Further properties of Hurwitz series rings and Hurwitz polynomials rings

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Abstract. For a commutative ring with unity \( R \), a commutative ring is defined called the ring of Hurwitz series, \( HR \). As a subring of this ring, \( HR \), the ring of Hurwitz polynomials is defined, \( hR \). In this paper, we characterize pure ideals in the ring \( hR \). Then we characterize when the ring \( hR \) is an almost \( P P \)-ring and a \( PF \)-ring. Finally, for the ring \( R \) satisfying \( x^n = x \) for a fixed positive integer \( n \), we prove that every prime ideal of the rings \( HR(hR) \) is maximal and so their spectrum is completely characterized.

Keywords: Hurwitz series ring, Hurwitz polynomial ring, pure ideal, \( PF \)-ring, almost \( PP \)-ring, spectrum of Hurwitz series ring, spectrum of Hurwitz polynomial ring, Zariski spectrum.

1. Introduction

All rings considered in this paper are commutative with unity \( 1 \neq 0 \). For a ring \( R \), a new ring, \( HR \), was defined in Keigher [13] as follows:

\( HR \) is the set of all infinite sequences of elements of \( R \), \((a_n)\), together with operations

\[
(a_n) + (b_n) = (a_n + b_n),
\]

\[
(a_n)\cdot (b_n) = (c_n),
\]

where \( c_n = \sum_{h+k=n}^n \binom{n}{k} a_h b_k \), where \( \binom{n}{k} \) is the binomial coefficient.

Now, \( HR \) is a commutative ring with unity \((1, 0, 0, 0, \ldots)\). This ring is called the ring of Hurwitz series over the ring \( R \). Let

\[
hR = \{(a_n) : a_n = 0 \text{ for all } n \geq k \text{ for some } k\}.
\]

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Then this is a subring of $HR$, called the ring of Hurwitz polynomials over $R$. The algebraic structure of this ring $HR$ was studied by several authors, e.g. Keigher [13], Benhissi [6, 7, 8], Abu Dayah [1]. Our aim is to study some properties of the ring of Hurwitz polynomials, $hR$.

Notice that two ring homomorphisms can be defined: \( \varepsilon : HR \rightarrow R \) defined by \( \varepsilon((a_n)) = a_0 \) and \( \lambda : R \rightarrow HR \) defined by \( \lambda(a) = (a, 0, 0, 0, \ldots) \).

Also, \( R \) is called a differential ring with a derivation \( d : R \rightarrow R \) if the derivation satisfies Leibniz rule \( d(ab) = a d(b) + d(a) b \) for all \( a, b \in R \). An ideal \( I \) is differential ideal of \( R \), if for every \( x \in I, d(x) \in I \). \( HR \) is a differential ring with a derivation \( d : HR \rightarrow HR, d((a_n)) = (a_{n+1}) \) for every \( (a_n) \in HR \).

Recall that an ideal \( I \) of a ring \( R \) is called pure ideal if for \( a \in I \), there exists \( b \in I \) such that \( ab = a \). Pure ideals are interesting because they classify important families of rings, e.g. Von Neumman rings and PF-rings, see Al-Ezeh [2, 3, 4, 5], Jondrup [12], DeMarco [10]. Also, a ring \( R \in \mathcal{C} \), if for any pure ideal \( I \) in \( R \), \( I \) is generated by a family of idempotents. Moreover, every pure ideal is a differential ideal.

In this paper we characterize pure ideals in the ring \( hR \) and give its relationship to pure ideals of the ring \( R \).

A ring \( R \) is called a PF-ring if every principal ideal of \( R \) is a flat \( R \)-module. These rings were studied by several authors. Al-Ezeh proved in [4] that a ring \( R \) is a PF-ring if and only if the annihilator of each \( a \in R, \text{ann}_R(a) \), is a pure ideal in \( R \). We characterize in this paper when the ring \( hR \) is a PF-ring. Recall that a ring \( R \) is called an almost PP-ring if the annihilator of each \( a \in R, \text{ann}_R(a) \), is generated by a set of idempotents. These rings were studied in literature as a generalization of PP-rings, i.e. rings in which every principal ideal is a projective \( R \)-module, see Al-Ezeh [5] and Burgess [9]. In this paper, we prove that \( hR \) is an almost PP-ring if and only if \( R \) is an almost PP-ring and \( R \) is torsion free. Moreover, we prove that \( hR \in \mathcal{C} \).

Finally, for the ring \( R \) satisfying \( x^n = x \) for some positive integer \( n \geq 2 \), we characterize the Zariski spectrum of the ring, \( hR \).

2. Pure ideals in \( hR \) ring

Note that \( HR(hR) \) means the ring \( HR \) or \( hR \). Abu Deyah [1] has given a partial characterization of the pure ideals in \( HR \). In this section we characterize the pure ideals in a ring \( hR \).

**Theorem 2.1.** Let \( I \) be an ideal in a ring \( hR \). Then \( I \) is pure ideal in \( hR \) if and only if \( I = [\lambda(\varepsilon(I))] \) and \( \varepsilon(I) \) is pure ideal in \( R \).

**Proof.** Suppose that \( I \) is pure ideal in \( hR \). Let \( (a_n) = (a_0, a_1, \ldots, a_n) \in I \). But \( I \) is a pure ideal, so there exists \( (b_m) = (b_0, b_1, \ldots, b_m) \in I \) such that
Suppose that \( (a_n)(b_m) = (a_n) \). Now, \( a_0 \) and \( b_0 \) in \( \varepsilon(I) \), so we have inductively

\[
\begin{align*}
    a_1 &= a_0b_1 + a_1b_0 \in \varepsilon(I) \\
    a_2 &= a_0b_2 + 2a_1b_1 + a_2b_0 \in \varepsilon(I) \\
    &\vdots \\
    a_n &= a_0b_n + \cdots + a_nb_0 \in \varepsilon(I).
\end{align*}
\]

Thus, \( \lambda(a_0), \lambda(a_1), \ldots, \lambda(a_n) \in \lambda(\varepsilon(I)) \). Hence, we have

\[
\begin{align*}
    \lambda(a_0) &= \lambda(a_0)(1) \in [\lambda(\varepsilon(I))] \\
    (0, a_1) &= \lambda(a_1)(0,1) \in [\lambda(\varepsilon(I))] \\
    &\vdots \\
    (0,0,\ldots,a_n) &= \lambda(a_n)(0,0,\ldots,1) \in [\lambda(\varepsilon(I))].
\end{align*}
\]

So, \( (a_n) = \lambda(a_0)+(0,a_1)+\ldots+(0,0,\ldots,a_n) \in [\lambda(\varepsilon(I))] \) and hence \( I \subseteq [\lambda(\varepsilon(I))] \).

Now let \( \alpha \in \varepsilon(I) \) such that \( \lambda(\alpha) \) is any generator for the ideal \( [\lambda(\varepsilon(I))] \). So there exist \( t_1, t_2, \ldots, t_n \in R \) such that \( (\alpha, t_1, \ldots, t_n) \in I \). Let \( \overline{\alpha} = (\alpha, t_1, \ldots, t_n) \).

However, since \( I \) is pure ideal, \( I \) is a differential ideal. Thus, inductively we get

\[
\begin{align*}
    d(\overline{\alpha}) &= (t_1, \ldots, t_n) \in I \\
    d^2(\overline{\alpha}) &= (t_2, \ldots, t_n) \in I \\
    &\vdots \\
    d^n(\overline{\alpha}) &= \lambda(t_n) \in I.
\end{align*}
\]

So, \( (0,0,\ldots, t_n) = \lambda(t_n)(0,0,\ldots,1) \in I \). Hence, \( \overline{\alpha}_{n-1} = \overline{\alpha} - (0,0,\ldots,t_n) \in I \). In the same way we can get \( \overline{\alpha}_{n-2}, \overline{\alpha}_{n-3}, \ldots, \overline{\alpha}_0 \) in \( I \). But \( \overline{\alpha}_0 = \lambda(\alpha) \in I \). Therefore \( I = [\lambda(\varepsilon(I))] \).

The other side of the theorem can be proved using the fact if \( S \subseteq T \) two rings such that \( I \) is a pure ideal in \( S \), then the ideal generated by \( I \), \( [I] \), is pure in \( T \).

**Corollary 2.2.** Suppose that \( I \) is an ideal in \( hR \) such that for any \( (a_n) = (a_0, a_1, \ldots, a_n) \in I \) we have \( a_0 = 0 \). Then

1. \( I = \{0\} \) if \( I \) is a pure ideal.
2. \( I \) is never pure ideal if there exist at least \( a_i \neq 0, i = 1, 2, \ldots, n \).

**Proof.** Clear from previous theorem.

**Lemma 2.3.** Let \( (a_n) = (a_0, a_1, \ldots, a_n) \in hR \). Then \( I = \text{ann}_{hR}((a_n)) \) is pure in \( hR \) if and only if \( J = \text{ann}_R((a_0, \ldots, a_n)) \) is pure in \( R \) and for any \( (b_m) \in I \), \( \lambda(b_i) \in I, i = 0, \ldots, m \).
Proof. Let \( b \in J \), so \( ba_i = 0, i = 0, \ldots, n \). This implies \( \lambda(b) \in I \). But \( I \) is pure ideal, so there exists \( (c_0, \ldots, c_m) \in I \) such that \( \lambda(b)(c_0, \ldots, c_m) = \lambda(b) \). Hence \( c_0 = b \). From Theorem 2.1, \( c_0 \in \varepsilon(I) \) and \( \lambda(c_0) \in I \). Thus \( c_0 a_i = 0, i = 0, 1, \ldots, n \). Therefore \( c_0 b = b \) and \( c_0 \in J \).

Conversely, let \( (b_m) = (b_0, \ldots, b_m) \in I \). By assumption \( \lambda(b_i) \in I, i = 0, \ldots, m \). So \( b_0, b_1, \ldots, b_m \in J \). But \( J \) is pure, so there exist \( w_i \in J \) such that \( b_i w_i = b_i, i = 0, 1, \ldots, m \). So, there exists \( \hat{w} \in J \) such that \( b_i \hat{w} = b_i, i = 0, 1, \ldots, m \). Also, \( \lambda(\hat{w}) \in I \) and \( (b_0, \ldots, b_m) \lambda(\hat{w}) = (b_0, \ldots, b_m) \).

3. When \( hR \) and \( hR \) are almost \( PP \)-rings (\( PF \)-rings)

**Theorem 3.1.** Suppose that \( hR(HR) \) is an almost \( PP \)-ring. Then \( R \) is an almost \( PP \)-ring and \( R \) is a torsion free as \( Z \)-module.

**Proof.** Let \( a \in R \). Suppose that \( r \in ann_R(a) \). Now \( ann_R(a) \subseteq ann_{hR}(\lambda(a)) \), so \( \lambda(r) \in ann_{hR}(\lambda(a)) \). However \( hR \) is an almost \( PP \)-ring, so from [2] there exist \( e \in Bool(hR) \) with \( re = e \) and hence \( R \) is an almost \( PP \)-ring. Also, since \( hR \) is an almost \( PP \)-ring, then \( hR \) is a \( PF \)-ring. From [7] \( R \) is torsion free.

Consequently, the same proof works for the ring \( HR \). □

**Theorem 3.2.** The ring \( hR \) is an almost \( PP \)-ring if and only if \( R \) is an almost \( PP \)-ring and \( R \) is a torsion free as \( Z \)-module.

**Proof.** Let \( f = (a_0, \ldots, a_n) \in hR \) and suppose that \( g = (b_0, \ldots, b_m) \in ann_{hR}(f) \). From [7] we get \( a_i b_j = 0, i = 0, \ldots, n, j = 0, \ldots, m \). But \( R \) is an almost \( PP \)-ring, so there exists

\[
e_0 \in ann_R(a_i) \quad \text{with} \quad b_0 e_0 = b_0, i = 0, \ldots, n,
\]

\[
e_1 \in ann_R(a_i) \quad \text{with} \quad b_1 e_1 = b_1, i = 0, \ldots, n,
\]

\[\vdots\]

\[
e_m \in ann_R(a_i) \quad \text{with} \quad b_m e_m = b_m, i = 0, \ldots, n.
\]

and \((e_k)\) \(e_k = e_k, i = 0, \ldots, n, k = 0, \ldots, m\).

Let

\[
\overline{e_0} = e_0 e_0 \ldots e_0,
\]

\[
\overline{e_1} = e_1 e_1 \ldots e_1,
\]

\[\vdots\]

\[
\overline{e_m} = e_m e_m \ldots e_m.
\]

Then \( \overline{e_i} \in ann_R(a_i) \) and \( \overline{e_i} = b_j \), where \( i = 0, \ldots, n, j = 0, \ldots, m \). So, there exist \( \hat{e} \in ann_R(a_i) \), \( i = 0, \ldots, n \) and \( \hat{e} \hat{e} = b_j, j = 0, \ldots, m \). Hence \( \lambda(\hat{e}) \in ann_{hR}(f) \) with \( g \lambda(\hat{e}) = g \). Therefore, \( hR \) is an almost \( PP \)-ring. □

**Theorem 3.3.** The ring \( hR \) is a \( PF \)-ring if and only if for any \( f = (a_0, \ldots, a_n) \in hR \), we have \( ann_R(a_0, \ldots, a_n) \) is pure ideal in \( R \) and \( R \) is torsion free.
Since for any $r \in R$, $\lambda(r) \in hR$, then $\text{ann}_R(r)$ is pure ideal in $R$. From [7], $hR$ is a PF-ring. Conversely, suppose that $hR$ is a PF-ring. Let $f = (a_0, \ldots, a_n) \in hR$ and assume $r \in \text{ann}_R\{a_0, \ldots, a_n\}$, so $\lambda(r) \in I = \text{ann}_hR(f)$. But $I$ is pure ideal, so there exists $(b_0, \ldots, b_m) \in I$ such that $\lambda(r)(b_0, \ldots, b_m) = \lambda(r)$. This implies $rb_0 = r$ and $rb_1 = rb_2 = \cdots = rb_m = 0$. From Lemma 2.3, we have $\lambda(b_i) \in I$ and hence $b_0 \in \text{ann}_R\{a_0, \ldots, a_n\}$ with $b_0r = r$.

**Theorem 3.4.** Suppose that $R$ be a Noetherian ring. Then $hR$ is a PF-ring if and only if $\text{ann}_R\{a_0, a_1, \ldots\}$ is pure ideal and $R$ is a torsion free as $\mathbb{Z}$-module, for every $(a_0, a_1, \ldots, a_n) \in hR$.

**Proof.** Let $b \in \text{ann}_R\{a_0, a_1, \ldots\}$, so $\lambda(b) \in \text{ann}_{hR}((a_0, a_1, \ldots))$. But, $hR$ is a PF-ring, so by [7] $R$ is reduced and torsion free. Moreover, there exists $c \in R$ such that, $cb = c$ and $ca_i = 0$, $i = 0, 1, 2, \ldots$, which implies $\text{ann}_R\{a_0, a_1, \ldots\}$ is pure ideal.

Conversely, let $r \in R$, so $\text{ann}_R\{r\}$ is pure ideal. Since $R$ is a torsion free and $R$ is Noetherian, from [7] $hR$ is a PF-ring.

Finally, we prove the following theorem:

**Theorem 3.5.** If $R \in \mathcal{C}$, then $hR \in \mathcal{C}$.

**Proof.** Let $I$ be a pure in $hR$, so $\varepsilon(I)$ is pure ideal in $R$. However $R \in \mathcal{C}$. So $\varepsilon(I)$ is generated by idempotents say $\{e_i : i \in \Delta\}$. Thus $\{\lambda(e_i) : i \in \Delta\}$ is the set of idempotent generators for the ideal $[\lambda(\varepsilon(I))]$ in $hR$. But, $I = [\lambda(\varepsilon(I))]$ using Theorem 2.1. Hence $I$ is generated by idempotents in $hR$ which implies $hR \in \mathcal{C}$.

4. Some results on the ring $hR(hR)$ over a ring $R$ satisfying $x^n = x$

In this section we only consider rings for every $x \in R$ we have $x^n = x$ for some fixed $n$ unless otherwise stated. It is well known that $R$ has a finite characteristic, $R$ is reduced, and every prime ideal is maximal ideal, see [11].

The next theorem was mentioned without proof in [13]. The theorem was proved by Abu Dayah [1]. His proof is also true for the ring $hR$.

**Theorem 4.1.** Let $R$ be a ring with $\text{ch}(R) = m$. So,

1. If $P$ is prime ideal of $hR(hR)$, then $\varepsilon(P)$ is prime ideal of $R$.
2. If $P$ is prime ideal of $hR(hR)$, then $P = \varepsilon^{-1}(\varepsilon(P))$.

**Lemma 4.2.** Let $x = (a_0, a_1, \ldots, a_t, \ldots) \in hR(or x = (a_0, a_1, \ldots, a_t) \in hR)$. Then $x$ is nilpotent if and only if $a_0 = 0$.

**Proof.** Suppose $x^r = 0$ for some positive integer $r$. Then $a_0^r = 0$ and since $R$ is reduced, $a_0 = 0$. Conversely, let $\text{char}(R) = m$, so $x^m = 0$ and hence $x$ is nilpotent.
Theorem 4.3. Every prime ideal in $hR(HR)$ is maximal ideal.

Proof. Assume that $\text{char}(R) = m$ and suppose that $P$ is a prime ideal in $hR$ with

$$P \subseteq M \subseteq hR,$$

where $M$ is an ideal in $hR$.

Case 1: Every $x \in M$ has the form $x = (0, x_1, x_2, \ldots, x_t)$. Now,

$$\varepsilon(P) \subseteq \varepsilon(M) \subseteq \varepsilon(hR).$$

However, $\varepsilon(M) = \{0\}$, so $\varepsilon(P) = \{0\}$. Since every prime ideal in $R$ is maximal, then by Theorem 4.1, we have $\varepsilon(P) = \{0\}$ is a maximal ideal in $R$. So, $R$ is a field and again using Theorem 4.1 we have $P = \varepsilon^{-1}(\{0\}) = M$.

Case 2: $M$ has an element $x = (x_0, x_1, \ldots, x_t)$ with $x_0 \neq 0$ and $x \notin P$.

So, $x^m = (x_0^m, 0, \ldots, 0) \notin P$ ans $x^m \in M$, hence $x_0^m \neq 0$. Let $y = \lambda(x_0^m)$, so $y^n = y$ and $y \notin P$. Thus,

$$0 = y - y^n = y(1 - y^{n-1}) \in P \subseteq M.$$

So, $1 - y^{n-1} \in P$. Hence $1 - y^{n-1} \in M$ and $y^{n-1} \in M$, so $M = hR$. Therefore $P$ is maximal ideal in $hR$.

Case 3: For each element $x = (x_0, x_1, \ldots, x_t)$ in $M$ with $x_0 \neq 0$, we have $x_0 \in P$.

From the assumption, we get $\varepsilon(P) = \varepsilon(M)$. By Theorem 4.1, we have $M = \varepsilon^{-1}(\varepsilon(M)) = \varepsilon^{-1}(\varepsilon(P)) = P$.

Consequently, the same proof works for the ring $HR$.

Theorem 4.4.

$$\text{Spec}(HR) = \{ \varepsilon^{-1}(P) : P \text{ is prime in } R \}.$$

Proof. Let $P$ be a prime ideal in $R$. So, $P$ is maximal ideal in $R$. Using [6], we get $\varepsilon^{-1}(P)$ is maximal in $HR$ and hence it is prime in $HR$. Conversely, let $Q$ be a prime ideal in $\text{Spec}(HR)$. By Theorem 4.1, $\varepsilon(Q)$ is a prime ideal in $R$. But every prime ideal in $R$ is maximal, so using Theorem 4.1 and [6], $Q = \varepsilon^{-1}(\varepsilon(Q))$ and so it is maximal in $HR$. Therefore, $Q = \varepsilon^{-1}(P), P = \varepsilon(Q)$ is prime in $R$.

Theorem 4.5.

$$\text{Spec}(hR) = \{ \varepsilon^{-1}(P) : P \text{ is prime in } R \}.$$

Proof. Note that if $M$ is a maximal ideal in $R$. Then $\varepsilon^{-1}(M)$ is maximal in $hR$. Because, it is easy to see that $hR/\varepsilon^{-1}(M) \simeq R/M$. So we can use the same proof of Theorem 4.4.
References


Accepted: 17.06.2019