leq k \leq \frac{n}{2}+1)$. Then for $k = 1$, we have $r(a_1b_1|\{a_1, c_1\}) = r(a_1b_n|\{a_1, c_1\}) = (0, 1)$. For $2 \leq k \leq \frac{n}{2}$, we have $r(a_1b_n|\{a_1, c_k\}) = r(a_1c_1|\{a_1, c_k\}) = (0, 2k-1)$ and for $k = \frac{n}{2}+1$, we have $r(a_1b_n|\{a_1, c_{\frac{n}{2}+1}\}) = r(a_1b_1|\{a_1, c_{\frac{n}{2}+1}\}) = (0, 2n)$, a contradiction.

• One vertex is in the set $\{b_i : 1 \leq i \leq n\}$ and the other one is in the set of interior vertices $\{c_i : 1 \leq i \leq n\}$. Without loss of generality, we can suppose that one resolving vertex is b_1. Suppose that the second resolving vertex is c_k $(1 \leq k \leq \frac{n}{2}+1)$. Then for $k = 1$, we have $r(x_1y_1|\{b_1, c_1\}) = r(x_1y_n|\{b_1, c_1\}) = (3, 1)$. For $2 \leq k \leq \frac{n}{2}$, we have $r(a_1c_1|\{b_1, c_k\}) = r(a_1b_n|\{b_1, c_k\}) = (1, 2k-1)$ and for $k = \frac{n}{2}+1$, we have $r(c_2x_2|\{b_1, c_{\frac{n}{2}+1}\}) = r(a_nb_n|\{b_1, c_{\frac{n}{2}+1}\}) = (2, n-1)$, a contradiction.

(5) One vertex is in the outer cycle and the other one is in the set of interior vertices. Due to the symmetry of the graph, this case is analogous to case **(4)**.

(6) One vertex is in the inner cycle and the other one is in the outer cycle. Here are the following subcases.

• One vertex is in the set $\{a_i : 1 \leq i \leq n\}$ and the other one is in the set $\{x_i : 1 \leq i \leq n\}$. Without loss of generality, we can suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is x_k $(1 \leq k \leq \frac{n}{2}+1)$. Then for $k = 1$, we have $r(a_1b_1|\{a_1, x_1\}) = r(a_1b_n|\{a_1, x_1\}) = (0, 2)$. For $2 \leq k \leq \frac{n}{2}+1$, we have $r(a_1b_1|\{a_1, x_k\}) = r(a_1c_1|\{a_1, x_k\}) = (0, 2k-1)$, a contradiction.

• One vertex is in the set $\{a_i : 1 \leq i \leq n\}$ and the other one is in the set $\{y_i : 1 \leq i \leq n\}$. Without loss of generality, we can suppose that one resolving vertex is a_1. Suppose that the second resolving vertex is y_k $(1 \leq k \leq \frac{n}{2}+1)$. Then for $k = 1$, we have $r(a_1b_1|\{a_1, y_1\}) = r(a_1b_n|\{a_1, y_1\}) = (0, 3)$. For $2 \le k \le \frac{n}{2}$, we have $r(a_1b_1|\{a_1, y_k\}) = r(a_1c_1|\{a_1, y_k\}) = (0, 2k)$ and for $k = \frac{n}{2} + 1$, we have $r(a_1b_1|\{a_1, y_{\frac{n}{2}+1}\}) = r(a_1c_1|\{a_1, y_{\frac{n}{2}+1}\}) = (0, n + 1)$, a contradiction.

• One vertex is in the set $\{b_i : 1 \le i \le n\}$ and the other one is in the set $\{y_i : 1 \le i \le n\}$. Without loss of generality, we can suppose that one resolving vertex is b_1. Suppose that the second resolving vertex is y_k $(1 \le k \le \frac{n}{2} + 1)$. Then for $k = 1$, we have $r(a_1b_1|\{b_1, y_1\}) = r(a_2b_1|\{b_1, y_1\}) = (0, 3)$. For $k = 2$, we have $r(x_1y_1|\{b_1, y_2\}) = r(a_3c_3|\{b_1, y_2\}) = (3, 2)$. For $3 \le k \le \frac{n}{2} + 1$, we have $r(x_2y_2|\{b_1, y_k\}) = r(a_3c_3|\{b_1, y_k\}) = (3, 2k - 4)$, a contradiction.

Hence from above it follows that there is no resolving set with two vertices for $V(BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2)))$ implying that $edim(BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2))) \ne 2$ in this case. Therefore, $edim(BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2))) = 3$.

Case 2. When n is odd.

Let $R = \{a_1, b_{\lceil \frac{n}{2} \rceil}, a_n\} \subset V(BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2)))$. We show that R is a resolving set for $BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2))$ in this case. For this we give representations of any edge of $E(BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2)))$ with respect to R.

Representations for the edges of the inner cycle of $BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2))$ are

$$r(a_ib_i|R) = \begin{cases} (2i-2, n+1-2i, 2i), & \text{for } 1 \le i \le \lceil \frac{n}{2} \rceil - 1 \\ (2i-2, n+1-2i, 2n-2i-1), & \text{for } i = \lceil \frac{n}{2} \rceil \\ (2n-2i+1, 2i-n-2, 2n-2i-1), & \text{for } \lceil \frac{n}{2} \rceil + 1 \le i \le n-1 \\ (1, n-2, 0), & \text{for } i = n. \end{cases}$$

and

$$r(b_ia_{i+1}|R) = \begin{cases} (2i-1, n-2i, 2i+1), & \text{for } 1 \le i \le \lceil \frac{n}{2} \rceil - 2 \\ (2i-1, n-2i, 2n-2i-2), & \text{for } i = \lceil \frac{n}{2} \rceil - 1 \\ (2n-2i, 2i-n-1, 2n-2i-2), & \text{for } \lceil \frac{n}{2} \rceil \le i \le n-1 \\ (0, n-1, 1), & \text{for } i = n \end{cases}$$

Representations for the edges on the outer cycle of $BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2))$ are

$$r(x_iy_i|R) = \begin{cases} (2i, n+3-2i, 2i+2), & \text{for } 1 \le i \le \lceil \frac{n}{2} \rceil - 1 \\ (2i, n-2i+3, 2n-2i+1), & \text{for } i = \lceil \frac{n}{2} \rceil \\ (2n-2i+3, 2i-n, 2n-2i+1), & \text{for } \lceil \frac{n}{2} \rceil + 1 \le i \le n-1 \\ (3, n, 2), & \text{for } i = n \end{cases}$$

and

$$r(y_ix_{i+1}|R) = \begin{cases} (2i+1, n-2i+2, 2i+3), & \text{for } 1 \le i \le \lceil \frac{n}{2} \rceil - 2 \\ (2i+1, n-2i+2, 2n-2i), & \text{for } i = \lceil \frac{n}{2} \rceil - 1 \\ (2n-2i+2, 3, 2n-2i), & \text{for } i = \lceil \frac{n}{2} \rceil \\ (2n-2i+2, 2i-n+1, 2n-2i), & \text{for } \lceil \frac{n}{2} \rceil + 1 \le i \le n-1 \\ (2, n+1, 3), & \text{for } i = n \end{cases}$$

Representations for the set of interior edges of $BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2))$ are

$$r(a_ic_i|R) = \begin{cases} (2i-2, n-2i+2, 2i), & \text{for } 1 \le i \le \lceil \frac{n}{2} \rceil - 1 \\ (2i-2, n-2i+2, 2n-2i), & \text{for } i = \lceil \frac{n}{2} \rceil \\ (2n-2i+2, 2i-n-2, 2n-2i), & \text{for } \lceil \frac{n}{2} \rceil + 1 \le i \le n \end{cases}$$

$$r(c_ix_i|R) = \begin{cases} (2i-1, n-2i+3, 2i+1), & \text{for } 1 \le i \le \lceil \frac{n}{2} \rceil - 1 \\ (2i-1, n-2i+3, 2n-2i+1), & \text{for } i = \lceil \frac{n}{2} \rceil \\ (2n-2i+3, 2i-n-1, 2n-2i+1), & \text{for } \lceil \frac{n}{2} \rceil + 1 \le i \le n \end{cases}$$

Again we see that there are no two vertices having the same representations which implies that $edim(BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2))) \le 3$.

On the other hand, suppose that $edim(BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2))) = 2$, then there are the same possibilities as in Case (1) and contradictions can be deduced analogously. This implies that $edim(BS(Cay(\mathbb{Z}_n \oplus \mathbb{Z}_2))) = 3$ in this case, which completes the proof.

□

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