# Further properties of Hurwitz series rings and Hurwitz polynomials rings

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**Abstract.** For a commutative ring with unity $R$, a commutative ring is defined called the ring of Hurwitz series, $HR$. As a subring of this ring, $HR$, the ring of Hurwitz polynomnials is defined, $hR$. In this paper, we characterize pure ideals in the ring $hR$. Then we characterize when the ring $hR$ is an almost $PP$-ring and a $PF$-ring. Finally, for the ring $R$ satisfying $x^n = x$ for a fixed positive integer $n$, we prove that every prime ideal of the rings $HR(hR)$ is maximal and so their spectrum is completely characterized.
**Keywords:** Hurwitz series ring, Hurwitz polynomial ring, pure ideal, $PF$-ring, almost $PP$-ring, spectrum of Hurwitz series ring, spectrum of Hurwitz polynomial ring, Zariski spectrum.

## 1. Introduction

All rings considered in this paper are commutative with unity $1 \neq 0$. For a ring $R$, a new ring, $HR$, was defined in Keigher [13] as follows:

$HR$ is the set of all infinite sequences of elements of $R$, $(a_n)$, together with operations

$$(a_n) + (b_n) = (a_n + b_n),$$
$$(a_n).(b_n) = (c_n),$$

where $c_n = \sum_{h+k=n}^{n} \binom{n}{h} a_h b_k$, where $\binom{n}{h}$ is the binomial coefficient.

Now, $HR$ is a commutative ring with unity $(1, 0, 0, 0, \ldots)$. This ring is called the ring of Hurwitz series over the ring $R$. Let

$$hR = \{(a_n) : a_n = 0 \text{ for all } n \geq k \text{ for some } k\}.$$

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Then this is a subring of $HR$, called the ring of Hurwitz polynomials over $R$. The algebraic structure of this ring $HR$ was studied by several authors, e.g Keigher [13], Benhissi [6, 7, 8], Abu Dayah [1]. Our aim is to study some properties of the ring of Hurwitz polynomials, $hR$.

Notice that two ring homomorphisms can be defined : $\varepsilon : HR \to R$ defined by $\varepsilon((a_n)) = a_0$ and $\lambda : R \to HR$ defined by $\lambda(a) = (a, 0, 0, 0, \ldots)$.

Also, $R$ is called a differential ring with a derivation $d : R \longrightarrow R$ if the derivation satisfies Leibniz rule $d(ab) = a\ d(b) + d(a)\ b$ for all $a, b \in R$. An ideal $I$ is differential ideal of $R$, if for every $x \in I$, $d(x) \in I$. $HR$ is a differential ring with a derivation $d : HR \longrightarrow HR$, $d((a_n)) = (a_{n+1})$ for every $(a_n) \in HR$.

Recall that an ideal $I$ of a ring $R$ is called pure ideal if for $a \in I$, there exists $b \in I$ such that $a.b = a$. Pure ideals are interesting because they classify important families of rings, e.g. Von Neumman rings and $PF$-rings, see Al-Ezeh [2, 3, 4, 5], Jondrup [12], DeMarco [10]. Also, a ring $R \in \mathscr{C}$, if for any pure ideal $I$ in $R$, $I$ is generated by a family of idempotents. Moreover, every pure ideal is a differential ideal.

In this paper we characterize pure ideals in the ring $hR$ and give its relationship to pure ideals of the ring $R$.

A ring $R$ is called a $PF$-ring if every principal ideal of $R$ is a flat $R$-module. These rings were studied by several authors. Al-Ezeh proved in [4] that a ring $R$ is a $PF$-ring if and only if the annihilator of each $a \in R$, $ann_R(a)$, is a pure ideal in $R$. We characterize in this paper when the ring $hR$ is a $PF$-ring. Recall that a ring $R$ is called an almost $PP$-ring if the annihilator of each $a \in R$, $ann_R(a)$, is generated by a set of idempotents. These rings were studied in literature as a generalization of $PP$-rings, i.e. rings in which every principal ideal is a projective $R$-module, see Al-Ezeh [5] and Burgess [9]. In this paper, we prove that $hR$ is an almost $PP$-ring if and only if $R$ is an almost $PP$-ring and $R$ is a torsion free. Moreover, we prove that $hR \in \mathscr{C}$

Finally, for the ring $R$ satisfying $x^n = x$ for some positive integer $n \geq 2$, we characterize the Zariski spectrum of the ring, $hR$.

## 2. Pure ideals in $hR$ ring

Note that $HR(hR)$ means the ring $HR$ or $hR$. Abu Deyah [1] has given a partial characterization of the pure ideals in $HR$. In this section we characterize the pure ideals in a ring $hR$.

**Theorem 2.1.** *Let $I$ be an ideal in a ring $hR$. Then $I$ is pure ideal in $hR$ if and only if $I = [\lambda(\varepsilon(I))]$ and $\varepsilon(I)$ is pure ideal in $R$.*

**Proof.** Suppose that $I$ is pure ideal in $hR$. Let $(a_n) = (a_0, a_1, \ldots, a_n) \in I$. But $I$ is a pure ideal, so there exists $(b_m) = (b_0, b_1, \ldots, b_m) \in I$ such that

$(a_n)(b_m) = (a_n)$. Now, $a_0$ and $b_0$ in $\varepsilon(I)$, so we have inductively

$$\begin{cases} a_1 &= a_0b_1 + a_1b_0 \in \varepsilon(I) \\ a_2 &= a_0b_2 + 2a_1b_1 + a_2b_0 \in \varepsilon(I) \\ \vdots & \\ a_n &= a_0b_n + \cdots\cdots + a_nb_0 \in \varepsilon(I). \end{cases}$$

Thus, $\lambda(a_0), \lambda(a_1), \ldots, \lambda(a_n) \in \lambda(\varepsilon(I))$. Hence, we have

$$\begin{cases} \lambda(a_0) &= \lambda(a_0)(1) \in [\lambda(\varepsilon(I))] \\ (0, a_1) &= \lambda(a_1)(0,1) \in [\lambda(\varepsilon(I))] \\ & \vdots \\ (0, 0, \ldots, a_n) &= \lambda(a_n)(0, \ldots, 1) \in [\lambda(\varepsilon(I))]. \end{cases}$$

So, $(a_n) = \lambda(a_0)+(0, a_1)+\ldots+(0, 0, \ldots, a_n) \in [\lambda(\varepsilon(I))]$ and hence $I \subseteq [\lambda(\varepsilon(I))]$. Now let $\alpha \in \varepsilon(I)$ such that $\lambda(\alpha)$ is any generator for the ideal $[\lambda(\varepsilon(I))]$. So there exist $t_1, t_2, \ldots, t_n \in R$ such that $(\alpha, t_1, \ldots, t_n) \in I$. Let $\overline{\alpha_n} = (\alpha, t_1, \ldots, t_n)$. However, since $I$ is pure ideal, $I$ is a differential ideal. Thus, inductively we get

$$\begin{cases} d(\overline{\alpha_n}) &= (t_1, \ldots, t_n) \in I \\ d^2(\overline{\alpha_n}) &= (t_2, \ldots, t_n) \in I \\ & \vdots \\ d^n(\overline{\alpha_n}) &= \lambda(t_n) \in I. \end{cases}$$

So, $(0, 0, \ldots, t_n) = \lambda(t_n)(0, 0, \ldots, 1) \in I$. Hence, $\overline{\alpha_{n-1}} = \overline{\alpha_n} - (0, 0, \ldots, t_n) \in I$. In the same way we can get $\overline{\alpha_{n-2}}, \overline{\alpha_{n-3}}, \ldots, \overline{\alpha_0}$ in $I$. But $\overline{\alpha_0} = \lambda(\alpha) \in I$. Therefore $I = [\lambda(\varepsilon(I))]$.

The other side of the theorem can be proved using the fact if $S \subseteq T$ two rings such that $I$ is a pure ideal in $S$, then the ideal generated by $I$, $[I]$, is pure in $T$. □

**Corollary 2.2.** *Suppose that $I$ is an ideal in $hR$ such that for any $(a_n) = (a_0, a_1, \ldots, a_n) \in I$ we have $a_0 = 0$. Then*

1. *$I = \{0\}$ if $I$ is a pure ideal.*
2. *$I$ is never pure ideal if there exist at least $a_i \neq 0, i = 1, 2, \ldots, n$.*

**Proof.** Clear from previous theorem. □

**Lemma 2.3.** *Let $(a_n) = (a_0, a_1, \ldots, a_n) \in hR$. Then $I = ann_{hR}((a_n))$ is pure in $hR$ if and only if $J = ann_R(\{a_0, \ldots a_n\})$ is pure in $R$ and for any $(b_m) \in I$, $\lambda(b_i) \in I, i = 0, \ldots, m$.*

**Proof.** Let $b \in J$, so $ba_i = 0, i = 0, \dots, n$. This implies $\lambda(b) \in I$. But $I$ is pure ideal, so there exists $(c_0, \dots, c_m) \in I$ such that $\lambda(b)(c_0, \dots, c_m) = \lambda(b)$. Hence $bc_0 = b$. From Theorem 2.1, $c_0 \in \varepsilon(I)$ and $\lambda(c_0) \in I$. Thus $c_0 a_i = 0, i = 0, 1, \dots, n$. Therefore $c_0 b = b$ and $c_0 \in J$.

Conversely, let $(b_m) = (b_0, \dots, b_m) \in I$. By assumption $\lambda(b_i) \in I, i = 0, \dots, m$. So $b_0, b_1, \dots, b_m \in J$. But $J$ is pure, so there exist $w_i \in J$ such that $b_i w_i = b_i, i = 0, 1, \dots, m$. So, there exists $\widehat{w} \in J$ such that $b_i \widehat{w} = b_i, i = 0, 1, \dots, m$. Also, $\lambda(\widehat{w}) \in I$ and $(b_0, \dots, b_m)\lambda(\widehat{w}) = (b_0, \dots, b_m)$. □

## 3. When HR and hR are almost $PP$-rings ($PF$-rings)

**Theorem 3.1.** *Suppose that $hR(HR)$ is an almost $PP$-ring. Then $R$ is an almost $PP$-ring and $R$ is a torsion free as $\mathbb{Z}$-module.*

**Proof.** Let $a \in R$. Suupose that $r \in ann_R(a)$. Now $ann_R(a) \subseteq ann_{hR}(\lambda(a))$, so $\lambda(r) \in ann_{hR}(\lambda(a))$. However $hR$ is an almost $PP$-ring, so from [2] there exist $e \in Bool(hR)$ with $re = e$ and hence $R$ is an almost $PP$-ring. Also, since $hR$ is an almost $PP$-ring, then $hR$ is a $PF$-ring. From [7] $R$ is torsion free.

Consequently, the same proof works for the ring $HR$. □

**Theorem 3.2.** *The ring $hR$ is an almost $PP$-ring if and only if $R$ is an almost $PP$-ring and $R$ is a torsion free as $\mathbb{Z}$-module.*

**Proof.** Let $f = (a_0, \dots, a_n) \in hR$ and suppose that $g = (b_0, \dots, b_m) \in ann_{hR}(f)$. From [7] we get $a_i b_j = 0, i = 0, \dots, n, j = 0, \dots, m$, so $\lambda(b_i) \in ann_{hR}(f)$. Hence $b_j \in ann_R(a_i), i = 0, \dots, n, j = 0, \dots, m$. But $R$ is an almost $PP$-ring, so there exist

$$\begin{gathered} e_{0i} \in ann_R(a_i) \text{ with } b_0 e_{0i} = b_0, i = 0, \dots, n, \\ e_{1i} \in ann_R(a_i) \text{ with } b_1 e_{1i} = b_1, i = 0, \dots, n, \\ \vdots \\ e_{mi} \in ann_R(a_i) \text{ with } b_m e_{mi} = b_m, i = 0, \dots, n. \end{gathered}$$

and $(e_{ki})^2 = e_{ki}, i = 0, \dots, n, k = 0, \dots, m$.

Let

$$\begin{gathered} \overline{e_0} = e_{00} e_{01} \cdots e_{0n}, \\ \overline{e_1} = e_{10} e_{11} \cdots e_{1n}, \\ \vdots \\ \overline{e_m} = e_{m0} e_{m1} \cdots e_{mn}. \end{gathered}$$

Then $\overline{e_j} \in ann_R(a_i)$ and $\overline{e_j} b_j = b_j$, where $i = 0, \dots, n, j = 0, \dots, m$. So, there exist $\widehat{e} \in ann_R(a_i), i = 0, \dots, n$ and $b_j \widehat{e} = b_j, j = 0, \dots, m$. Hence $\lambda(\widehat{e}) \in ann_{hR}(f)$ with $g\lambda(\widehat{e}) = g$. Therefore, $hR$ is an almost $PP$-ring. □

**Theorem 3.3.** *The ring $hR$ is a $PF$-ring if and only if for any $f = (a_0, \dots, a_n) \in hR$, we have $ann_R\{a_0, \dots, a_n\}$ is pure ideal in $R$ and $R$ is torsion free.*

**Proof.** Since for any $r \in R$, $\lambda(r) \in hR$, then $ann_R(r)$ is pure ideal in $R$. From [7], $hR$ is a $PF$-ring. Conversely, suppose that $hR$ is a $PF$-ring. Let $f = (a_0, \ldots, a_n) \in hR$ and assume $r \in ann_R\{a_0, \ldots, a_n\}$, so $\lambda(r) \in I = ann_{hR}(f)$. But $I$ is pure ideal, so there exists $(b_0, \ldots, b_m) \in I$ such that $\lambda(r)(b_0, \ldots, b_m) = \lambda(r)$. This implies $rb_0 = r$ and $rb_1 = rb_2 = \cdots = rb_m = 0$. From Lemma 2.3, we have $\lambda(b_i) \in I$ and hence $b_0 \in ann_R\{a_0, \ldots, a_n\}$ with $b_0 r = r$. □

**Theorem 3.4.** *Suppose that $R$ be a Notherian ring. Then $HR$ is a $PF$-ring if and only if $ann_R\{a_0, a_1, \ldots\}$ is pure ideal and $R$ is a torsion free as $\mathbb{Z}$-moduel, for every $(a_0, a_1, \ldots,) \in HR$.*

**Proof.** Let $b \in ann_R\{a_0, a_1, \ldots\}$, so $\lambda(b) \in ann_{HR}((a_0, a_1, \ldots))$. But, $HR$ is a $PF$-ring, so by [7] $R$ is reduced and torsion free. Moreover, there exists $c \in R$ such that, $cb = c$ and $ca_i = 0, i = 0, 1, 2, \ldots$, which implies $ann_R\{a_0, a_1, \ldots\}$ is pure ideal.

Conversely, let $r \in R$, so $ann_R\{r\}$ is pure ideal. Since $R$ is a torsion free and $R$ is Notherian, from [7] $HR$ is a $PF$-ring. □

Finally, we prove the following theorem :

**Theorem 3.5.** *If $R \in \mathscr{C}$, then $hR \in \mathscr{C}$.*

**Proof.** Let $I$ be a pure in $hR$, so $\varepsilon(I)$ is pure ideal in $R$. However $R \in \mathscr{C}$. So $\varepsilon(I)$ is generated by idempotents say $\{e_i : i \in \Delta\}$. Thus $\{\lambda(e_i) : i \in \Delta\}$ is the set of idempotent generators for the ideal $[\lambda(\varepsilon(I))]$ in $hR$. But, $I = [\lambda(\varepsilon(I))]$ using Theorem 2.1. Hence $I$ is generated by idempotents in $hR$ which implies $hR \in \mathscr{C}$ □

## 4. Some results on the ring $hR(HR)$ over a ring $R$ satisfying $x^n = x$

In this section we only consider rings for every $x \in R$ we have $x^n = x$ for some fixed $n$ unless otherwise stated. It is well known that $R$ has a finite characteristic, $R$ is reduced, and every prime ideal is maximal ideal, see [11].

The next theorem was mentioned without proof in [13]. The theorem was proved by Abu Dayah [1]. His proof is also true for the ring $hR$

**Theorem 4.1.** *Let $R$ be a ring with $ch(R) = m$. So,*

1. *If $P$ is prime ideal of $HR(hR)$, then $\varepsilon(P)$ is prime ideal of $R$.*
2. *If $P$ is prime ideal of $HR(hR)$, then $P = \varepsilon^{-1}(\varepsilon(P))$.*

**Lemma 4.2.** *Let $x = (a_0, a_1, \ldots, a_t, \ldots) \in HR$(or $x = (a_0, a_1, \ldots, a_t) \in hR$). Then $x$ is nilpotent if and only if $a_0 = 0$.*

**Proof.** Suppose $x^r = 0$ for some positive integer $r$. Then $a_0^r = 0$ and since $R$ is reduced, $a_0 = 0$. Conversely, let $char(R) = m$, so $x^m = 0$ and hence $x$ is nilpotent. □

**Theorem 4.3.** *Every prime ideal in $hR(HR)$ is maximal ideal.*

**Proof.** Assume that $char(R) = m$ and suppose that $P$ is a prime ideal in $hR$ with

$$P \subseteq M \subseteq hR,$$

where $M$ is an ideal in $hR$.

**Case** 1 : Every $x \in M$ has the form $x = (0, x_1, x_2, \ldots, x_t)$. Now,

$$\varepsilon(P) \subseteq \varepsilon(M) \subseteq \varepsilon(hR).$$

However, $\varepsilon(M) = \{0\}$, so $\varepsilon(P) = \{0\}$. Since every prime ideal in $R$ is maximal,then by Theorem 4.1, we have $\varepsilon(P) = \{0\}$ is a maximal ideal in $R$. So, $R$ is a field and again using Theorem 4.1 we have $P = \varepsilon^{-1}(\{0\}) = M$.

**Case** 2 : $M$ has an element $x = (x_0, x_1, \ldots, x_t)$ with $x_0 \neq 0$ and $x \notin P$. So, $x^m = (x_0^m, 0, \ldots, 0) \notin P$ ans $x^m \in M$, hence $x_0^m \neq 0$. Let $y = \lambda(x_0^m)$, so $y^n = y$ and $y \notin P$. Thus,

$$0 = y - y^n = y(1 - y^{n-1}) \in P \subseteq M.$$

So, $1 - y^{n-1} \in P$. Hence $1 - y^{n-1} \in M$ and $y^{n-1} \in M$, so $M = hR$. Therefore $P$ is maximal ideal in $hR$.

**Case** 3 : For each element $x = (x_0, x_1, \ldots, x_t)$ in $M$ with $x_0 \neq 0$, we have $x_0 \in P$.

From the assumption , we get $\varepsilon(P) = \varepsilon(M)$. By Theorem 4.1, we have

$$M = \varepsilon^{-1}(\varepsilon(M)) = \varepsilon^{-1}(\varepsilon(P)) = P.$$

Consequently, the same proof works for the ring $HR$. □

**Theorem 4.4.**

$$Spec(HR) = \{\varepsilon^{-1}(P) : P \textit{ is prime in } R\}.$$

**Proof.** Let $P$ be a prime ideal in $R$. So, $P$ is maximal ideal in $R$. Using [6], we get $\varepsilon^{-1}(P)$ is maximal in $HR$ and hence it is prime in $HR$. Conversely, let $Q$ be a prime ideal in $Spec(HR)$. By Theorem 4.1, $\varepsilon(Q)$ is a prime ideal in $R$. But every prime ideal in $R$ is maximal, so using Theorem 4.1 and [6], $Q = \varepsilon^{-1}(\varepsilon(Q))$ and so it is maximal in $HR$. Therefore, $Q = \varepsilon^{-1}(P), P = \varepsilon(Q)$ is prime in $R$. □

**Theorem 4.5.**

$$Spec(hR) = \{\varepsilon^{-1}(P) : P \textit{ is prime in } R\}.$$

**Proof.** Note that if $M$ is a maximal ideal in $R$. Then $\varepsilon^{-1}(M)$ is maximal in $hR$. Because, it is easy to see that $hR/\varepsilon^{-1}(M) \simeq R/M$. So we can use the same proof of Theorem 4.4.

□

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