On the power of simulation map for almost $\mathbb{Z}-$ contraction in $\mathcal{G}$-metric space with applications to the solution of the integral equation

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Abstract. In this manuscript, we explore the presence and uniqueness of a fixed point of almost $\mathbb{Z}-$contraction by means of simulation map in the framework of $\mathcal{G}-$metric spaces. Also, an illustrative example and an application to solve integral equation are given to help accessibility of the got outcomes.

Keywords: $\mathcal{G}-$metric space, almost $\mathbb{Z}-$contraction, simulation map.

1. Introduction

The Banach contraction rule is a one of the predominant outcomes in analysis and has continuously been at the front line of making and providing remarkable speculations for its researchers. Numerous authors have summed up and used the Banach principle in their relevant research ([1-4]). Along these lines, we can without much of a stretch presume that the biggest part of the fixed point theory has been involved by different speculations of the Banach contraction rule. Further, Khojasteh and Shukla [6] presented an alternate thought of simulation map by utilizing an idea of [3] and explored some fundamental

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properties. Subsequently, Argoubi [7] demonstrated fixed point results for non linear contraction in the casing of metric space.

Throughout the paper, $\mathcal{N}$ denotes set of natural numbers, $\mathcal{R}$ denotes set of real numbers.

Next, we review some essential definitions about contraction and the outcomes from the writing. Wardowski [1] defined the F-contraction as follows:

Definition 1.1. Let (X, d) be a metric space and let $f : X \to X$ be a self-mapping. Then f is called an $F-contraction$ on (X, d) if there exist $F \in \Im$ for all $x, y \in X,\ d(fx, fy) > 0 \Rightarrow \gamma + F(d(fx, fy)) \leq F(d(x, y))$, where $\Im$ be class of all mappings $F : (0, \infty) \to R$ such that:

(F1) F is strictly increasing function, that is, for all $a, b \in (0, \infty)$, if $a < b$,then $F(a) < F(b)$.

(F2) For every sequence $\{a_n\}$ of natural numbers, $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} F(a_n) = -\infty$.

(F3) There exists $q \in (0, 1)$ such that $\lim_{a\to 0^+}(a^q F(a)) = 0$.

F-weak contraction was established by Wardowski and Dung [2] in 2014 which is defined as follows:

Definition 1.2. [2] Let (X, d) be a metric space and $\mathcal{T} : X \to X$ be a function. $\mathcal{T}$ is known as F-weak contraction on (X, d) if there exist $F \in \Im$ and $\gamma > 0$ such that for all $x, y \in X$, $d(\mathcal{T}x, \mathcal{T}y) > 0 \Rightarrow \gamma + F(d(\mathcal{T}x, \mathcal{T}y)) \leq F(max\{d(x, y), d(\mathcal{T}x, x), d(y, \mathcal{T}y), \frac{d(x,\mathcal{T}y)+d(y,\mathcal{T}x)}{2}\})$.

Definition 1.3 ([3, 4])**.** Let (X, d) be a metric space and let $\sigma : X \to X$ be a self-mapping. Then σ is called an almost $\mathbb{Z}-$ contraction on (X, d) if there exist $\beta \geq 0$ and $\beta_1 < 1$ such that for all $x, y \in X$, $d(\sigma x, \sigma y) \leq \beta_1 d(x, y) + \beta f(x, y)$.

Definition 1.4 ([5])**.** The mapping $\theta : [0, \infty) \times [0, \infty) \to \mathcal{R}$ is said to be a simulation function, if the following properties hold: $(\theta_1)\theta(0, 0) = 0$, $(\theta_2)\theta(a, b) < b - a$, for all $a, b > 0$, (θ_3) if $\{a_n\}, \{b_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty}\{a_n\} = \lim_{n\to\infty}\{b_n\} = \ell \in (0, \infty)$, then $\lim_{n\to\infty} \sup \theta(a_n, b_n) < 0$.

The family of all simulation functions $\theta : [0, \infty) \times [0, \infty) \to \mathcal{R}$ is denoted by $\mathbb{Z}$.

Definition 1.5 ([6])**.** Let (X, d) be a metric space and $\theta \in \mathbb{Z}$. The self map $\sigma : X \to X$ is $\mathbb{Z}-$contraction with respect to θ if for each $x, y \in X$,

$$\theta(d(\sigma x, \sigma y), d(x, y)) \geq 0. \tag{1.1}$$

Definition 1.6 ([8])**.** Let (X, d) be a metric space and $\theta \in \mathbb{Z}$. Then, the self map $\sigma : X \to X$ is said to be almost $\mathbb{Z}-$contraction if for each $x, y \in X$, we can find a positive constant β such that

$$\theta(d(\sigma x, \sigma y), d(x, y) + \beta m(x, y)) \geq 0, \tag{1.2}$$

where

$$m(x, y) = \min\{d(x, \sigma x), d(y, \sigma y), d(x, \sigma y), d(y, \sigma x)\}.$$

Definition 1.7 ([9]). Let X be a non void set and $\mathcal{G} : X^3 \to [0, \infty)$ be a map which fulfils the accompanying conditions:

(i) $\mathcal{G}(x_1, y_1, z_1) = 0$ if $x_1 = y_1 = z_1$;

(ii) $0 < \mathcal{G}(x_1, x_1, y_1)$ whenever $x_1 \neq y_1$, for all $x_1, y_1 \in X$;

(iii) $\mathcal{G}(x_1, x_1, y_1) \leq \mathcal{G}(x_1, y_1, z_1)$, $y_1 \neq z_1$;

(iv) $\mathcal{G}(x_1, y_1, z_1) = \mathcal{G}(x_1, z_1, y_1) = \mathcal{G}(y_1, x_1, z_1) = \mathcal{G}(z_1, x_1, y_1)$
$= \mathcal{G}(y_1, z_1, x_1) = \mathcal{G}(z_1, y_1, x_1)$;

(v) $\mathcal{G}(x_1, y_1, z_1) \leq \mathcal{G}(x_1, a_1, a_1) + \mathcal{G}(a_1, y_1, z_1)$;

for every $x_1, y_1, z_1, a_1 \in X$, then the function $\mathcal{G}$ is said to be $\mathcal{G}-$metric on X and $(X, \mathcal{G})$ is known as $\mathcal{G}-$metric space.

In this paper, we consolidate the view of simulation map, $\mathcal{G}-$metric space and indispensable nature of contractive mappings to construct fixed point hypotheses in the casing of generalized metric space.

The principle point of our examination is to talk about the reasonability of application to solve the integral equation with the assistance of our main theorem.

2. Main results

Definition 2.1. Let $(X, \mathcal{G})$ be a $\mathcal{G}-$metric space and $\theta \in \mathbb{Z}$. Then, the self map $\sigma : X \to X$ is said to be almost $\mathbb{Z}-$contraction if for each $x, y, z \in X$, we can find a positive constant β such that

$$\theta(\mathcal{G}(\sigma x, \sigma y, \sigma z), \mathcal{G}(x, y, z) + \beta m(x, y, z)) \geq 0, \tag{2.1}$$

where

$$\begin{aligned} m(x, y, z) = \min\{&\mathcal{G}(x, \sigma y, \sigma y), \mathcal{G}(y, \sigma x, \sigma x), \mathcal{G}(y, \sigma z, \sigma z), \mathcal{G}(z, \sigma y, \sigma y), \\ &\mathcal{G}(z, \sigma x, \sigma x), \mathcal{G}(x, \sigma z, \sigma z)\}. \end{aligned} \tag{2.2}$$

Remark 2.2. If σ is almost $\mathbb{Z}-$contraction with respect to $\theta \in \mathbb{Z}$, then we have $\mathcal{G}(\sigma x, \sigma y, \sigma z) < \mathcal{G}(x, y, z) + \beta m(x, y, z)$, where $x, y, z \in X$.

Theorem 2.3. *Let $(X, \mathcal{G})$ be a complete $\mathcal{G}$ metric space and $\sigma : X \to X$ be an almost $\mathbb{Z}$ contraction with respect to $\theta \in \mathbb{Z}$. Then, σ has a fixed point. Moreover, the sequence $\{\sigma^s z_0\}$ converges to fixed point of σ for each $z_0 \in X$.*

Proof. Step 1. Suppose that, $z_{s+1} = \sigma^s z_0 = \sigma z_s$, where $s \in \mathcal{N}$ and $z_0 \in X$.

If $\exists\, s \in \mathcal{N}$ such that $z_{s+1} = z_s$, then $\sigma z_s = z_s$. So, z_s is fixed point of σ.

Let us suppose that $z_s \neq z_{s+1}$ for each $s \in \mathcal{N}$. Then, $\mathcal{G}(z_s, z_{s+1}, z_{s+1}) > 0$ for all $s \in \mathcal{N}$.

Firstly, we prove that $\mathcal{G}(z_s, z_{s+1}, z_{s+1}) = 0$. On account of (2.2), we get that

$$\begin{aligned}
f(z_{s-1}, z_s, z_s) &= \min\{\mathcal{G}(z_{s-1}, \sigma z_s, \sigma z_s), \mathcal{G}(z_s, \sigma z_{s-1}, \sigma z_{s-1}), \mathcal{G}(z_s, \sigma z_s, \sigma z_s),\\
&\quad \mathcal{G}(z_s, \sigma z_s, \sigma z_s), \mathcal{G}(z_s, \sigma z_{s-1}, \sigma z_{s-1}), \mathcal{G}(z_{s-1}, \sigma z_s, \sigma z_s)\}\\
&= \min\{\mathcal{G}(z_{s-1}, z_{s+1}, z_{s+1}), \mathcal{G}(z_s, z_s, z_s),\\
&\quad \mathcal{G}(z_s, z_{s+1}, z_{s+1}), \mathcal{G}(z_s, z_{s+1}, z_{s+1}), \mathcal{G}(z_s, z_s, z_s), \mathcal{G}(z_{s-1}, z_{s+1}, z_{s+1})\}\\
&= 0. \qquad (2.3)
\end{aligned}$$

On account of inequality (2.1), we get that

$$\theta(\mathcal{G}(\sigma z_{s-1}, \sigma z_s, \sigma z_s), \mathcal{G}(z_{s-1}, z_s, z_s) + \beta f(z_{s-1}, z_s, z_s)) \geq 0.$$

Taking (2.3) into account, we obtain

$$\begin{aligned}
0 &\leq \theta(\mathcal{G}(\sigma z_{s-1}, \sigma z_s, \sigma z_s), \mathcal{G}(z_{s-1}, z_s, z_s))\\
&= \theta(\mathcal{G}(z_s, z_{s+1}, z_{s+1}), \mathcal{G}(z_{s-1}, z_s, z_s))\\
&< \mathcal{G}(z_{s-1}, z_s, z_s) - \mathcal{G}(z_s, z_{s+1}, z_{s+1})
\end{aligned}$$

which implies that

$$\mathcal{G}(z_s, z_{s+1}, z_{s+1}) < \mathcal{G}(z_{s-1}, z_s, z_s).$$

Therefore, $\{\mathcal{G}(x_{s+1}, x_s, x_s)\}$ is non negative decreasing sequence of real numbers where $s \in \mathcal{N}$. So, $\exists\, \lambda \geq 0$ such that

$$\lim_{s\to\infty} \mathcal{G}(z_s, z_{s+1}, z_{s+1}) = \lambda.$$

Now, we will indicate that

$$\lim_{s\to\infty} \mathcal{G}(z_s, z_{s+1}, z_{s+1}) = 0.$$

Suppose that $\lambda > 0$. Let $\{c_s\}$ and $\{d_s\}$ be sequences such that $\mathcal{G}(z_s, z_{s+1}, z_{s+1}) = c_s$ and $\mathcal{G}(z_{s-1}, z_s, z_s) = d_s$.

Now,

$$\lim_{s\to\infty} c_s = \lim_{s\to\infty} d_s = \lambda.$$

Accordingly, from θ_3 we deduce that

$$0 \leq \lim_{n\to\infty} \sup \theta((\mathcal{G}(z_s, z_{s+1}, z_{s+1}), \mathcal{G}(z_{s-1}, z_s, z_s)) < 0,$$

which is a contradiction. Thus, we have

$$\lim_{n\to\infty} \mathcal{G}(z_s, z_{s+1}, z_{s+1}) = 0. \qquad (2.4)$$

Step 2. We assert that $\{z_s\}$ is bounded.

Suppose, on the contrary, that $\{z_s\}$ is unbounded. So, there exists a subsequence $\{z_{s_u}\}$ so that $s_1 = 1$ and s_{u+1} is the smallest integer larger than s_u such that $\mathcal{G}(z_{s_{u+1}}, z_{s_u}, z_{s_u}) > 1$ and $\mathcal{G}(z_u, z_r, z_r) \leq 1$, for every $r \in [s_u, s_{u+1} - 1]$.

With the assistance of triangle inequality, we obtain

$$\begin{aligned} 1 &< \mathcal{G}(z_{s_{u+1}}, z_{s_u}, z_{s_u}) \\ &\leq \mathcal{G}(z_{s_{u+1}}, z_{s_{u+1}-1}, z_{s_{u+1}-1}) + \mathcal{G}(z_{s_{u+1}-1}, z_{s_u}, z_{s_u}) \\ &\leq 1 + \mathcal{G}(z_{s_{u+1}}, z_{s_{u+1}-1}, z_{s_{u+1}-1}). \end{aligned}$$

Making $u \to \infty$ and using(2.4), we get

$$\lim_{u\to\infty} \mathcal{G}(z_{s_{u+1}}, z_{s_u}, z_{s_u}) = 1. \tag{2.5}$$

Since, σ is an almost $\mathbb{Z}$ contraction with respect to θ, we conclude that

$$\mathcal{G}(z_{s_u+1}, z_{s_u}, z_{s_u}) \leq \mathcal{G}(z_{s_{u+1}-1}, z_{s_u-1}, z_{s_u-1}).$$

On account of triangle inequality, we obtain

$$\begin{aligned} 1 &< \mathcal{G}(z_{s_{u+1}}, z_{s_u}, z_{s_u}) \\ &\leq \mathcal{G}(z_{s_{u+1}-1}, z_{s_u-1}, z_{s_u-1}) \\ &\leq \mathcal{G}(z_{s_{u+1}-1}, z_{s_u}, z_{s_u}) + \mathcal{G}(z_{s_u}, z_{s_u-1}, z_{s_u-1}) \\ &\leq 1 + \mathcal{G}(z_{s_u}, z_{s_u-1}, z_{s_u-1}). \end{aligned}$$

Letting $u \to \infty$ and using (2.4), we get

$$\lim_{u\to\infty} \mathcal{G}(z_{s_{u+1}-1}, z_{s_u-1}, z_{s_u-1}) = 1. \tag{2.6}$$

Now, taking(2.2) into account

$$\begin{aligned} f(z_{s_{u+1}-1}, z_{s_u-1}, z_{s_u-1}) = \min\{&\mathcal{G}(z_{s_{u+1}-1}, \sigma z_{s_u-1}, \sigma z_{s_u-1}), \\ &\mathcal{G}(z_{s_u-1}, \sigma z_{s_{u+1}-1}, \sigma z_{s_{u+1}-1}), \\ &\mathcal{G}(z_{s_u-1}, \sigma z_{s_u-1}, \sigma z_{s_u-1}), \mathcal{G}(z_{s_u-1}, \sigma z_{s_u-1}, \sigma z_{s_u-1}), \\ &\mathcal{G}(z_{s_u-1}, \sigma z_{s_{u+1}-1}, \sigma z_{s_{u+1}-1}), \mathcal{G}(z_{s_{u+1}-1}, \sigma z_{s_u-1}, \sigma z_{s_u-1})\}. \end{aligned}$$

Letting $u \to \infty$ and using(2.4), we get

$$f(z_{s_{u+1}-1}, z_{s_u-1}, z_{s_u-1}) = 1. \tag{2.7}$$

With the assistance of (2.1) and θ_2, we obtain

$$\begin{aligned} 0 \leq \theta(&\mathcal{G}(\sigma z_{s_{u+1}-1}, \sigma z_{s_u-1}, \sigma z_{s_u-1}), \\ &\mathcal{G}(z_{s_{u+1}-1}, z_{s_u-1}, z_{s_u-1}) + \beta f(z_{s_{u+1}-1}, \\ &z_{s_u-1}, z_{s_u-1})) < \mathcal{G}(z_{s_{u+1}-1}, z_{s_u-1}, z_{s_u-1}) \\ &+ \beta f(z_{s_{u+1}-1}, z_{s_u-1}, z_{s_u-1}) - \mathcal{G}(\sigma z_{s_{u+1}-1}, \sigma z_{s_u-1}, \sigma z_{s_u-1}) \\ &= \mathcal{G}(z_{s_{u+1}-1}, z_{s_u-1}, z_{s_u-1}) + \beta f(z_{s_{u+1}-1}, z_{s_u-1}, z_{s_u-1}) - \mathcal{G}(z_{s_{u+1}}, z_{s_u}, z_{s_u}) \end{aligned}$$

which yields that

$$\mathcal{G}(z_{s_{u+1}}, z_{s_u}, z_{s_u}) < \mathcal{G}(z_{s_{u+1}-1}, z_{s_u-1}, z_{s_u-1}) + \beta f(z_{s_{u+1}-1}, z_{s_u-1}, z_{s_u-1}).$$

Let $\{w_u\}$ and $\{v_u\}$ be sequences such that $\mathcal{G}(z_{s_{u+1}}, z_{s_u}, z_{s_u}) = w_u$ and $\mathcal{G}(z_{s_{u+1}-1}, z_{s_u-1}, z_{s_u-1}) + \beta f(z_{s_{u+1}-1}, z_{s_u-1}, z_{s_u-1}) = v_u$.

Accordingly, from (2.5), (2.6) and (2.7), we get $\lim_{u\to\infty} \mathcal{G}(z_{s_{u+1}}, z_{s_u}, z_{s_u})$=1= $\lim_{u\to\infty} \mathcal{G}(z_{s_{u+1}-1}, z_{s_u-1}, z_{s_u-1})+\beta f(z_{s_{u+1}-1}, z_{s_u-1}, z_{s_u-1})$. On account of assumption of θ_3, we obtain $\lim_{u\to\infty} \sup \theta(\mathcal{G}(z_{s_{u+1}}, z_{s_u}, z_{s_u}), \mathcal{G}(z_{s_{u+1}-1}, z_{s_u-1}, z_{s_u-1}) +\beta f(z_{s_{u+1}-1}, z_{s_u-1}, z_{s_u-1}))$<0 which is a contradiction. Therefore, $\{z_s\}$ is bounded.

Step 3. We assert that $\{z_s\}$ is a cauchy sequence. Let us suppose

$$\mathcal{O}_s = \sup\{\mathcal{G}(z_c, z_d, z_d) : c, d \geq s\}.$$

Now, $\{\mathcal{O}_s\}$ is decreasing sequence of positive entries, therefore, there exists $\mathcal{O} \geq 0$ such that

$$\lim_{s\to\infty} \mathcal{O}_s = \mathcal{O}.$$

When $\mathcal{O} > 0$, then applying the definition of $\mathcal{O}_s$, there exists r_u, s_u satisfying $r_u > s_u \geq u$ and $\mathcal{O}_s - \frac{1}{s} < \mathcal{G}(z_{r_u}, z_{s_u}, z_{s_u}) \leq \mathcal{O}_s$ which yields that

$$\mathcal{O} = \lim_{s\to\infty} \mathcal{G}(z_{r_u}, z_{s_u}, z_{s_u}) \tag{2.8}$$

Again, with the assistance of triangle inequality, we obtain $\mathcal{G}(z_{r_u}, z_{s_u}, z_{s_u}) \leq \mathcal{G}(z_{r_u-1}, z_{s_u-1}, z_{s_u-1}) \leq \mathcal{G}(z_{r_u-1}, z_{r_u}, z_{r_u})+\mathcal{G}(z_{r_u}, z_{s_u}, z_{s_u})+\mathcal{G}(z_{s_u}, z_{s_u-1}, z_{s_u-1})$. Letting $u \to \infty$ and taking (2.6), (2.8) into account, we obtain

$$\mathcal{G}(z_{r_u-1}, z_{s_u-1}, z_{s_u-1}) = \mathcal{O}. \tag{2.9}$$

With the aid of remark(2.2), we have

$$\mathcal{G}(z_{r_u}, z_{s_u}, z_{s_u}) < \mathcal{G}(z_{r_u-1}, z_{s_u-1}, z_{s_u-1}) + \beta f(z_{r_u-1}, z_{s_u-1}, z_{s_u-1}). \tag{2.10}$$

Accordingly, from (2.4), we get

$$\lim_{u\to\infty} f(z_{r_u-1}, z_{s_u-1}, z_{s_u-1}) = 0. \tag{2.11}$$

Let $\{w_u\}$ and $\{v_u\}$ be sequences such that $\mathcal{G}(z_{r_u}, z_{s_u}, z_{s_u}) = w_u$ and $\mathcal{G}(z_{r_u-1}, z_{s_u-1}, z_{s_u-1}) + \beta f(z_{r_u-1}, z_{s_u-1}, z_{s_u-1}) = v_u$.

Using (2.8) to (2.11), we obtain

$$\lim_{u\to\infty} \mathcal{G}(z_{r_u}, z_{s_u}, z_{s_u}) = \mathcal{O} = \lim_{u\to\infty} \mathcal{G}(z_{r_u-1}, z_{s_u-1}, z_{s_u-1}) + \beta f(z_{r_u-1}, z_{s_u-1}, z_{s_u-1}).$$

On account of assumption of θ_3, we obtain

$$\lim_{u\to\infty} \sup \theta(\mathcal{G}(z_{r_u}, z_{s_u}, z_{s_u}), \mathcal{G}(z_{r_u-1}, z_{s_u-1}, z_{s_u-1}) + \beta f(z_{r_u-1}, z_{s_u-1}, z_{s_u-1})) < 0.$$

which is a contradiction. Therefore, $\mathcal{O} = 0$, which yields that $\{z_s\}$ is a Cauchy sequence. Due to completeness of $\mathcal{G}$ metric space, there exists $\varrho \in X$ so that $\lim_{s\to\infty} z_s = \varrho$.

Step 4. In the last step, we assert that ϱ is fixed point of σ.

Let $\sigma\varrho \neq \varrho$. Thus, $\mathcal{G}(\varrho, \sigma\varrho, \sigma\varrho) > 0$. With the assistance of (2.1), θ_2 and θ_3, we obtain

$$\begin{aligned}
0 &\leq \lim_{u\to\infty} \sup \theta[\mathcal{G}(\sigma z_s, \sigma\varrho, \sigma\varrho), \mathcal{G}(z_s, \varrho, \varrho) + \beta f(z_s, \varrho, \varrho)] \\
&< \lim_{u\to\infty} \sup[\mathcal{G}(z_s, \varrho, \varrho) + \beta f(z_s, \varrho, \varrho) - \mathcal{G}(\sigma z_s, \sigma\varrho, \sigma\varrho)] = -\mathcal{G}(\varrho, \sigma\varrho, \sigma\varrho),
\end{aligned}$$

which yields that $\mathcal{G}(\varrho, \sigma\varrho, \sigma\varrho) = 0$. Thus, $\sigma\varrho = \varrho$, which proves that ϱ is fixed point of σ. □

Theorem 2.4. *Let $(X, \mathcal{G})$ be a complete $\mathcal{G}$ metric space and $\sigma : X \to X$ be an almost $\mathbb{Z}$ contraction with respect to $\theta \in \mathbb{Z}$. If, σ has a fixed point, then it is unique.*

Proof. In view of Theorem 2.3, we ensure the presence of settled point of map σ, that is $\varrho = \sigma\varrho$. Next, we claim that if ϱ_1 and ϱ_2 are the fixed points of σ, then $\varrho_1 = \varrho_2$. On account of (2.1), we acquire that

$$\begin{aligned}
0 &\leq \theta[\mathcal{G}(\sigma\varrho_1, \sigma\varrho_2, \sigma\varrho_2), \mathcal{G}(\varrho_1, \varrho_2, \varrho_2) + \beta f(\varrho_1, \varrho_2, \varrho_2)] \\
&= \theta[\mathcal{G}(\sigma\varrho_1, \sigma\varrho_2, \sigma\varrho_2), \mathcal{G}(\varrho_1, \varrho_2, \varrho_2) + \beta \min\{\mathcal{G}(\varrho_1, \sigma\varrho_2, \sigma\varrho_2), \mathcal{G}(\varrho_2, \sigma\varrho_1, \sigma\varrho_1), \\
&\qquad \mathcal{G}(\varrho_2, \sigma\varrho_2, \sigma\varrho_2), \mathcal{G}(\varrho_2, \sigma\varrho_2, \sigma\varrho_2), \mathcal{G}(\varrho_2, \sigma\varrho_1, \sigma\varrho_1), \mathcal{G}(\varrho_1, \sigma\varrho_2, \sigma\varrho_2)\} \\
&= \theta[\mathcal{G}(\sigma\varrho_1, \sigma\varrho_2, \sigma\varrho_2), \mathcal{G}(\varrho_1, \varrho_2, \varrho_2)] \\
&< \mathcal{G}(\sigma\varrho_1, \sigma\varrho_2, \sigma\varrho_2) - \mathcal{G}(\sigma\varrho_1, \sigma\varrho_2, \sigma\varrho_2) = 0,
\end{aligned}$$

which is contradiction. Consequently, $\varrho_1 = \varrho_2$, which indicates that fixed point of σ is unique. □

Illustrative Example 2.5. Let $X = [0, 1]$ and $\mathcal{G}$ be defined as $\mathcal{G}(x_1, y_1, z_1) =\mid x_1 - y_1 \mid + \mid y_1 - z_1 \mid + \mid z_1 - x_1 \mid$.

Now, we define $\sigma : X \to X$ as $\sigma z = \frac{1}{2} - z$, for each $z \in X$. We shall prove that $\sigma : X \to X$ is an almost $\mathbb{Z}$ contraction with respect to $\theta \in \mathbb{Z}$, but σ is not $\mathbb{Z}$ contraction with respect to $\theta \in \mathbb{Z}$, where for each $a, b > 0$, $\theta(a, b) = \delta b - a$, where $\delta \in [0, 1)$. For each distict elements z_1, z_2 of X,

$$\begin{aligned}
\theta(\mathcal{G}(\sigma z_1, \sigma z_2, \sigma z_2), \mathcal{G}(z_1, z_2, z_2)) &= \delta\mathcal{G}(z_1, z_2, z_2) - \mathcal{G}(\sigma z_1, \sigma z_2, \sigma z_2) \\
&= \delta 2 \mid z_1 - z_2 \mid -2\left|\frac{1}{2} - z_1 - (\frac{1}{2} - z_2)\right| \\
&= 2(\delta \mid z_1 - z_2 \mid - \mid z_1 - z_2 \mid) \\
&< 2(\mid z_1 - z_2 \mid - \mid z_1 - z_2 \mid) \\
&= 2 \times 0 = 0.
\end{aligned}$$

which yields that σ is not $\mathbb{Z}$ contraction with respect to $\theta \in \mathbb{Z}$, but σ has a unique fixed point $z = \frac{1}{4}$. Now, $\mathcal{G}(z_1, \sigma z_2, \sigma z_2) = 2|z_1 - \sigma z_2| = 2|z_1 - (\frac{1}{2} - z_2)| = 2|z_1 + z_2 - \frac{1}{2}|$. Thus,

$$\begin{aligned} f(z_1, z_2, z_3) = \min\{&\mathcal{G}(z_1, \sigma z_2, \sigma z_2), \mathcal{G}(z_2, \sigma z_1, \sigma z_1), \mathcal{G}(z_2, \sigma z_3, \sigma z_3), \\ &\mathcal{G}(z_3, \sigma z_2, \sigma z_2), \mathcal{G}(z_3, \sigma z_1, \sigma z_1), \mathcal{G}(z_1, \sigma z_3, \sigma z_3)\} \\ = \min\{&2|z_1 + z_2 - \frac{1}{2}|, 2|z_1 + z_2 - \frac{1}{2}|, 2|z_3 + z_2 - \frac{1}{2}|, \\ &2|z_3 + z_2 - \frac{1}{2}|, 2|z_1 + z_3 - \frac{1}{2}|, 2|z_1 + z_3 - \frac{1}{2}|\}. \end{aligned}$$

Further,

$$\begin{aligned} &\theta(\mathcal{G}(\sigma z_1, \sigma z_2, \sigma z_2), \mathcal{G}(z_1, z_2, z_2) + \beta f(z_1, z_2, z_3)) \\ &= \delta[\mathcal{G}(z_1, z_2, z_2) + \beta f(z_1, z_2, z_3)] - \mathcal{G}(\sigma z_1, \sigma z_2, \sigma z_2) \\ &= \delta[2 \mid z_1 - z_2 \mid + \beta \min\{2|z_1 + z_2 - \frac{1}{2}|, 2|z_1 + z_2 - \frac{1}{2}|, \\ &\quad 2|z_3 + z_2 - \frac{1}{2}|, 2|z_3 + z_2 - \frac{1}{2}|, 2|z_1 + z_3 - \frac{1}{2}|, \\ &\quad 2|z_1 + z_3 - \frac{1}{2}|\}] - 2 \mid z_1 - z_2 \mid . \end{aligned}$$

Two cases arise:

Case 1. When $z_1 = z_2 = z_3$, then

$$\theta(\mathcal{G}(\sigma z_1, \sigma z_2, \sigma z_2), \mathcal{G}(z_1, z_2, z_2) + \beta f(z_1, z_2, z_3)) = 2\delta\beta|2z_1 - \frac{1}{2}| \geq 0.$$

Case 2. Let $z_1 > z_2 > z_3$, then $\theta(\mathcal{G}(\sigma z_1, \sigma z_2, \sigma z_2), \mathcal{G}(z_1, z_2, z_2) + \beta f(z_1, z_2, z_3)) = 2\delta|z_1 - z_2| + 2\delta\beta|z_2 + z_3 - \frac{1}{2}| - 2|z_1 - z_2|$. Now, choose $\beta = \frac{1}{3}$ and $\delta = 15$, we acquire $\theta(\mathcal{G}(\sigma z_1, \sigma z_2, \sigma z_2), \mathcal{G}(z_1, z_2, z_2) + \beta f(z_1, z_2, z_3)) = 30 \mid z_1 - z_2 \mid + 2 \times \frac{1}{3} \times 15|z_2 + z_3 - \frac{1}{2}| - 2 \times \mid z_1 - z_2 \mid = 28 \mid z_1 - z_2 \mid + 10|z_2 + z_3 - \frac{1}{2}| \geq 0$. Therefore, all the assumptions of Theorem (2.3) are satisfied. Consequently, σ has a unique fixed point $\sigma = \frac{1}{4}$.

Corollary 2.6 ([10]). *Let $(X, \mathcal{G})$ be a $\mathcal{G}$ metric space where $\theta \in \mathbb{Z}$, $\beta < 1$ and the self map σ satisfies the following condition*

$$\theta(\mathcal{G}(\sigma x, \sigma y, \sigma z), \mathcal{G}(x, y, z)) \geq 0, \tag{2.12}$$

for each $x, y, z \in X$, Then, σ has a unique fixed pint.

Proof. The result follows if $\delta = 0$ and $\theta(a, b) = \beta b - a$ in Theorem 2.3. □

Corollary 2.7 ([10]). *Let $(X, \mathcal{G})$ be a $\mathcal{G}$ metric space where $\theta \in \mathbb{Z}$, α is a self map which is upper semi continuous fulfilling $\alpha(s) < s$ and the self map σ satisfies the following condition*

$$\theta(\mathcal{G}(\sigma x, \sigma y, \sigma z), \alpha(\mathcal{G}(x, y, z)) \geq 0, \tag{2.13}$$

for each $x, y, z \in X$. Then, σ has a unique fixed pint.

3. Application

Let self map σ be defined as

$$\sigma\alpha(\gamma) = \kappa(\gamma) + \beta \int_m^n S(\gamma, x, x)\eta(x, \alpha(x), \alpha(x))dx, \tag{3.1}$$

where $\gamma \in [m, n]$. Let X be equipped with the metric $\mathcal{G}$ which is defined as $\mathcal{G}(\alpha, \kappa, \kappa) = 2 \sup |\alpha(\gamma) - \kappa(\gamma)|$.

Theorem 3.1. *The integral equation* (3.1) *has a unique solution if the following conditions are fulfilled:*
 (i) $\sup \int_m^n S(\gamma, x, x)dx \leq \frac{1}{2n-2m}$*;*
 (ii) $\mathcal{G}(x, \alpha, \alpha) - \mathcal{G}(x, \kappa, \kappa) \leq \tau(|\alpha - \kappa|)$*;*
 (iii) $|\beta| \leq 1$*,*
where τ is non decreasing continuous map having $\tau(s) < s$, for each $s > 0$.

Proof. Now,

$$\begin{aligned}
&\mathcal{G}(\sigma\alpha_1, \sigma\alpha_2, \sigma\alpha_2) = 2 \sup |\sigma\alpha_1(\gamma) - \sigma\alpha_2(\gamma)| \\
&= 2 \sup |\kappa(\gamma) + \beta \int_m^n S(\gamma, x, x)\eta(x, \alpha_1(x), \alpha_1(x))dx - \kappa(\gamma) - \\
&\quad \beta \int_m^n S(\gamma, x, x)\eta(x, \alpha_2(x), \alpha_2(x))dx| \\
&= 2|\beta| \sup \left| \int_m^n S(\gamma, x, x)(\eta(x, \alpha_1(x), \alpha_1(x)) - \eta(x, \alpha_2(x), \alpha_2(x)))dx \right| \\
&\leq 2|\beta| \sup \left[\int_m^n S(\gamma, x, x)dx \int_m^n (\eta(x, \alpha_1(x), \alpha_1(x)) - \eta(x, \alpha_2(x), \alpha(x)))|dx \right] \\
&\leq 2|\beta| \times \frac{1}{2n - 2m} \left[\int_m^n \tau(|\alpha_1(x) - \alpha_2(x)|)dx \right] \\
&\leq |\beta| \times \frac{1}{n - m} \left[\int_m^n \tau(\mathcal{G}(\alpha_1, \alpha_2, \alpha_2))dx \right] \\
&= |\beta| \times \frac{1}{n - m} \tau(\mathcal{G}(\alpha_1, \alpha_2, \alpha_2)) \times n - m \\
&= |\beta|\tau(\mathcal{G}(\alpha_1, \alpha_2, \alpha_2)) \leq \tau(\mathcal{G}(\alpha_1, \alpha_2, \alpha_2)).
\end{aligned}$$

Therefore, σ has a unique solution in X, which means that (3.1) has a unique solution in X. □

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