Different characterizations of a game-theoretical solution and its application on sports management

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Abstract. In this paper, we firstly propose some axiomatic results of a game-theoretical solution, and further investigate the relations among these game-theoretical results and the field of sports management. The main investigation methods are as follows. Different from the pre-existing results, we provide different axioms and reduction to characterize of a game-theoretical solution. By applying some rules of management sciences in the real-world, we assign several reinterpretations to these axioms and related game-theoretical results. Finally, these axioms and related game-theoretical results are applied to the framework of sports management.
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1. Introduction

Due to the constant renovation of the trend of real-world management systems, management methods of a combination of different theoretical results and related applications, including sports management, have become the main notion

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in the field of management science. In the framework of sports management, the use of several management methods could promote the management efficiency no matter for training of sports skills or management of sports industry. Related results could be found in as Chen and Su (2011), Mullin (1980), and so on. On the other hand, the axiomatic results of game-theoretical solutions could be always adopted to analyze various interaction relationship and related models among agents and coalitions by applying mathematical results. In addition to theoretical analysis, game-theoretical results also could be applied to provide optimal results or equilibrium conditions for some real-word models.

In a traditional coalitional transferable-utility (TU) game, each player is either fully involved or not involved at all in participation with some other agents, while in a multi-choice TU game, each player is allowed to participate with finite many different activity levels. The solutions on multi-choice TU games could be applied in many fields such as economics, political sciences and even sports management. The *core* is, perhaps, the most intuitive solution concept in game theory. There are several extended core for multi-choice TU games in literature. Here we apply the *unit-level-core* proposed by Hwang and Liao (2010).

Consistency, originally introduced by Harsanyi (1959) under the name of bilateral equilibrium, is a crucial characteristic of viable solutions in the axiomatic formulation of cooperative games. The idea behind this kind of consistency is as follows: for a given game, agents may elaborate expectations of the game and may be willing to allow the computation of their payments to be based upon these expectations. The solution concept is consistent if it gives the same payments to players in the original game as it does to players of the imaginary reduced game. Thus, consistency is a requirement of the internal "*robustness*" of compromises. The fundamental property of solutions has been investigated in various classes of problems by applying *reduced games* always. Various definitions of a reduced game have been proposed, depending upon exactly how the agents outside of the subgroup should be paid off. Hwang and Liao (2010) extended the reduction introduced by Davis and Maschler (1965) to multi-choice games and provided related consistency property to characterize the unit-level-core. Later, Liao (2009) extended the reduction introduced by Moulin (1985) and Voorneveld and van den Nouweland (1998) to multi-choice TU games respectively and provided related consistency properties to characterize the unit-level-core.

In this paper, we firstly propose new game-theoretical results of the unit-level-core, and further investigate several relations and applications among these game-theoretical results and sports management. Here we build on the results of Hwang and Liao (2010) and Liao (2009). By considering both game-theoretical approach and sports management, the main results of this paper are as follows.

1. Different from the results of Hwang and Liao (2010) and Liao (2009), several characterizations are proposed in this paper.

- Inspired by Peleg (1986), we adopt the reduction proposed by Liao (2009) to show that the unit-level-core is the only solution satisfying *non-emptiness*, *individually rationality*, *consistency* and *superadditivity* on the domain of all balanced multi-choice TU games.
- By restricting the application of the non-emptiness property to balanced games, we adopt a weakening of non-emptiness to extend the results of Peleg (1986) to the domain of all multi-choice TU games.
- Inspired by Serrano and Volij (1998), we propose an alternative reduction to characterize the unit-level-core on the domain of all multi-choice TU games.

2. In section 4, we adopt the framework of multi-choice TU games and the characterizations of the unit-level-core to build the model of sports management systems, and thus to apply related results to the field of sports management. Further, some more relations among the unit-level-core and sports management would be proposed in detail.

2. Notions and definitions

Let U be the universe of players. For $i \in U$ and $m_i \in \mathbb{N}$, we set $M_i = \{0, 1, \cdots, m_i\}$ as the action space of player i, where the action 0 means not participating, and $M_i^+ = M_i \setminus \{0\}$. For $N \subseteq U \setminus \{\emptyset\}$, let $M^N = \prod_{i \in N} M_i$ be the product set of the action spaces for players N and $L^N = \{(i, j) \mid i \in N, j \in M_i^+\}$. Denote the zero vector in $\mathbb{R}^N$ by 0_N.

A **multi-choice TU game** is a pair (N, v), where N is a non-empty and finite set of players, $m = (m_i)_{i \in N}$ is a vector that describes the number of activity levels for each player, and $v : M^N \to \mathbb{R}$ is a characteristic function which assigns to each action vector $\alpha = (\alpha_i)_{i \in N} \in M^N$ the worth that the players can obtain when each player i plays at activity level $\alpha_i \in M_i$ with $v(0_N) = 0$.

Denote the class of all multi-choice TU games by Γ. Given $(N, v) \in \Gamma$, $\alpha \in M^N$ and $T \subseteq N$, we define $S(\alpha) = \{i \in N | \alpha_i \neq 0\}$ and denote $\alpha_T \in \mathbb{R}^T$ to be the restriction of α to T. Let $i \in N$. For convenience we introduce the substitution notation α_{-i} to stand for $\alpha_{N \setminus \{i\}}$ and let $\beta = (\alpha_{-i}, j) \in \mathbb{R}^N$ be defined by $\beta_{-i} = \alpha_{-i}$ and $\beta_i = j$.

Given $S \subseteq N$, let $|S|$ be the number of elements in S and let $e^S(N)$ be the binary vector in $\mathbb{R}^N$ whose component $e_i^S(N)$ satisfies

$$e_i^S(N) = \begin{cases} 1, & \text{if } i \in S, \\ 0, & \text{otherwise.} \end{cases}$$

Note that if no confusion can arise $e^S(N)$ will be denoted by e^S.

Given $(N, v) \in \Gamma$. A **payoff vector** of (N, v) is a function $x : N \to \mathbb{R}$ where the number x_i represents the per unit payoff that player i receives for all

$i \in N$, hence $m_i \cdot x_i$ is the total payoff that player i receives at (N, v). Given $(N, v) \in \Gamma$, $x \in \mathbb{R}^N$, $\alpha \in M^N$ and $S \subseteq N$, we denote $x(S) = \sum_{i \in S} x_i$, and $x(\alpha) = \sum_{i \in N} \alpha_i \cdot x_i$. Then

- a payoff vector x of $(N, v) \in \Gamma$ is **efficient (EFF)** if $x(m) = v(m)$.
- a payoff vector x of $(N, v) \in \Gamma$ is **individually rational (IR)** if for all $i \in N$ and for all $j \in M_i$, $x(je^{\{i\}}) \geq v(je^{\{i\}})$.

Moreover, x is an **imputation** of (N, v) if it is EFF and IR. And the set of **feasible payoff vectors** of (N, v) is denoted by

$$X^*(N, v) = \{x \in \mathbb{R}^N | \ x(m) \leq v(m) \ \},$$

whereas

$$X(N, v) = \{x \in \mathbb{R}^N \mid x \text{ is EFF}\}$$

is the set of **preimputations** of (N, v) and the set of imputations of (N, v) is denoted by $I(N, v)$.

A **solution on** Γ is a function σ which associates with each $(N, v) \in \Gamma$ a subset $\sigma(N, v)$ of $X^*(N, v)$.

Definition 1. *The* **unit-level-core (Hwang and Liao, 2010)** $C(N, v)$ *of* $(N, v) \in \Gamma$ *consists of all* $x \in X(N, v)$ *that satisfy for all* $\alpha \in M^N$, $x(\alpha) \geq v(\alpha)$, *i.e.,*

$$C(N, v) = \{x \in X(N, v) \mid x(\alpha) \geq v(\alpha) \textit{ for all } \alpha \in M^N\}.$$

3. Game-theoretical results: alternative characterizations of the unit-level-core

In this section, we adopt the reduced game proposed by Liao (2009) to characterize the unit-level-core. Further, an alternative reduction and related characterizations are also proposed.

Liao (2009) extended the reduced game introduced by Voorneveld and van den Nouweland (1998) to multi-choice TU games as follows.

Definition 2 ([5]). *Let* $(N, v) \in \Gamma$ *and* $S \in 2^N \setminus \{\emptyset\}$. *The* **reduced game with respect to** S **and** x *is the game* $(S, v_{S,x})$ *where*

$$v_{S,x}(\alpha) = \begin{cases} 0, & \alpha = 0_S, \\ v(m) - \sum\limits_{i \in N \setminus S} m_i \cdot x_i, & \alpha = m_S, \\ \max\limits_{\beta \in M^{N \setminus S} \setminus \{0_{N \setminus S}\}} \{ \ v(\alpha, \beta) - \sum\limits_{i \in N \setminus S} \beta_i \cdot x_i \ \}, & \textit{otherwise.} \end{cases}$$

In the reduced game $(S, v_{S,x})$, the players in S are regathered to play. The players in $N \setminus S$ are required to cooperate in at least one participation level when some players in S take nonzero actions. Of course, if each player in S does not participate in playing, the account S can achieve is zero. If all the players in S play at full steam, the players in $N \setminus S$ also cooperate at full steam. Assume that some players in S do not play at full steam, all the coalitions comprised by the participation levels of $N \setminus S$ should be considered to cooperate so that the maximal benefit can be reached. Since the players in $N \setminus S$ accept the payoff according to x, they receive the deserved payoff and leave then.

Consistency requires that if x is prescribed by σ for a game (N, v), then the projection of x to S should be prescribed by σ for the reduced game with respect to S and x for all S. Thus, the projection of x to S should be consistent with the expectations of the members of S as reflected by their reduced game. *Converse consistency* requires that if the projection of an efficient payoff vector x to every proper S is consistent with the expectations of the members of S as reflected by their reduced game then x itself should be recommended for whole game.

- **Consistency (CON)**: For all $(N, v) \in \Gamma$, for all $S \in 2^N \setminus \{\emptyset\}$, and for all $x \in \sigma(N, v)$, $(S, v_{S,x}) \in \Gamma$ and $x_S \in \sigma(S, v_{S,x})$.
- **Converse consistency (CCON)**: For all $(N, v) \in \Gamma$ with $|N| \geq 2$ and for all $x \in X(N, v)$, if for all $S \subset N$ such that $0 < |S| < |N|$, $(S, v_{S,x}) \in \Gamma$ and $x_S \in \sigma(S, v_{S,x})$, then $x \in \sigma(N, v)$.
- **Weak converse consistency (WCCON)**[1]: For all $(N, v) \in \Gamma$ with $|N| \geq 2$ and for all $x \in I(N, v)$, if for all $S \subset N$ such that $0 < |S| < |N|$, $(S, v_{S,x}) \in \Gamma$ and $x_S \in \sigma(S, v_{S,x})$, then $x \in \sigma(N, v)$.

Liao (2009) characterized the unit-level-core by means of CON and WCCON. Subsequently, we provide alternative characterizations of the unit-level-core. We will make use of the following axioms. We say that the multi-choice game (N, v) is *balanced* [2] if $C(N, v) \neq \emptyset$. Let Γ_c denote the set of all balanced multi-choice games. Let $\Gamma' \subseteq \Gamma$ and σ be a solution on Γ'. σ satisfies **non-emptiness (NE)** if for all $(N, v) \in \Gamma'$, $\sigma(N, v) \neq \emptyset$. σ satisfies **non-emptiness for balanced games (NEB)** if for all $(N, v) \in \Gamma_c$, $\sigma(N, v) \neq \emptyset$. σ satisfies **individually rationality (IR)** if for all $(N, v) \in \Gamma'$, $\sigma(N, v) \subseteq I(N, v)$. σ satisfies **efficiency (EFF)** if for all $(N, v) \in \Gamma'$, $\sigma(N, v) \subseteq X(N, v)$. σ satisfies **superadditivity (SUPA)**[3] if for all $(N, v) \in \Gamma'$, $\sigma(N, v + w) \supseteq \sigma(N, v) + \sigma(N, w)$, where for all $\alpha \in M^N$, $(v + w)(\alpha) = v(\alpha) + w(\alpha)$.

Lemma 1 ([5]). *1. On both Γ_c and Γ, the unit-level-core satisfies CON and WCCON.*

2. Let $\Gamma' \subset \Gamma$ and let σ be a solution on Γ'. If σ satisfies IR and CON, then it also satisfies EFF.

1. The following axiom is a weakening of the previous axiom, since it requires that x be individually rational as well.
2. The balancedness of the unit-level-core was proposed by Hwang and Liao (2010).
3. If $N \subseteq U$ and $A, B \subset \mathbb{R}^N$, then $A + B = \{\ a + b \mid a \in A \text{ and } b \in B\ \}$.

Lemma 2. *Let* $\Gamma' \subset \Gamma$ *and let* σ *be a solution on* Γ'. *If* σ *satisfies IR and CON, then for all* $(N, v) \in \Gamma'$, $\sigma(N, v) \subseteq C(N, v)$.

Proof. Let $(N, v) \in \Gamma'$. The proof proceeds by induction on the number $|N|$. If $|N| = 1$, then by IR of σ and C, $\sigma(N, v) \subseteq C(N, v)$. Assume that $\sigma(N, v) \subseteq C(N, v)$ if for $|N| \leq k - 1$, where $k \geq 2$.
The case $|N| = k$:
Since σ satisfies IR and CON, by Lemma 1, σ satisfies EFF. Hence, $\sigma(N, v) \subseteq I(N, v)$. Let $x \in \sigma(N, v)$. Since σ satisfies CON, for all $S \subseteq N$ with $0 < |S| < |N|$, $x_S \in \sigma(S, v_{S,x})$. Hence, by the induction hypotheses, $x_S \in \sigma(S, v_{S,x}) \subseteq C(S, v_{S,x})$. Since C satisfies WCCON, $x \in C(N, v)$. □

Theorem 1.

1. *On* Γ_c, *the unit-level-core is the only solution satisfying NE, IR, SUPA and CON.*

2. *Let* $\Gamma' \subset \Gamma$. *On* Γ', *the unit-level-core is the only solution satisfying NEB, IR, SUPA and CON.*

Proof. By Lemma 1, the unit-level-core satisfies CON. Clearly, the unit-level-core satisfies NE, NEB, IR and SUPA.

To prove uniqueness of (1), assume that a solution σ satisfies NE, IR, SUPA and CON on Γ_c. Let$(N, v) \in \Gamma_c$. Two cases may be distinguished:
Case 1 : Assume that $|N| \geq 3$. Let $x \in C(N, v)$. Define $(N, w) \in \Gamma$ by the following rule:

$$w(t) = \begin{cases} v(t), & t = je^{\{i\}} \text{ for all } (i, j) \in L^N, \\ x(t), & \text{otherwise.} \end{cases}$$

As the reader can easily to verify that $C(N, w) = \{x\}$. Thus, by NE of σ and Lemma 2, $\sigma(N, w) = \{x\}$. Now let $u = v - w$. Clearly, for all $(i, j) \in L^N$, $u(je^{\{i\}}) = 0$. And for all $t \in M^N$, $u(m) = 0$ and $u(t) \leq 0$. Hence, $C(N, u) = \{0_N\}$. By NE of σ and Lemma 2, $\sigma(N, w) = \{0_N\}$. Since $v = u + w$ and σ satisfies SUPA,

$$\sigma(N, v) \supseteq \sigma(N, u) + \sigma(N, w) = \{x\}.$$

Thus, $\sigma(N, v) \supseteq C(N, v)$. By Lemma 2, $\sigma(N, v) \subseteq C(N, v)$. Hence, $\sigma(N, v) = C(N, v)$.
Case 2 : Assume that $|N| \leq 2$. If $|N| = 1$, then by NE and IR of σ and C, $\sigma(N, v) = C(N, v)$. Thus, let $|N| = 2$. Denote that $N = \{i, k\}$ and let $p \in U \setminus N$. Define $(H, u) \in \Gamma$ with $H = \{i, k, p\}$ in the following rule. For all $t \in M^H$, $u(t) = v(t_N)$. Let $x \in C(N, v)$. By definitions of u and $u_{N,(x,0)}$, it is easy to verify that $(x, 0) \in C(H, u)$ and $(N, u_{N,(x,0)}) = (N, v)$. Since $|H| = 3$, by Case 1, $\sigma(H, u) = C(H, u)$. Thus, $(x, 0) \in \sigma(H, u)$. By CON of σ, $x = (x, 0)_N \in \sigma(N, u_{N,(x,0)}) = \sigma(N, v)$. Hence, $C(N, v) \subseteq \sigma(N, v)$. By Lemma 2, $\sigma(N, v) \subseteq C(N, v)$. Hence, $\sigma(N, v) = C(N, v)$.

This proof of uniqueness of (2) is a copy of (1) except “NEB and Γ'” instead of “NE and Γ_c”; hence, we omit it.

□

Different from the reductions proposed by Hwang and Liao (2010) and Liao (2009), we propose an extension of the reduction proposed by Serrano and Volij (1998) as follows.

Definition 3. *Let* $(N, v) \in \Gamma$ *and* $S \in 2^N \setminus \{\emptyset\}$*. The* **revised reduced game with respect to** S **and** x *is the game* $(S, v^R_{S,x})$ *where*

$$v^R_{S,x}(\alpha) = \begin{cases} 0, & \alpha = 0_S, \\ \max\limits_{\beta \in M^{N\setminus S}\setminus\{0_{N\setminus S}\}} \{\ v(\alpha,\beta) - \sum\limits_{i\in N\setminus S} \beta_i \cdot x_i\ \}, & \textit{otherwise.} \end{cases}$$

The only difference between this and the reduced game in Definition 2 is the fact that the coalition S is also allowed to imagine potential interaction with any of the subset of $N \setminus S$. Informally, in order to reach the maximal benefit, all the coalitions comprised by the members of $N \setminus S$ should to be considered to cooperate with S. “Revised reduction” instead of “reduction”, we also introduce **R-consistency (RCON)** and **converse R-consistency (CRCON)**.

Lemma 3. *On both* Γ_c *and* Γ*, the unit-level-core satisfies CRCON.*

Proof. We first show that on Γ, the unit-level-core satisfies CRCON. And the proof is also applied to the domain Γ_c. Let $(N, v) \in \Gamma$ with $|N| \geq 2$ and let $x \in X(N, v)$. Suppose for all $S \subset N$ such that $0 < |S| < |N|$, $(S, v^R_{S,x}) \in \Gamma$ and $x_S \in C(S, v^R_{S,x})$. We will show that $x \in C(N, v)$, i.e., for all $\alpha \in M^N$, $\sum_{i\in N} \alpha_i x_i \geq v(\alpha)$. Let $i \in N$ and $\alpha \in M^N$. Consider the reduced game $(\{i\}, v^R_{\{i\},x})$. Then

$$\begin{cases} \alpha_i x_i & \geq v^R_{\{i\},x}(\alpha_i) \quad \Big(\text{Since } x_i \in C(\{i\}, v^R_{\{i\},x})\Big) \\ & = \max\limits_{\beta\in M^{N\setminus\{i\}}\setminus\{0_{N\setminus\{i\}}\}} \{v(\alpha_i,\beta) - \sum\limits_{j\in N\setminus\{i\}} \beta_j x_j\} \\ & \geq v(\alpha_i, \alpha_{N\setminus\{i\}}) - \sum\limits_{j\in N\setminus\{i\}} \alpha_j x_j \\ & = v(\alpha) - \sum\limits_{j\in N\setminus\{i\}} \alpha_j x_j. \end{cases}$$

Hence, $\sum_{k\in N} \alpha_k x_k \geq v(\alpha)$. □

Lemma 4.

1. *On both* Γ_c *and* Γ*, the unit-level-core satisfies RCON.*
2. *Let* $\Gamma' \subset \Gamma$ *and let* σ *be a solution on* Γ'*. If* σ *satisfies IR and RCON, then it also satisfies EFF.*

3. *Let* $\Gamma' \subset \Gamma$ *and let* σ *be a solution on* Γ'. *If* σ *satisfies IR and RCON, then for all* $(N, v) \in \Gamma'$, $\sigma(N, v) \subseteq C(N, v)$.

Proof. The proof is similar to Lemmas 1, 2, hence we omit it. □

Next, we provide alternative characterizations by means of RCON.

Theorem 2.

1. *On* Γ_c, *the unit-level-core is the only solution satisfying NE, IR, SUPA and RCON.*
2. *Let* $\Gamma' \subset \Gamma$. *On* Γ', *the unit-level-core is the only solution satisfying NEB, IR, SUPA and RCON.*

Proof. The proof is similar to Theorem 1, hence we omit it. □

The following examples show that each of the axioms used in Theorems 1, 2 is logically independent of the others.

Example 1. Let $\sigma(N, v) = \emptyset$ for all $(N, v) \in \Gamma_c$. Then σ satisfies IR, SUPA and CON (RCON), but it violates NE (NEB).

Example 2. Let $\sigma(N, v) = X(N, v)$ for all $(N, v) \in \Gamma_c$. Then σ satisfies NE (NEB), SUPA and CON (RCON), but it violates IR.

Example 3. Let $\sigma(N, v) = I(N, v)$ for all $(N, v) \in \Gamma_c$. Then σ satisfies NE (NEB), IR and SUPA, but it violates CON (RCON).

Example 4. For all $(N, v) \in \Gamma_c$, we define a solution σ on Γ_c to be

$$\sigma(N, v) = \begin{cases} C(N, v), & \text{if for all } x \in C(N, v) \text{ with } x_i \neq 0 \\ \{x \in C(N, v) \mid x_i = 0\}, & \text{otherwise.} \end{cases}$$

Then σ satisfies NE (NEB), IR and CON (RCON), but it violates SUPA.

4. Application on sports management

In this section, we provide an application of multi-choice TU games and the unit-level-core in the setting of "allocation of the utilities for sports management systems", such as the National Basketball Association (NBA), the Major League Baseball (MLB) and so on.

In a sports organization, such as the NBA, each department of the sports organization may take several operation strategies to manage. Besides competing in sports games, all departments should develop to raise entire utilities of whole the sports organization also, such as security, box office, affiliated products and so on. This kind of problem could be formulated as follows: Let $N = \{1, 2, \cdots, n\}$ be a set of all departments of the sports organization that could be provided jointly by some coalitions and let $v(\alpha)$ be the profit of providing the strategical vector $\alpha = (\alpha_i)_{i\in N}$ in N jointly. For all department i in this sports organization, the activity level α_i could be treated as one of the operation strategies of the department i. The function v could be treated as an utility function which assigns to each operation strategical vector α the worth that the departments can obtain when each department i takes at operation strategy $\alpha_i \in M_i$. Modeled in this way, the sports management system of a sports organization could be considered as a cooperative multi-choice TU game, with v being its characteristic function. The unit-level-core could provide "optimal utilities" among all departments, in the sense that the sports organizations could get utilities from each combination of strategies of all departments under sports management systems. An illustrative example as follows.

Example 5. Let $(N, v) \in \Gamma$, where $N = \{a, b, c\}$ is a sports organization with departments a, b, c, $m = (2, 2, 1)$is a vector that describes the number of operational strategies for departments a, b, c respectively, and $v : M^N \to \mathbb{R}$ is the utility function which assigns to each strategy vector $\alpha = (\alpha_a, \alpha_b, \alpha_c)$ the worth that the departments can obtain when each department i operates at strategy α_i with $v(2, 2, 1) = 9$, $v(2, 1, 1) = 4$, $v(1, 2, 1) = 2$, $v(1, 1, 1) = 3$, $v(0, 2, 1) = 4$, $v(0, 1, 1) = 2$, $v(2, 0, 1) = 1$, $v(1, 0, 1) = -2$, $v(2, 2, 0) = 5$, $v(2, 1, 0) = 3$, $v(1, 2, 0) = 2$, $v(1, 1, 0) = 0$, $v(0, 0, 1) = -2$, $v(0, 2, 0) = -1$, $v(0, 1, 0) = -7$, $v(2, 0, 0) = 1$, $v(1, 0, 0) = -2$, $v(0, 0, 0) = 0$. By definition of the unit-level-core, it is easy to determine that

$$\begin{aligned} C(N, v) = &\{(x_a, x_b, x_c)|\tfrac{1}{2} \le x_a \le \tfrac{3}{2},\ \tfrac{5-2x_a}{2} \le x_b \le 4,\ x_c = -2x_a - 2x_b + 9\} \\ &\cup\{(x_a, x_b, x_c)|\tfrac{3}{2} < x_a \le \tfrac{5}{2},\ \tfrac{5-2x_a}{2} \le x_b \le 7 - 2x_a,\ x_c = -2x_a - 2x_b + 9\}. \end{aligned}$$

Pick $x = (1, 4, -1)$, $y = (2, 1, 3)$ and $z = (2, 2, 1)$. Clearly, $x,\ y,\ z \in C(N, v)$ respectively.

Due to chances of the rise of sports competence and its additional values such as commercial profits, social resources and nation images, the arrangements of sports management systems meets have developed diversely under this context. On the other hand, the existence of globalization has brought shocks to the world and cities' politics, economics, and culture to a significant extent. However, it also led to increase the association as well as dependence among sports organizations. Several researches have found that the critical factor of sports management systems successfully should be that concern of "strategies and effectiveness". More precisely, the crucial notions of success in order are drafting the conditions of "efficient strategic operation", "balanced cooperation", "stable processes", and "resource sharing".

In order to promote the effectiveness of sports management systems, we firstly propose some relations among the game-theoretical axioms and sports management systems for sports organizations follows.

1. The property **non-emptiness (non-emptiness for balanced games)** asserts that *the efficient cooperation doesn't come to naught*: There might

be several coalitions in a sports organizations. In order to successfully operate sports management systems, a efficient strategic plan should be cohered with all coalitions simultaneously.

2. The property **individually rationality** asserts that *the efficiency could be promoted by balanced cooperation.*: The payoff (or utility) of each coalition assigned by balanced cooperation may be better than the utility of each coalition when it works alone.

3. The property **consistency (R-consistency)** asserts that *the operational processes should be stable and identical*: Consistency allows us to deduce, from the desirability of an outcome for some utility allocation problem, the desirability of its restriction to each coalition of a sports organization for the associated reduced condition the coalition faces.

4. The property **superadditivity** asserts that *the effectiveness could be promoted by resource sharing*: In order to successfully operate sports management systems, the utilities of all the cooperative coalitions or groups must be integrated successfully by resource sharing.

Based on Theorems 1, 2, the unit-level-core is the only solution satisfying these game-theoretical axioms. Based previous relations and statements among these game-theoretical axioms and sports management systems for sports organizations, the unit-level-core could be considered as an useful theoretical technique for utilities allocation in sports management systems. The conclusion of this paper also could point out that other game-theoretical methods could be considered to applied in the field of sports management.

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