On Roman domination stability in some simple graphs

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Abstract. The Roman domination stability of a graph G, denoted by $st_{\gamma R}(G)$, is the minimum number of vertices whose removal changes the Roman domination number of G. In this paper, we continue the study of this concept, and determine the Roman domination stability of some classes of graphs, including paths, cycles, complete bipartite graphs and some Cartesian products of paths and complete graphs.
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1. Introduction

Let $G = (V(G), E(G))$ be a simple graph of order n with vertex set $V(G)$ and edge set $E(G)$. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) = \{u|uv \in E\}$, and its *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. The *degree* $\deg(v)$ of a vertex v is $|N(v)|$. We refer to $\Delta(G)$ and $\delta(G)$ as the *maximum degree* and the *minimum degree* among the vertices of G, respectively. With K_n we denote the *complete graph* on n vertices, with C_n the *cycle* on n vertices, and with P_n the *path* on n vertices. The *cartesian product* of two graphs G_1 and G_2 is the graph $G_1 \Box G_2$ with vertex set $V(G_1) \times V(G_2)$, where two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if either $u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or $u_2 = v_2$ and $u_1v_1 \in E(G_1)$. An *r-partite* graph G is a graph whose vertex set $V(G)$ can be partitioned into r sets of pairwise non-adjacent vertices. For positive integers $p_1, p_2, ..., p_r$, the complete r- partite graph $K_{p_1,p_2,...,p_r}$ is the r-partite graph with partition $V(G) = V_1 \cup V_2 \cup ... \cup V_r$ such $|V_i| = p_i$ for $1 \leq i \leq r$ and such that every two vertices belonging to different partite sets are adjacent to each other. A set $S \subseteq V(G)$ is a *domination set* of G, if every vertex not in S is adjacent to a vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a domination set in G. A domination set of cardinality $\gamma(G)$ is called a $\gamma(G)$-set.

The concept of domination stability in graphs has been introduced by Bauer at al [2] . The *domination stability* of a graph G is the minimum number of ver-

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tices whose removal changes the domination number. We denote the domination stabilitay of a graph G by $st_{\gamma}(G)$.

A function $f : V \longrightarrow \{0,1,2\}$ having the property that for every vertex $v \in V$ with $f(v) = 0$, there exists a vertex $u \in N(v)$ with $f(u) = 2$, is called a *Roman dominating function* or just an RDF. The *weight* of an RDF f is the sum $f(V) = \sum_{v\in V} f(v)$. The minimum weight of an RDF on G is called the *Roman domination number* of G and is denoted $\gamma_R(G)$. An RDF on G with weight $\gamma_R(G)$ is called a γ_R-function of G. For an RDF f in a graph G, we denote by V_i (or V_i^f to refer to f) the set of all vertices of G with label i under f. Thus an RDF f can be represented by a triple (V_0, V_1, V_2), and we can use the notation $f = (V_0, V_1, V_2)$. The concept of Roman domination is studied in, for example, [3, 5, 6] .

Amraee et al. [1] studied the concept of Roman domination stability in graphs. The *Roman domination stability* of a graph G, denoted by $st_{\gamma R}(G)$, is the minimum number of vertices whose removal changes the Roman domination number of G. In this paper, we continue the study of this concept, and determine the Roman domination stability of some classes of graphs, including paths, cycles, complete bipartite graphs and cartesian products $P_2\square P_n$ and $K_p\square K_q$.

The following are useful.

Theorem 1 ([3]). *For paths and cycles,*

$$\gamma_R(P_n) = \gamma_R(C_n) = \lceil \frac{2n}{3} \rceil.$$

Theorem 2 ([4]). *For $n \geq 2$, $\gamma_R(P_2\square P_n) = n + 1$.*

2. Results

We begin with determining the Roman domination stability of paths.

Theorem 3. *For a path P_n,* $St_{\gamma R}(P_n) = \begin{cases} 2, & n \equiv 0 \pmod 3 \\ 1, & n \equiv 1 \text{ or } 2 \pmod 3 \end{cases}$.

Proof. Assume first that $n \equiv 0 \pmod 3$. Let $n = 3k$, for some integer $k \geq 1$. By Theorem 1, $\gamma_R(P_n) = \lceil \frac{2(3k)}{3} \rceil = 2k$. Clearly removal of a leaf and a support vertex of P_n produces the path P_{n-2}, and by Theorem 1, $\gamma_R(P_{n-2}) = \lceil \frac{2(3k-2)}{3} \rceil = \lceil \frac{6k-4}{3} \rceil = 2k - 1 < \gamma_R(P_n)$. Thus $st_{\gamma_R}(P_n) \leq 2$. Let v be a vertex of P_n. If v is a leaf of P_n, then $P_n - v = P_{n-1}$, and so by Theorem 1, we obtain that $\gamma_R(P_n - v) = \gamma_R(P_n)$. Thus assume that v is not a leaf of P_n. Then $P_n - v$ has two components P_{n_1} and P_{n_2}, where $n_1 + n_2 = n - 1$. Then $\gamma_R(P_n - v) = \gamma_R(P_{n_1}) + \gamma_R(P_{n_2})$, and it is straightforward to see that $\gamma_R(P_n - v) = \gamma_R(P_n)$. We deduce that $st_{\gamma_R}(P_n) \geq 2$. Consequently $st_{\gamma_R}(P_n) = 2$.

Next assume that $n \equiv 1 \pmod 3$. Let $n = 3k + 1$, where k is a positive integer. By Theorem 1, $\gamma_R(P_n) = 2k+1$. Since removal of a leaf of P_n produces the path P_{n-1}, and $\gamma_R(P_{n-1}) = 2k$ by Theorem 1, we conclude that $st_{\gamma R}(P_n) = 1$.

The proof for the case $n \equiv 2 \pmod 3$ is similar. □

We next determine the Roman domination stability of a cycle C_n.

Theorem 4. *For a cycle C_n, $st_{\gamma R}(C_n) = \begin{cases} 2 & n \equiv 0 \pmod 3 \\ 1 & n \equiv 1 \text{ or } 2 \pmod 3 \end{cases}$.*

Proof. Assume that $n \equiv 0 \pmod 3$. Let $n = 3k$ for some integer k. Then by Theorem 1, $\gamma_R(C_n) = 2k$. Clearly removal of a vertex leaves a path P_{n-1}, and $\gamma_R(P_{n-1}) = 2k$ by Theorem 1. Thus $st_{\gamma_R}(C_n) \geq 2$. On the other hand removal of two adjacent vertices of C_n leaves a path P_{n-2} with Roman domination number $2k-1$ by Theorem 1. Thus $st_{\gamma_R}(C_n) \leq 2$. Consequently, $st_{\gamma_R}(C_n) = 2$.

Next assume that $n \equiv 1$ or $2 \pmod 3$. Then Removal of a vertex of C_n produces a path P_{n-1} with Roman domination number less than $\gamma_R(C_n)$, by Theorem 1. Consequently, $st_{\gamma_R}(C_n) = 1$. □

Theorem 5. *For complete bipartite graph $K_{m,n}$ with $m \leq n$, we have:*

$$St_{\gamma_R}(K_{m,n}) = \begin{cases} m-2, & m > 3 \\ 2, & m = 1, n = 2 \\ 1, & o.w \end{cases}$$

Proof. Let X and Y be the partite sets of $K_{m.n}$, where $|X| = m$ and $|Y| = n$. Assume that $|X| = 1$. Clearly $\gamma_R(K_{m,n}) = 2$. If $|Y| = 1$, then removal of the vertex of X produces a K_1 with Roman domination number one, and so $st_{\gamma_R}(K_{m,n}) = 1$. If $|Y| \geq 3$, then removal of the vertex of X produces a graph $\overline{K_n}$ with $n \geq 3$, and note that $\gamma_R(\overline{K_n}) > 2$. Consequently, $st_{\gamma_R}(K_{m,n}) = 1$. It remains to assume that $|Y| = 2$. It can be easily seen that removal of any vertex of $K_{m,n}$ does not change the Roman domination number, and so $st_{\gamma_R}(K_{m,n}) \geq 2$. On the other hand, removal of Y leaves the graph K_1 with Roman domination number one, and thus $st_{\gamma_R}(K_{m,n}) \leq 2$. Consequently, $st_{\gamma_R}(K_{m,n}) = 2$.

Next assume that $|X| = 2$. Then $\gamma_R(K_{m,n}) = 3$, while $\gamma_R(K_{m,n} - v) = 2$, where v is a vertex of X. Consequently, $st_{\gamma_R}(K_{m,n}) = 1$.

If $|X| = 3$, then $\gamma_R(K_{m,n}) = 4$, while $\gamma_R(K_{m,n} - v) = 3$, where v is a vertex of X. Consequently, $st_{\gamma_R}(K_{m,n}) = 1$.

Now we assume that $|X| \geq 4$. Clearly, $\gamma_R(K_{m,n}) = 4$. Note that removal of $m-2$ vertices of X produces the graph $K_{2,n}$ with Roman domination number 3, and so $st_{\gamma_R}(K_{m,n}) \leq m-2$. On the other hand removal of at most $m-3$ vertices of $K_{m,n}$ leaves at least three vertices in X and at least three vertices in Y, and thus the remaining graph has Roman domination number at least 4. Thus, $st_{\gamma_R}(K_{m,n}) \geq m-2$. Consequently, $st_{\gamma_R}(K_{m,n}) = m-2$. □

Theorem 6. *For every $n \in \mathbb{N}$, $st_{\gamma R}(P_2 \square P_n) = 1$.*

Proof. By Theorem 2, $\gamma_R(P_2\Box P_n) = n+1$. Since $P_2\Box P_1 \cong K_2$, we see that $st_{\gamma R}(P_2\Box P_1) = 1$. Since $P_2\Box P_2 \cong C_4$, we can see that $st_{\gamma R}(P_2\Box P_2) = 1$. Thus assume that $n \geq 3$. Assume that n is even. Since $\gamma_R(P_2\Box P_n) = n+1$ is odd, for any $\gamma_R(P_2\Box P_n)$-function f, there is a vertex v with $f(v) = 1$. Then Removal of v produces a graph with Roman domination number at most n, and thus $st_{\gamma R}(P_2\Box P_n) = 1$. Next assume that n is odd. By re-labeling of the vertices of $P_2\Box P_n$, assume that $v_0, v_1, ..., v_{n-1}$ are the vertices on the top row and $w_0, w_1, ..., w_{n-1}$ are the vertices on the bottom row. Let $H = P_2\Box P_n - v_0$. We show that $\gamma_R(H) \leq n$. If $n \overset{4}{\equiv} 1$, so, $n = 4k+1$, $k \in \mathbb{N}$, then it is straightforward to see that the function $f : V(H) \to \{0,1,2\}$ defined by $f(v_i) = \begin{cases} 2, & i \overset{4}{\equiv} 3, \\ 0, & o.w \end{cases}$ and $f(w_i) = \begin{cases} 2, & i \overset{4}{\equiv} 1, \\ 1, & i = n, \\ 0, & o.w \end{cases}$ is an RDF for H, implying that

$$\gamma_R(H) \leq \sum_{v\in V(H)} f(v) = 2(\frac{4k}{4}) + 2(\frac{4k}{4}) + 1 = 4k+1 = n.$$

If $n \overset{4}{\equiv} 3$, so, $n = 4k+3$, $k \in \mathbb{N}$, then the function $f : V(H) \to \{0,1,2\}$ defined by $f(v_i) = \begin{cases} 2, & i \overset{4}{\equiv} 3, \\ 1, & i = n, \\ 0, & o.w \end{cases}$ and $f(w_i) = \begin{cases} 2, & i \overset{4}{\equiv} 1, \\ 0, & o.w \end{cases}$ is an RDF for H, implying that

$$\gamma_R(H) \leq \sum_{v\in V(H)} f(v) = 2(\frac{4k}{4}) + 1 + 2(\frac{4(k+1)}{4}) = 4k+3 = n.$$

Consequently, $st_{\gamma R}(P_2\Box P_n) = 1$. □

Theorem 7. *If $q \leq p$, then* $st_{\gamma R}(K_p\Box K_q) = \begin{cases} 1, & |p-q| \leq 1, \\ p-q, & p-q \geq 2 \end{cases}$.

Proof. By Proposition 3 of [4] we have $\gamma_R(K_p\Box K_q) = \begin{cases} 2q, & p \neq q, \\ 2q-1, & p = q \end{cases}$. If $v \in V(K_p\Box K_q)$ then according to the proof of Proposition 3 of [4],

$$\gamma_R(K_P\Box K_q - v) = \begin{cases} 2q, & p-q \geq 2, \\ 2q-1, & p-q = 1, \\ 2q-2, & p = q \end{cases}.$$

Thus, $st_{\gamma R}(K_P\Box K_q) = 1$ if $p = q$ or $|p-q| = 1$. Next assume that $|p-q| \geq 2$. We will show that $st_{\gamma R}(K_p\Box K_q) = p-q$.

By re-labeling the vertices of $K_P \Box K_q$, assume that

$$V(K_p \Box K_q) = \{v_1^{(1)}, ..., v_p^{(1)}, ..., v_1^{(\alpha)}, ..., v_p^{(\alpha)}, ..., v_1^{(q)}, ..., v_p^{(q)}\}.$$

For an integer α with $1 \leq \alpha \leq q$, it is straightforward to check that the function f defined by

$$f(v_i^{(j)}) = \begin{cases} 2, & 1 \leq i, j \leq q, i = j, i \neq \alpha, \\ 1, & i = j = \alpha, \\ 0, & o.w \end{cases}$$

is an RDF for $K_p \Box K_q - \{v_{q+1}^{(\alpha)}, ..., v_p^{(\alpha)}\}$. Since $w(f) = 2(q-1) + 1 = 2q - 1$, we obtain that $\gamma_R(K_p \Box K_q - \{v_{q+1}^{(\alpha)}, ..., v_p^{(\alpha)}\}) \leq 2q - 1$ and so $st_{\gamma R}(K_P \Box K_q) \leq p - q$. We show that $st_{\gamma R}(K_P \Box K_q) = p - q$. For this purpose, we show that the removal of less than $p - q$ vertices of $K_p \Box K_q$ does not change $\gamma R(K_p \Box K_q)$. Let H be a graph obtained from $K_p \Box K_q$ by removing x vertices, where $x \leq p - q - 1$. Clearly each column has at least two vertices. For column 1, we assign 2 to a vertex of this column and 0 to any other vertex of this column. Assume the vertex of the column 1 valued 2 is in the row i_1. For column 2, we assign 2 to a vertex of this column on a row $i_2 \leq i_1$ and 0 to any other vertex of this column. For column 3, we assign 2 to a vertex of this column on a row $i_3 \notin \{i_1, i_2\}$ and 0 to any other vertex of this column. We continue this process to obtain an RDF of weight $2q$ in H, and so $\gamma_R(H) \leq 2q$. Suppose that $\gamma_R(H) < 2q$. Let f be a $\gamma_R(H)$-function. Then there is a row i such that the sum of labels of vertices of this row under f is at most one. Using this row, since any vertex valued 0 is adjacent to a vertex valued 2, we obtain that $w(f) \geq 2q$, a contradiction. We thus deduce that $\gamma_R(H) = 2q$. Consequently, $st_{\gamma R}(K_p \Box K_q) = p - q$. □

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