A note on dimensions in N-groups

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Abstract. The concepts essential and finite dimension played an important role in the development of the dimension theory of modules over rings. Finite dimension, essential, strictly essential, and some related concepts were studied in nearrings and N-groups by Reddy-Satyanarayana [11], Satyanarayana [14, 20], Satyanarayana-Syam Prasad [22, 23, 24, 25]. In this paper, the authors introduced the concept finite 1-dimension and studied the relationship between finite dimension and finite 1-dimension. Some related examples are also provided.

Keywords: nearring, N-Group, essential ideal, uniform ideal, finite dimension.

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1. Introduction

A nearring is a set N together with two binary operations $+$ and $\cdot$ such that

(1) $(N, +)$ is a group (not necessarily abelian)

(2) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$,

(3) $(a + b) \cdot c = a \cdot c + b \cdot c$ for all $a, b, c \in N$.

In view of (3), N satisfies the right distributive law, and so it is called a right nearring. It is evident that $0 \cdot n = 0$ for all $n \in N$. However, $n \cdot 0$ need not be equal to 0, in general. We denote $N_0 = \{n \in N : n \cdot 0 = 0\}$, the zero-symmetric part of the right nearring. If $N = N_0$ then we say that the nearring N is zero-symmetric.

Let $(G, +)$ be a group. By an N-group, we mean a mapping $N \times G \to G$ (the image of $(n, g) \in N \times G$ is denoted by ng), satisfying the following two conditions

(1) $(n + n^1)g = ng + n^1g$, and

(2) $(nn^1)g = n(n^1g)$ for all $g \in G$ and $n, n^1 \in N$.

We will denote this N-group by $_NG$ or simply by G. For preliminary definitions and results we refer to Pilz [8], and Satyanarayana-Syam Prasad [29]. An ideal I of G is said to be essential (strictly essential, resp.) in an ideal J of G if it satisfies the condition: K is an ideal (N-subgroup of G, resp.) of G, $I \cap K = (0), K \subseteq J$ imply $K = (0)$.

2. Finite dimension in N-Groups

In this section we introduce the concept "N-group with finite dimension" and provide few examples for such type of N-group.

Definition 2.1. Let H be an ideal of G. Then H is said to have finite dimension if for any increasing chain $A_1 \subseteq A_2 \subseteq ...$ of ideals of G contained in H, there corresponds a positive integer k such that A_i is essential in A_{i+1} for all $i \geq k$.

Definition 2.2 ([14])**.** Let H be an ideal of G. Then H is said to have Finite Goldie dimension (FGD, in short) if H do not contain an infinite number of non-zero ideals of G whose sum is direct.

Theorem 2.1 ([14])**.** *Let H be an ideal of G. Then the following are equivalent:*

(i) H has finite Goldie dimension.

(ii) For any increasing sequence $H_0 \subseteq H_1 \subseteq ...$ of ideals of G with each $H_i \subseteq H$, there exists an integer k such that H_i is essential in H_{i+1} for $i \geq k$.

Note 2.1. From Theorem 2.1 we conclude that the following two conditions are equivalent for an ideal H.

(i) H have finite dimension.

(ii) H has finite Goldie dimension.

Proposition 2.1. *Every finite N-group G must have finite dimension.*

Proof. Let G be a finite N-group. Then the number of elements in G is finite. Let $k = |G|$. In a contrary way, suppose that G does not have finite dimension. Then there exists an ascending infinite sequence $A_1 \subseteq A_2 \subseteq ...$ of ideals of G such that for any positive integer n, there exist an integer i such that $i \geq n$ and A_i is not essential in A_{n_i+1}. Now we use mathematical induction to create an infinite number of elements in G. For $n = 1$, there exists an integer i_1 such that A_{i_1} is not essential in A_{i_1+1}. Now it is clear that $A_{i_1} \subsetneq A_{i_1+1}$ and so there exists $a_1 \in A_{i_1+1} \setminus A_{i_1}$. For $n = i_1 + 1$ there exist an integer i_2 such that $i_2 \geq i_1 + 1$ and A_{i_2} is not essential in A_{i_2+1}. It is clear that $A_{i_2} \subsetneq A_{i_2+1}$ and so there exist $a_2 \in A_{i_2+1} \setminus A_{i_2}$. Note that $i_2 \geq i_1 + 1 \geq i_1$. So $A_{i_1} \subseteq A_{i+1} \subseteq A_{i_2} \subseteq A_{i_2+1}$. Since $a_2 \in A_{i_2+1} \setminus A_{i_2}$ and $a_1 \in A_{i_1+1} \subseteq A_{i_2}$, it follows that $a_1 \neq a_2$. After selecting $i_1 \leq i_2 \leq ... \leq i_s$, we select $i_s + 1$ as follows: For $n = i_s$, there exists an integer $i_{(s+1)}$ such that $i_{(s+1)} \geq n = i_s$ and $A_{i_{(s+1)}}$ is not essential in $A_{i_{(s+1)}+1}$. It is clear that $A_{i_{(s+1)}} \subsetneq A_{i_{(s+1)}+1}$ and so there exist $a_{s+1} \in A_{i_{(s+1)}+1} \setminus A_{i_{(s+1)}}$ with $a_{s+1} \notin \{a_1, a_2, ..., a_s\}$. By mathematical induction we get an infinite number of elements $a_1, a_2, ..., a_s, ...$ in G, a contradiction to the fact that G is a finite set. □

Proposition 2.2. *Every N-group with ascending chain condition on its ideals must have finite dimension.*

Proof. Let G be an N-group with ACC on its ideals. Let $A_1 \subseteq A_2 \subseteq ...$ be an increasing chain of ideals of G. Since G have ACC on its ideals, this chain $A_1 \subseteq A_2 \subseteq ...$ must be stationary after some stage. That is, there exists a positive integer s such that $A_s = A_{s+1} = A_{s+2} = ...$ Now it is clear that A_i is essential in A_{i+1} for $i \geq s$. Hence G has finite dimension. □

Definition 2.3 ([11])**.** An N-group G is said to be completely reducible if G is equal to the sum of all of its simple ideals. If G contains infinite number of simple ideals whose sum is direct, then G has no finite dimension. In this case dimension of G is infinite.

Proposition 2.3. *Every N-group G which is completely reducible and which can be written a finite sum of simple ideals of G, must have finite dimension.*

Proof. Let G be an N-group which is completely reducible and which can be written as finite sum of simple ideals of G. Suppose $G = H_1 \oplus H_2 \oplus ... \oplus H_s$ where s is a positive integer and $H_i, 1 \leq i \leq s$ are ideals of G.

Write $B_i = H_1 \oplus H_2 \oplus ... \oplus H_s$ for $1 \leq i \leq s$. Since the sum is direct and by Theorem 2.3.5 of Satyanarayana-Syam Prasad [29] we get $B_{i+1}/B_i =$

$\dfrac{(H_1 + ... + H_{i+1})}{(H_1 + ... + H_i)} \cong \dfrac{H_{i+1}}{(H_1 + ... + H_i) \cap H_{i+1}} \cong \dfrac{H_{i+1}}{(0)} \cong H_{i+1}$. Hence by Proposition 2.38 of Pilz [8], we get that $(0) \subseteq B_1 \subseteq B_2 \subseteq ... \subseteq B_s = G$ is a composition series of G. Since this composition series is finite, by Theorem 2.41 of Pilz [8], G has ACC and DCC on its ideals. Now by Proposition 2.2, we have that G have finite dimension. □

Note 2.2. Every module M over a ring R with finite Goldie dimension is an N-group G with finite dimension, by considering R as a nearring N and M as G.

Proposition 2.4. *If G has finite dimension, then every ideal H of G must have finite dimension.*

Proof. Suppose G has finite dimension. let H be an ideal of G. To verify that H have finite dimension, let $A_1 \subseteq A_2 \subseteq ...$ be an infinite sequence of ideals of G contained in H. Since $A_1 \subseteq A_2 \subseteq ...$ is an ascending chain of ideals in G and G have finite dimension, we can get a positive integer s such that A_i is essential in A_{i+1} for all $i \geq s$. This sows that H have finite dimension. □

3. Finite 1-dimension in N-groups

In this section, we study the concept strictly essential ideal of an N-group G and study finite 1-dimension in N-groups. We also observe the relationship between finite dimension studied in section-1 and finite 1-dimension.

Definition 3.1 ([17])**.** An ideal H of G is said to be strictly essential in G if $H \cap A = (0)$, A is an N-subgroup of G imply $A = (0)$.

(ii) Let H_1 and H_2 be two ideals of G such that $H_1 \subseteq H_2$. Then H_1 is said to be strictly essential (with respect to H_2) in H_2 if $H_1 \cap A = (0), A \subseteq H_2$, A is an N-subgroup of H_2 imply $A = (0)$.

Lemma 3.1. *Suppose that N is a zero-symmetric nearring.*

(i) Every strictly essential ideal of G is an essential ideal of G.

(ii) Let H_1 and H_2 be two ideals of G such that $H_1 \subseteq H_2$. Then "H_1 is strictly essential in H_2" (with respect to H_2) imply that "H_1 is essential in H_2" (with respect to H_2) when we consider H_2 as N-group. Hence H_1 is also essential in H_2.

Proof. (i) Let H be a strictly essential ideal of G. To verify that H is an essential ideal, take an ideal A of G such that $H \cap A = (0)$. Since N is a zero symmetric, by Proposition 1.34 (b) of Pilz [8], we have A is an N-subgroup of G. Since $H \cap A = (0)$, A is an N-subgroup, and H is strictly essential, it follows that $A = (0)$. Therefore, H is an essential ideal of G.

(ii) Suppose that H_1 is strictly essential in H_2. Now we wish to verify that H_1 is essential in H_2. Suppose $H_1 \cap A = (0)$, A is an ideal of H_2 (consider H_2 as N-group). Note that N is a zero symmetric nearring, A is an N-subgroup of H_2. Now, since $H_1 \cap A = (0)$, A is an N-subgroup of H_2 and H_1 is strictly essential in H_2, it follows that $A = (0)$. This shows that "H_1 is essential in H_2". □

The following example shows that every essential ideal of G need not be strictly essential.

Example 3.1. Let G be the symmetric group (written additively) on three elements. Then G can be considered as a N-group, where $N = Z$, the nearring of integers.

(i) Let P be the alternating subgroup of G. Then P is only the proper ideal of G. So P is essential in G.

(ii) Note that $P = \{f, f^2, e\}$, where $f = (123) = (12)(13); f^2 = (132) = (13)(12); f^3 = e = (1)$. Write Q = {g, e} where $g = (12)$. Then $g^2 = e$. So Q is an N-subgroup of G.

(iii) Since $P \cap Q = (0)$, Q is an N-subgroup of G such that $Q \neq (0)$, it follows that P is not strictly essential in G.

Note 3.1. From the Example 3.1, we conclude that, in general, every essential ideal need not be a strictly essential ideal.

Definition 3.2. An ideal H of G is said to have finite 1-dimension, if for any increasing chain $A_1 \subseteq A_2 \subseteq ...$ of ideals of G contained in H there exists a positive integer i such that A_i is strictly essential in A_{i+1}.

Theorem 3.1. *Let $N = N_0$ and H be an ideal of G. If H have finite 1-dimension, then H must have finite dimension.*

Proof. Suppose H is having finite 1-dimension. To verify that H have finite dimension, take an increasing sequence $A_1 \subseteq A_2 \subseteq ...$ of ideals of G contained in H. Since H have finite 1-dimension, we can conclude that there exists a positive integer k such that A_i is strictly essential in A_{i+1} for all $i \geq k$. By Lemma 3.1 (ii), we conclude that A_i is essential in A_{i+1} for all $i \geq k$. Thus we have proved that given an increasing sequence $A_1 \subseteq A_2 \subseteq ...$ of ideals of G contained in H, there corresponds a positive integer k such that A_i is essential in A_{i+1} for all $i \geq k$. Hence H has finite dimension. □

Example 3.2. As in Example 3.1, let G be the symmetric group (written additively) on three symbols 1, 2 and 3. Then G can be considered as an N-group where $N = Z$, the nearring of integers. Let P be the alternating subgroup and Q be the subgroup $\{g, e\}$, as given in Example 3.1.

(i) Write $G_i = G, P_i = P, Q_i = Q$ for all positive integers i. Consider the direct sum G of the N-groups $\{G_i\}_{i \in Z^+}$.

Then $G = G_1 \oplus G_2 \oplus ....$

(ii) $\sum_{i=1}^{\infty} P_i \subseteq G_1 \oplus \sum_{i=2}^{\infty} P_i \subseteq G_1 \oplus G_2 \oplus \sum_{i=3}^{\infty} P_i \subseteq ...$ is an increasing sequence of ideals of G.

(iii) Since P_1 is essential in G_1, we have that $\sum_{i=1}^{\infty} P_i = P_1 \oplus \sum_{i=2}^{\infty} P_i$ is essential in $G_1 \oplus \sum_{i=1}^{\infty} P_i$. Since P_2 is essential in G_2, we have that $\sum_{i=2}^{\infty} P_i = P_2 \oplus \sum_{i=3}^{\infty} P_i$ is essential in $G_2 \oplus \sum_{i=3}^{\infty} P_i$, and so $G_1 \oplus \sum_{i=2}^{\infty} P_i$ is essential in $G_1 \oplus G_2 \oplus \sum_{i=3}^{\infty} P_i$.

In this way we can verify that for all $n \geq 1$, $G_1 \oplus G_2 \oplus ... \oplus G_n \oplus \sum_{i=n+1}^{\infty} P_i$ is essential in the next term $G_1 \oplus G_2 \oplus ... \oplus G_n \oplus G_{n+1} \oplus \sum_{i=n+2}^{\infty} P_i$. So the sequence considered satisfies the property given in the definition "finite dimension".

(iv) Since $P_1 \cap Q_1 = (0)$, Q_1 is an N-subgroup of G_1 and $Q_1 \neq (0)$, we have that P_1 is not strictly essential in G_1.

Since $(P_1 \oplus \sum_{i=2}^{\infty} P_i) \cap Q_1 = (0), Q_1 \neq (0)$, Q_1 is an N-subgroup of $G_1 \oplus \sum_{i=2}^{\infty} P_i$, it follows that $P_1 \oplus \sum_{i=2}^{\infty} P_i = \sum_{i=1}^{\infty} P_i$, is not essential in $G_1 \oplus \sum_{i=2}^{\infty} P_i$.

(v) Since $P_2 \cap Q_2 = (0)$, Q_2 is an N-subgroup of G_2 and $Q_2 \neq (0)$, we have that P_2 is not strictly essential in G_2.

Since $(P_2 \oplus \sum_{i=3}^{\infty} P_i) \cap Q_2 = (0), Q_2 \neq (0)$, Q_2 is an N-subgroup of $G_2 \oplus \sum_{i=3}^{\infty} P_i$, it follows that $P_2 \oplus \sum_{i=3}^{\infty} P_i$, is not essential in $G_2 \oplus \sum_{i=3}^{\infty} P_i$, and hence $G_1 \oplus \sum_{i=2}^{\infty} P_i = G_1 \oplus P_2 \oplus \sum_{i=3}^{\infty} P_i$ is not essential in $G_1 \oplus G_2 \oplus \sum_{i=3}^{\infty} P_i$.

(vi) If we continue as in (iv) and (v), we can understand that for all $n \geq 1$, $G_1 \oplus ... \oplus G_n \oplus \sum_{i=n+1}^{\infty} P_i$ is not strictly essential in the next term $G_1 \oplus ... \oplus G_n \oplus G_{n+1} \oplus \sum_{i=n+2}^{\infty} P_i$.

So this sequence considered does not satisfy the property given in the definition "finite 1-dimension".

(vii) From the discussion in (vi), we can conclude that G does not have finite 1-dimension.

(viii) Now we verify that G does not have finite dimension.

Consider the increasing sequence $P_1 \subseteq P_1 \oplus P_2 \subseteq P_1 \oplus P_2 \oplus P_3 \subseteq ...$ of ideals of G. It is clear that $P_1 \oplus ... \oplus P_s$ is not essential in $P_1 \oplus ... \oplus P_s \oplus P_{s+1}$, since $(P_1 \oplus ... \oplus P_s) \cap P_{s+1} = (0)$ and $P_{s+1} \neq (0)$. Thus G does not have finite dimension.

Example 3.3. If G is an N-group satisfying DCC on its ideals, then G must have finite 1-dimension. Let G be an N-group satisfying DCC on its ideals. Let $A_1 \subseteq A_2 \subseteq ...$ be an increasing sequence of ideals of G. Since G has DCC on its ideals, there exists a positive integer s such that $A_s = A_{s+1} = ...$ Now A_{s+t} is strictly essential in A_{s+t+1} for all $t \geq 1$. Now we verified that for any increasing sequence of ideals $A_1 \subseteq A_2 \subseteq ...$, there exists a positive integer s such that A_i is strictly essential in A_{i+1} for all $i \geq s$. So we can conclude that G have the finite 1-dimension.

Example 3.4. A routine verification shows that every finite N-group G must have the finite 1-dimension.

Example 3.5. Every module with finite Goldie dimension must have finite 1-dimension.

For this, take a module M over a ring R. The ideals and N-subgroups of $G = M$ (as an N-group over $N = R$) coincides (as M is a module). So the concepts “finite dimension” and “finite 1-dimension” coincides for modules. By Note 2.2, we know that M have finite dimension. Hence M has finite 1-dimension.

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References

[1] F.W. Anderson, K.R. Fuller, *Rings and categories of modules*, SpringerVerlag, New York, 1974.

[2] V. Camillo, J. Zelmanowitz, *On the dimension of a sum of modules*, Communications in Algebra, 6 (1978), 345-352.

[3] V. Camillo, J. Zelmanowitz, *Dimension modules,* Pacific Jr. Math., 91 (1980), 249- 261.

[4] C.C. Ferrero, G. Ferrero, *Nearrings: some developments linked to semigroups and groups,* The Netherlands, Amsterdam, Kluwer Academic Publishers, 2002.

[5] P. Fleury, *A note on dualizing goldie dimension,* Canadian. Math. Bull., 17 (1974).

[6] J.D.P. Meldrum, *Near-rings and their links with groups,* Pitman Advanced Publishing Program, Boston-London-Melbourne, 1985.

[7] A. Oswald, *Near-rings in which every N-group is prinicpal*, Proc. London Math. Soc., 28 (1974), 67-88.

[8] G. Pilz, *Nearrings,* North Holland, 1983.

[9] Y.V. Reddy, Bh. Satyanarayana, *The f-prime radical in near-rings,* Indian J. Pure and Appl. Math., 17 (1986), 327-330.

[10] Reddy Y.V., Bh. Satyanarayana, *A note on modules,* Proc. Japan Academy, 63 (1987), 208-211.

[11] Reddy Y.V., Bh. Satyanarayana, *A note on N-groups,* Indian J. Pure and Appl. Math., 19 (1988), 842-845.

[12] Bh. Satyanarayana, *Tertiary decomposition in noetherian N-groups*, Communications in Algebra, 10 (1982), 1951-1963.

[13] Bh. Satyanarayana, *Primary decomposition in noetherian near rings,* Indian J. Pure and Appl. Math., 15 (1984), 127-130.

[14] Bh. Satyanarayana, *Contributions to nearring theory*, Doctoral Dissertation, Acharya Nagarjuna University, 1984.

[15] Bh. Satyanarayana, *A Note on E-direct and S-inverse systems*, Proc. of the Japan Academy, 64-A (1988), 292-295.

[16] Bh. Satyanarayana, *The injective hull of a module with FGD*, Indian J. Pure and Appl. Math., 20 (1989), 874-883.

[17] Bh. Satyanarayana, *Modules with finite spanning dimension*, J. Austral, Math. Society. (Series A), 57 (1994), 170-178.

[18] Bh. Satyanarayana, *On modules with finite goldie dimension*, J. Ramanujan Math. Society., 5 (1990), 61-75.

[19] Bh. Satyanarayana, *On essential E-irreducible submodules*, Proc., 4th Ramanujan Symposium on Algebra and its Applications, University of Madras, Feb 1-3 (1995), 127-129.

[20] Bh. Satyanarayana, *Contributions to near-ring theory,* VDM Verlag Dr Muller, Germany, 2010.

[21] Bh. Satyanarayana, Rao G. Koteswara, *On a class of modules and N-groups,* Journal of Indian Math. Society., 59 (1993), 3944.

[22] Bh. Satyanarayana, K. Syam Prasad, *A result on E-direct systems in N-groups,* Indian J. Pure and Appl. Math., 29 (1998), 285-287.

[23] Bh. Satyanarayana, K. Syam Prasad, *On direct and inverse systems in N-groups,* Indian J. Math. (BN Prasad Birth Commemoration Volume), 42 (2000), 183-192.

[24] Bh. Satyanarayana, K. Syam Prasad, *Linearly independent elements in N-groups with finite Goldie dimension*, Bulletin of the Korean Mathematical Society, 42 (2005), 433-441.

[25] Bh. Satyanarayana, K. Syam Prasad, *On finite goldie dimension of $M_n(N) - groupN^n$ nearrings and nearfields* (Editors: Hubert Kiechle, Alexander Kreuzer and Momme Johs Thomsen) (2005), (Proc. 18th International Conference on Nearrings and Nearfields, Universitat Bundeswar, Hamburg, Germany July 27-Aug 03, 2003), Springer Verlag, Netherlands, 2005, 301-310.

[26] Bh. Satyanarayana, K. Syam Prasad, *Discrete mathematics and graph theory*, Prentice Hall India Learning Ltd., 2014.

[27] Bh. Satyanarayana, K. Syam Prasad, D. Nagaraju, *A theorem on modules with finite Goldie dimension*, Soochow Journal of Mathematics, 32 (2006), 311-315.

[28] K. Syam Prasad, Bh. Satyanarayana, *Finite dimension in N-groups and fuzzy ideals of gamma nearrings*, VDM Verlag, Germany, 2011.

[29] Bh. Satyanarayana, and Syam Prasad K. Near Rings, *Fuzzy ideals, and graph theory*, Chapman and Hall, (2013), Taylor and Francis Group (London, New York).

[30] K. Syam Prasad, *Contributions to nearring theory II,* Doctoral Dissertation, Acharya Nagarjuna University, India, 2000.

[31] K. Syam Prasad, K.B. Srinivas, P.K. Harikrishnan, Bh. Satyanarayana (Editors), *Nearrings, nearfields and related topics*, Review Volume, World Scientific (Singapore), 2017.

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