Hesitant fuzzy soft sets over UP-algebras by means of anti-type

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Abstract. This paper aims to extend the concept of anti-type of hesitant fuzzy sets on UP-algebras to anti-type of hesitant fuzzy soft sets over UP-algebras by merging the concept of anti-type of hesitant fuzzy sets and soft sets. Further, we discuss the concepts of anti-hesitant fuzzy soft strongly UP-ideals, anti-hesitant fuzzy soft UP-ideals, anti-hesitant fuzzy soft UP-filters, and anti-hesitant fuzzy soft UP-subalgebras of UP-algebras and provide some properties.
Keywords: UP-algebra, anti-hesitant fuzzy soft UP-subalgebra, anti-hesitant fuzzy soft UP-filter, anti-hesitant fuzzy soft UP-ideal, anti-hesitant fuzzy soft strongly UP-ideal.

1. Introduction

The branch of the logical algebra, a UP-algebra was introduced by Iampan [6] in 2017, and it is known that the class of KU-algebras [16] is a proper subclass of the class of UP-algebras. It have been examined by several researchers, for example, Somjanta et al. [23] introduced the notion of fuzzy sets in UP-algebras, the notion of intuitionistic fuzzy sets in UP-algebras was introduced by Kesorn et al. [10], Kaijae et al. [9] introduced the notions of anti-fuzzy UP-ideals and anti-fuzzy UP-subalgebras of UP-algebras, the notion of Q-fuzzy sets in UP-algebras was introduced by Tanamoon et al. [26], Sripaeng et al. [25] introduced the notion anti Q-fuzzy UP-ideals and anti Q-fuzzy UP-subalgebras of UP-algebras, the notion of $\mathcal{N}$-fuzzy sets in UP-algebras was introduced by Songsaeng and Iampan [24], Senapati et al. [21, 22] applied cubic set and interval-valued intuitionistic fuzzy structure in UP-algebras, Romano [17] introduced the notion of proper UP-filters in UP-algebras, etc.

A soft set over a universe set is a parametized family of subsets of the universe set. Molodtsov [11] introduced the concept of soft sets over a universe set in 1999.

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A hesitant fuzzy set on a set is a function from a reference set to a power set of the unit interval. The concept of a hesitant fuzzy set on a set was first considered by Torra [27] in 2010. In UP-algebras, Mosrijai et al. [14] extended the concept of fuzzy sets in UP-algebras to hesitant fuzzy sets on UP-algebras, and Satirad et al. [20] considered level subsets of a hesitant fuzzy set on UP-algebras in 2017. The concept of partial constant hesitant fuzzy sets on UP-algebras was introduced by Mosrijai et al. [15] afterwards. Mosrijai and Iampan [12] introduced the notion of anti-type of hesitant fuzzy sets on UP-algebras in 2018.

The concept of hesitant fuzzy soft sets that is a link between classical soft sets and hesitant fuzzy sets is introduced by Babitha and John [3] in 2013. There exists some researchers, such as Jun et al. [8], applied hesitant fuzzy soft set theory to some algebraic structures, which are BCK and BCI algebras. Alshehri and Alshehri [1] introduced the notion of hesitant anti-fuzzy sets over BCK-algebras later. In UP-algebras, Mosrijai and Iampan [13] extended the concept of hesitant fuzzy sets on UP-algebras to hesitant fuzzy soft sets over UP-algebras.

In this paper, we extend the concept of anti-type of hesitant fuzzy sets on UP-algebras to anti-type of hesitant fuzzy soft sets over UP-algebras by merging the concept of anti-type of hesitant fuzzy sets and soft sets. Further, we discuss the concept of anti-hesitant fuzzy soft UP-subalgebras, anti-hesitant fuzzy soft UP-filters, anti-hesitant fuzzy soft UP-ideals and anti-hesitant fuzzy soft strongly UP-ideals of UP-algebras, and provide some properties.

2. Basic results on UP-algebras

Before we begin our study, we will introduce the definition of a UP-algebra.

Definition 2.1 ([6])**.** An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a *UP-algebra* where A is a nonempty set, $\cdot$ is a binary operation on A, and 0 is a fixed element of A (i.e., a nullary operation) if it satisfies the following axioms:

(UP-1) $(\forall x, y, z \in A)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0)$,

(UP-2) $(\forall x \in A)(0 \cdot x = x)$,

(UP-3) $(\forall x \in A)(x \cdot 0 = 0)$, and

(UP-4) $(\forall x, y \in A)(x \cdot y = 0, y \cdot x = 0 \Rightarrow x = y)$.

From [6], we know that the notion of UP-algebras is a generalization of KU-algebras (see [16]).

Example 2.2 ([19])**.** Let X be a universal set and let $\Omega \in \mathcal{P}(X)$. Let $\mathcal{P}_\Omega(X) = \{A \in \mathcal{P}(X) \mid \Omega \subseteq A\}$. Define a binary operation $\cdot$ on $\mathcal{P}_\Omega(X)$ by putting $A \cdot B = B \cap (A' \cup \Omega)$ for all $A, B \in \mathcal{P}_\Omega(X)$. Then $(\mathcal{P}_\Omega(X), \cdot, \Omega)$ is a UP-algebra and we shall call it the *generalized power UP-algebra of type 1 with respect to* Ω.

In particular, $(\mathcal{P}_{\emptyset}(X), \cdot, \emptyset)$ is the power UP-algebra of type 1.

Example 2.3 ([19]). Let X be a universal set and let $\Omega \in \mathcal{P}(X)$. Let $\mathcal{P}^{\Omega}(X) = \{A \in \mathcal{P}(X) \mid A \subseteq \Omega\}$. Define a binary operation $*$ on $\mathcal{P}^{\Omega}(X)$ by putting $A * B = B \cup (A' \cap \Omega)$ for all $A, B \in \mathcal{P}^{\Omega}(X)$. Then $(\mathcal{P}^{\Omega}(X), *, \Omega)$ is a UP-algebra and we shall call it the *generalized power UP-algebra of type 2 with respect to* Ω.

In particular, $(\mathcal{P}^{X}(X), *, X)$ is the power UP-algebra of type 2.

Example 2.4 ([4]). Let $\mathbb{N}$ be the set of all natural numbers with two binary operations $\circ$ and $\bullet$ defined by

$$(\forall x, y \in \mathbb{N}) \left(x \circ y = \begin{cases} y & \text{if } x < y, \\ 0 & \text{otherwise} \end{cases} \right)$$

and

$$(\forall x, y \in \mathbb{N}) \left(x \bullet y = \begin{cases} y & \text{if } x > y \text{ or } x = 0, \\ 0 & \text{otherwise} \end{cases} \right).$$

Then $(\mathbb{N}, \circ, 0)$ and $(\mathbb{N}, \bullet, 0)$ are UP-algebras.

For more examples of UP-algebras, see [2, 7, 18, 19].

In what follows, let A denote UP-algebras unless otherwise specified. The following proposition is very important for the study of UP-algebras.

Proposition 2.5 ([6, 7]). In a UP-algebra A, the following properties hold:

(1) $(\forall x \in A)(x \cdot x = 0)$,

(2) $(\forall x, y, z \in A)(x \cdot y = 0, y \cdot z = 0 \Rightarrow x \cdot z = 0)$,

(3) $(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow (z \cdot x) \cdot (z \cdot y) = 0)$,

(4) $(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow (y \cdot z) \cdot (x \cdot z) = 0)$,

(5) $(\forall x, y \in A)(x \cdot (y \cdot x) = 0)$,

(6) $(\forall x, y \in A)((y \cdot x) \cdot x = 0 \Leftrightarrow x = y \cdot x)$,

(7) $(\forall x, y \in A)(x \cdot (y \cdot y) = 0)$,

(8) $(\forall a, x, y, z \in A)((x \cdot (y \cdot z)) \cdot (x \cdot ((a \cdot y) \cdot (a \cdot z))) = 0)$,

(9) $(\forall a, x, y, z \in A)((((a \cdot x) \cdot (a \cdot y)) \cdot z) \cdot ((x \cdot y) \cdot z) = 0)$,

(10) $(\forall x, y, z \in A)(((x \cdot y) \cdot z) \cdot (y \cdot z) = 0)$,

(11) $(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow x \cdot (z \cdot y) = 0)$,

(12) $(\forall x, y, z \in A)(((x \cdot y) \cdot z) \cdot (x \cdot (y \cdot z)) = 0)$, and

(13) $(\forall a, x, y, z \in A)(((x \cdot y) \cdot z) \cdot (y \cdot (a \cdot z)) = 0)$.

On a UP-algebra $A = (A, \cdot, 0)$, we define a binary relation $\leq$ on A [6] as follows:

$$(\forall x, y \in A)(x \leq y \Leftrightarrow x \cdot y = 0).$$

Definition 2.6 ([6, 23, 5])**.** A nonempty subset S of a UP-algebra $(A, \cdot, 0)$ is called

(1) a *UP-subalgebra* of A if $(\forall x, y \in S)(x \cdot y \in S)$.

(2) a *UP-filter* of A if

  (i) the constant 0 of A is in S, and

  (ii) $(\forall x, y \in A)(x \cdot y \in S, x \in S \Rightarrow y \in S)$.

(3) a *UP-ideal* of A if

  (i) the constant 0 of A is in S, and

  (ii) $(\forall x, y, z \in A)(x \cdot (y \cdot z) \in S, y \in S \Rightarrow x \cdot z \in S)$.

(4) a *strongly UP-ideal* of A if

  (i) the constant 0 of A is in S, and

  (ii) $(\forall x, y, z \in A)((z \cdot y) \cdot (z \cdot x) \in S, y \in S \Rightarrow x \in S)$.

Guntasow et al. [5] proved that the notion of UP-subalgebras is a generalization of UP-filters, the notion of UP-filters is a generalization of UP-ideals, and the notion of UP-ideals is a generalization of strongly UP-ideals. Moreover, they also proved that a UP-algebra A is the only one strongly UP-ideal of itself.

3. Basic results on hesitant fuzzy sets

Definition 3.1 ([27])**.** Let X be a reference set. A *hesitant fuzzy set* on X is defined in term of a function $\mathrm{h_H}$ that when applied to X return a subset of $[0, 1]$, that is, $\mathrm{h_H}\colon X \to \mathcal{P}([0, 1])$. A hesitant fuzzy set $\mathrm{h_H}$ can also be viewed as the following mathematical representation:

$$\mathrm{H} := \{(x, \mathrm{h_H}(x)) \mid x \in X\}$$

where $\mathrm{h_H}(x)$ is a set of some values in $[0, 1]$, denoting the possible membership degrees of the elements $x \in X$ to the set H.

Definition 3.2 ([14])**.** A hesitant fuzzy set H on A is called

(1) a *hesitant fuzzy UP-subalgebra* of A if $(\forall x, y \in A)(\mathrm{h_H}(x \cdot y) \supseteq \mathrm{h_H}(x) \cap \mathrm{h_H}(y))$.

(2) a *hesitant fuzzy UP-filter* of A if

(1) $(\forall x \in A)(\mathrm{h_H}(0) \supseteq \mathrm{h_H}(x))$, and
(2) $(\forall x, y \in A)(\mathrm{h_H}(y) \supseteq \mathrm{h_H}(x \cdot y) \cap \mathrm{h_H}(x))$.

(3) a *hesitant fuzzy UP-ideal* of A if

(1) $(\forall x \in A)(\mathrm{h_H}(0) \supseteq \mathrm{h_H}(x))$, and
(2) $(\forall x, y, z \in A)(\mathrm{h_H}(x \cdot z) \supseteq \mathrm{h_H}(x \cdot (y \cdot z)) \cap \mathrm{h_H}(y))$.

(4) a *hesitant fuzzy strongly UP-ideal* of A if

(1) $(\forall x \in A)(\mathrm{h_H}(0) \supseteq \mathrm{h_H}(x))$, and
(2) $(\forall x, y, z \in A)(\mathrm{h_H}(x) \supseteq \mathrm{h_H}((z \cdot y) \cdot (z \cdot x)) \cap \mathrm{h_H}(y))$.

Mosrijai et al. [14] proved that the notion of hesitant fuzzy UP-subalgebras of UP-algebras is a generalization of hesitant fuzzy UP-filters, the notion of hesitant fuzzy UP-filters of UP-algebras is a generalization of hesitant fuzzy UP-ideals, and the notion of hesitant fuzzy UP-ideals of UP-algebras is a generalization of hesitant fuzzy strongly UP-ideals.

Definition 3.3 ([12]). A hesitant fuzzy set H on A is called

(1) an *anti-hesitant fuzzy UP-subalgebra* of A if $(\forall x, y \in A)(\mathrm{h_H}(x \cdot y) \subseteq \mathrm{h_H}(x) \cup \mathrm{h_H}(y))$.

(2) an *anti-hesitant fuzzy UP-filter* of A if

(1) $(\forall x \in A)(\mathrm{h_H}(0) \subseteq \mathrm{h_H}(x))$, and
(2) $(\forall x, y \in A)(\mathrm{h_H}(y) \subseteq \mathrm{h_H}(x \cdot y) \cup \mathrm{h_H}(x))$.

(3) an *anti-hesitant fuzzy UP-ideal* of A if

(1) $(\forall x \in A)(\mathrm{h_H}(0) \subseteq \mathrm{h_H}(x))$, and
(2) $(\forall x, y, z \in A)(\mathrm{h_H}(x \cdot z) \subseteq \mathrm{h_H}(x \cdot (y \cdot z)) \cup \mathrm{h_H}(y))$.

(4) an *anti-hesitant fuzzy strongly UP-ideal* of A if

(1) $(\forall x \in A)(\mathrm{h_H}(0) \subseteq \mathrm{h_H}(x))$, and
(2) $(\forall x, y, z \in A)(\mathrm{h_H}(x) \subseteq \mathrm{h_H}((z \cdot y) \cdot (z \cdot x)) \cup \mathrm{h_H}(y))$.

Mosrijai and Iampan [12] proved that the notion of anti-hesitant fuzzy UP-subalgebras of UP-algebras is a generalization of anti-hesitant fuzzy UP-filters, the notion of anti-hesitant fuzzy UP-filters of UP-algebras is a generalization of anti-hesitant fuzzy UP-ideals, and the notion of anti-hesitant fuzzy UP-ideals of UP-algebras is a generalization of anti-hesitant fuzzy strongly UP-ideals.

Proposition 3.4 ([12]). *Let* H *be an anti-hesitant fuzzy UP-filter (and also anti-hesitant fuzzy UP-ideal, anti-hesitant fuzzy strongly UP-ideal) of* A*. Then*

$$(\forall x, y \in A)(x \leq y \Rightarrow \mathrm{h_H}(x) \supseteq \mathrm{h_H}(y) \supseteq \mathrm{h_H}(x \cdot y)).$$

Definition 3.5 ([13])**.** Let Y be a nonempty subset of P. A hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over A is called a *hesitant fuzzy soft UP-subalgebra* (resp., hesitant fuzzy soft UP-filter, hesitant fuzzy soft UP-ideal, hesitant fuzzy soft strongly UP-ideal) based on $p \in Y$ (we shortly call a *p-hesitant fuzzy soft UP-subalgebra* (resp., p-hesitant fuzzy soft UP-filter, p-hesitant fuzzy soft UP-ideal, p-hesitant fuzzy soft strongly UP-ideal)) of A if the hesitant fuzzy set

$$\widetilde{\mathrm{H}}[p] := \{(a, \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(a)) \mid a \in A\}$$

on A is a hesitant fuzzy UP-subalgebra (resp., hesitant fuzzy UP-filter, hesitant fuzzy UP-ideal, hesitant fuzzy strongly UP-ideal) of A. If $(\widetilde{\mathrm{H}}, Y)$ is a p-hesitant fuzzy soft UP-subalgebra (resp., hesitant fuzzy soft UP-filter, hesitant fuzzy soft UP-ideal, hesitant fuzzy soft strongly UP-ideal) of A for all $p \in Y$, we state that $(\widetilde{\mathrm{H}}, Y)$ is a *hesitant fuzzy soft UP-subalgebra* (resp., hesitant fuzzy soft UP-filter, hesitant fuzzy soft UP-ideal, hesitant fuzzy soft strongly UP-ideal) of A.

4. Anti-hesitant fuzzy soft UP-subalgebras

Definition 4.1 ([3])**.** Let X be a reference set (or an initial universe set) and P be a set of parameters. Let $\mathrm{HFS}(X)$ be the set of all hesitant fuzzy sets on X and Y be a nonempty subset of P. A pair $(\widetilde{\mathrm{H}}, Y)$ is called a *hesitant fuzzy soft set* over X where $\widetilde{\mathrm{H}}$ is a mapping given by

$$\widetilde{\mathrm{H}} \colon Y \to \mathrm{HFS}(X), p \mapsto \widetilde{\mathrm{H}}[p].$$

Definition 4.2. Let Y be a nonempty subset of P. A hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over A is called an *anti-hesitant fuzzy soft UP-subalgebra* based on $p \in Y$ (we shortly call a *p-anti-hesitant fuzzy soft UP-subalgebra*) of A if the hesitant fuzzy set

$$\widetilde{\mathrm{H}}[p] := \{(a, \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(a)) \mid a \in A\}$$

on A is an anti-hesitant fuzzy UP-subalgebra of A. If $(\widetilde{\mathrm{H}}, Y)$ is a p-anti-hesitant fuzzy soft UP-subalgebra of A for all $p \in Y$, we state that $(\widetilde{\mathrm{H}}, Y)$ is an *anti-hesitant fuzzy soft UP-subalgebra* of A.

Theorem 4.3. *If $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft UP-subalgebra of A, then it satisfies the property:*

$$(\forall p \in Y \forall x \in A)(\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(0) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x)). \tag{4.1}$$

Proof. Assume that $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft UP-subalgebra of A. Let $p \in Y$ and $x \in A$. Then $\widetilde{\mathrm{H}}[p]$ is an anti-hesitant fuzzy UP-subalgebra of A. Therefore, $\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(0) = \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot x) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x) = \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x)$. □

Example 4.4. For any objects a and b, let $(\mathcal{P}_{\emptyset}(\{a,b\}), \cdot, \emptyset)$ be the power UP-algebra of type 1 with the following Cayley table:

$\cdot$	$\emptyset$	$\{a\}$	$\{b\}$	$\{a,b\}$
$\emptyset$	$\emptyset$	$\{a\}$	$\{b\}$	$\{a,b\}$
$\{a\}$	$\emptyset$	$\emptyset$	$\{b\}$	$\{b\}$
$\{b\}$	$\emptyset$	$\{a\}$	$\emptyset$	$\{a\}$
$\{a,b\}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Let $Y = \{p_1, p_2, p_3, p_4\}$ be a parameter set. We define a hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over $\mathcal{P}_{\emptyset}(\{a,b\})$ by the following table:

$\widetilde{\mathrm{H}}$	$\emptyset$	$\{a\}$	$\{b\}$	$\{a,b\}$
p_1	$\emptyset$	$\{0.1\}$	$\{0.2\}$	$\{0.1, 0.2\}$
p_2	$(0.6, 0.7)$	$(0.6, 0.7)$	$[0.6, 0.7]$	$[0.6, 0.7]$
p_3	$\{0.5\}$	$\{0.4, 0.5\}$	$[0.4, 0.5]$	$(0.2, 0.7)$
p_4	$\{0.9\}$	$[0.8, 0.9]$	$\{0.8, 0.9\}$	$\{0.8, 0.9\}$

Then $(\widetilde{\mathrm{H}}, Y)$ satisfies the property (4.1), but not an anti-hesitant fuzzy soft UP-subalgebra of $\mathcal{P}_{\emptyset}(\{a,b\})$ based on parameter p_4. Indeed,

$$\begin{aligned}\mathrm{h}_{\widetilde{\mathrm{H}}[p_4]}(\{b\} \cdot \{a,b\}) = \mathrm{h}_{\widetilde{\mathrm{H}}[p_4]}(\{a\}) = [0.8, 0.9] &\nsubseteq \{0.8, 0.9\} \\ &= \{0.8, 0.9\} \cup \{0.8, 0.9\} \\ &= \mathrm{h}_{\widetilde{\mathrm{H}}[p_4]}(\{b\}) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p_4]}(\{a,b\}).\end{aligned}$$

Theorem 4.5. *Let $(\widetilde{\mathrm{H}}, Y)$ be a hesitant fuzzy soft set over A which satisfies the condition:*

$$(4.2) \qquad (\forall p \in Y \forall x, y, z \in A)(z \leq x \cdot y \Rightarrow \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(z) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x)).$$

Then $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft UP-subalgebra of A.

Proof. Let $p \in Y$ and $x, y \in A$. By Proposition 2.5 (5) and (UP-3), we have $x \cdot (y \cdot (x \cdot y)) = x \cdot 0 = 0$ and thus $x \leq y \cdot (x \cdot y)$. It follows from (4.2) that

$$\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot y) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y).$$

Therefore, $\widetilde{\mathrm{H}}[p]$ is an anti-hesitant fuzzy UP-subalgebra of A. Hence, $(\widetilde{\mathrm{H}}, Y)$ is a p-anti-hesitant fuzzy soft UP-subalgebra of A. Since p is arbitrary, we know that $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft UP-subalgebra of A. □

Corollary 4.6. *If $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft set over A which satisfies the condition* (4.2), *then it satisfies the property* (4.1).

Proof. It is straightforward from Theorems 4.5 and 4.3. □

Theorem 4.7. *If* $(\widetilde{\mathrm{H}}, Y)$ *is an anti-hesitant fuzzy soft UP-subalgebra of* A *and* N *is a nonempty subset of* Y*, then* $(\widetilde{\mathrm{H}}|_N, N)$ *is an anti-hesitant fuzzy soft UP-subalgebra of* A*.*

Proof. Assume that $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft UP-subalgebra of A and $\emptyset \neq N \subseteq Y$. Since $N \subseteq Y$, we have $(\widetilde{\mathrm{H}}|_N, N)$ is a p-anti-hesitant fuzzy soft UP-subalgebra of A for all $p \in N$. Therefore, $(\widetilde{\mathrm{H}}|_N, N)$ is an anti-hesitant fuzzy soft UP-subalgebra of A. □

By Example 4.4, we have $(\widetilde{\mathrm{H}}, Y)$ is not an anti-hesitant fuzzy soft UP-subalgebra of A. But if we choose $N = \{p_1, p_2, p_3\}$, then $(\widetilde{\mathrm{H}}|_N, N)$ is an anti-hesitant fuzzy soft UP-subalgebra of A. We can conclude that there exists a nonempty subset N of Y such that $(\widetilde{\mathrm{H}}|_N, N)$ is an anti-hesitant fuzzy soft UP-subalgebra of A, but $(\widetilde{\mathrm{H}}, Y)$ is not an anti-hesitant fuzzy soft UP-subalgebra of A.

5. Anti-hesitant fuzzy soft UP-filters

Definition 5.1. Let Y be a nonempty subset of P. A hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over A is called a *anti-hesitant fuzzy soft UP-filter* based on $p \in Y$ (we shortly call a *anti-p-hesitant fuzzy soft UP-filter*) of A if the hesitant fuzzy set

$$\widetilde{\mathrm{H}}[p] := \{(a, \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(a)) \mid a \in A\}$$

on A is an anti-hesitant fuzzy UP-filter of A. If $(\widetilde{\mathrm{H}}, Y)$ is a p-anti-hesitant fuzzy soft UP-filter of A for all $p \in Y$, we state that $(\widetilde{\mathrm{H}}, Y)$ is an *anti-hesitant fuzzy soft UP-filter* of A.

From [12], we know that every anti-hesitant fuzzy UP-filter of A is an anti-hesitant fuzzy UP-subalgebra. Then we have the following Theorem:

Theorem 5.2. *Every* p*-anti-hesitant fuzzy soft UP-filter of* A *is a* p*-anti-hesitant fuzzy soft UP-subalgebra.*

The following example shows that the converse of Theorem 5.2 is not true in general.

Example 5.3. For any objects a and b, let $(\mathcal{P}_\emptyset(\{a, b\}), \cdot, \emptyset)$ be the power UP-algebra of type 1 with the Cayley table from Example 4.4, and let $Y = \{p_1, p_2, p_3, p_4\}$ be a parameter set. We define a hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over $\mathcal{P}_\emptyset(\{a, b\})$ by the following table:

$\widetilde{\mathrm{H}}$	$\emptyset$	$\{a\}$	$\{b\}$	$\{a, b\}$
p_1	$\emptyset$	$\{0\}$	$\{0.1\}$	$\{0, 0.1\}$
p_2	$\{0.2\}$	$\{0.2\}$	$\{0.2, 0.3\}$	$\{0.2, 0.3\}$
p_3	$\{0.4\}$	$\{0.4, 0.5\}$	$[0.4, 0.5]$	$[0.4, 0.6)$
p_4	$\{0.8\}$	$[0.7, 0.9]$	$[0.7, 0.9]$	$\{0.8\}$

Then $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft UP-subalgebra of A, but not an anti-hesitant fuzzy soft UP-filter of $\mathcal{P}_{\emptyset}(\{a,b\})$ based on parameters p_3 and p_4. Indeed,

$$\begin{aligned} \mathrm{h}_{\widetilde{\mathrm{H}}[p_3]}(\{a,b\}) = [0.4, 0.6) \nsubseteq [0.4, 0.5] &\\ &= [0.4, 0.5] \cup \{0.4, 0.5\} \\ &= \mathrm{h}_{\widetilde{\mathrm{H}}[p_3]}(\{b\}) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p_3]}(\{a\}) \\ &= \mathrm{h}_{\widetilde{\mathrm{H}}[p_3]}(\{a\} \cdot \{a,b\}) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p_3]}(\{a\}) \end{aligned}$$

and

$$\begin{aligned} \mathrm{h}_{\widetilde{\mathrm{H}}[p_4]}(\{b\}) = [0.7, 0.9] \nsubseteq \{0.8\} &\\ &= \{0.8\} \cup \{0.8\} \\ &= \mathrm{h}_{\widetilde{\mathrm{H}}[p_4]}(\emptyset) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p_4]}(\{a,b\}) \\ &= \mathrm{h}_{\widetilde{\mathrm{H}}[p_4]}(\{a,b\} \cdot \{b\}) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p_4]}(\{a,b\}). \end{aligned}$$

Theorem 5.4. *A hesitant fuzzy soft set* $(\widetilde{\mathrm{H}}, Y)$ *over* A *is an anti-hesitant fuzzy soft UP-filter of* A *if and only if it satisfies the condition* (4.2).

Proof. Assume that $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft UP-filter of A. Let $p \in Y$ and let $x, y, z \in A$ be such that $z \leq x \cdot y$. Then $\widetilde{\mathrm{H}}[p]$ is an anti-hesitant fuzzy UP-filter of A. By Proposition 3.4, we have $\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(z) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot y)$. Therefore,

$$\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot y) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(z) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x).$$

Conversely, assume that $(\widetilde{\mathrm{H}}, Y)$ satisfies the condition (4.2). Let $p \in Y$ and let $x \in A$. By Corollary 4.6, we have $\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(0) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x)$. Let $x, y \in A$. By Proposition 2.5 (1), we have $(x \cdot y) \cdot (x \cdot y) = 0$ and thus $x \cdot y \leq x \cdot y$. It follows from (4.2) that

$$\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot y) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x).$$

Therefore, $\widetilde{\mathrm{H}}[p]$ is an anti-hesitant fuzzy UP-filer of A. Hence, $(\widetilde{\mathrm{H}}, Y)$ is a p-anti-hesitant fuzzy soft UP-filter of A. Since p is arbitrary, we know that $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft UP-filter of A. □

Theorem 5.5. *Let* $(\widetilde{\mathrm{H}}, Y)$ *be a hesitant fuzzy soft set over* A *which satisfies the condition:*

(5.1) $(\forall p \in Y \forall w, x, y, z \in A)(x \leq w \cdot (y \cdot z) \Rightarrow \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot z) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y))$.

Then it is an anti-hesitant fuzzy soft UP-filter of A.

Proof. Assume that $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft set over A which satisfies the condition (5.1). Let $p \in Y$ and let $x, y \in A$. By Proposition 2.5 (1) and (UP-2), we have $0 \cdot ((x \cdot y) \cdot (x \cdot y)) = 0 \cdot 0 = 0$ and thus $0 \leq (x \cdot y) \cdot (x \cdot y)$. It follows from (5.1) that

$$\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y) = \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(0 \cdot y) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot y) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x).$$

Therefore, $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft UP-filter of A. □

Corollary 5.6. *If* $(\widetilde{\mathrm{H}}, Y)$ *is a hesitant fuzzy soft set over A which satisfies the condition* (5.1)*, then it satisfies the condition* (4.2)*.*

Proof. It is straightforward from Theorems 5.5 and 5.4. □

Example 5.7. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	3	3
2	0	1	0	0
3	0	1	2	0

Then $(A, \cdot, 0)$ is a UP-algebra. Let $Y = \{p_1, p_2, p_3, p_4\}$ be a parameter set. We define a hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over A by the following table:

$\widetilde{\mathrm{H}}$	0	1	2	3
p_1	$\{0.4\}$	$\{0.4, 0.5\}$	$\{0.4\}$	$\{0.4\}$
p_2	$\{0\}$	$\{0, 1\}$	$\{0, 1\}$	$\{0\}$
p_3	$\{0.9\}$	$\{0.9\}$	$[0.9, 1]$	$[0.9, 1]$
p_4	$(0, 1]$	$[0, 1]$	$[0, 1]$	$[0, 1]$

Then $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft UP-filter of A, But it does not satisfy the condition (5.1) because $3 \cdot (0 \cdot (1 \cdot 2)) = 0$ implies

$$\begin{aligned}\mathrm{h}_{\widetilde{\mathrm{H}}[p_3]}(3 \cdot 2) &= \mathrm{h}_{\widetilde{\mathrm{H}}[p_3]}(2) \\ &= [0.9, 1] \\ &\not\subseteq \{0.9\} \\ &= \mathrm{h}_{\widetilde{\mathrm{H}}[p_3]}(0) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p_3]}(1).\end{aligned}$$

Theorem 5.8. *Let* $(\widetilde{\mathrm{H}}, Y)$ *be a hesitant fuzzy soft set over A which satisfies the condition:*

(5.2) $(\forall p \in Y \forall w, x, y, z \in A)(w \leq x \cdot (y \cdot z) \Rightarrow \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot z) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y)).$

Then it is an anti-hesitant fuzzy soft UP-filter of A.

Proof. Assume that $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft set over A which satisfies the condition (5.2). Let $p \in Y$ and let $x, y \in A$. By Proposition 2.5 (1) and (UP-2), we have $(x \cdot y) \cdot (0 \cdot (x \cdot y)) = (x \cdot y) \cdot (x \cdot y) = 0$ and thus $x \cdot y \leq 0 \cdot (x \cdot y)$. It follows from (5.2) that

$$\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y) = \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(0 \cdot y) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot y) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x).$$

Therefore, $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft UP-filter of A. □

Corollary 5.9. *If* $(\widetilde{\mathrm{H}}, Y)$ *is a hesitant fuzzy soft set over A which satisfies the condition* (5.2)*, then it satisfies the condition* (4.2)*.*

Proof. It is straightforward from Theorems 5.8 and 5.4. □

6. Anti-hesitant fuzzy soft UP-ideals

Definition 6.1. Let Y be a nonempty subset of P. A hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over A is called a *anti-hesitant fuzzy soft UP-ideal* based on $p \in Y$ (we shortly call a *p-anti-hesitant fuzzy soft UP-ideal*) of A if the hesitant fuzzy set

$$\widetilde{\mathrm{H}}[p] := \{(a, \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(a)) \mid a \in A\}$$

on A is an anti-hesitant fuzzy UP-ideal of A. If $(\widetilde{\mathrm{H}}, Y)$ is a p-anti-hesitant fuzzy soft UP-ideal of A for all $p \in Y$, we state that $(\widetilde{\mathrm{H}}, Y)$ is an *anti-hesitant fuzzy soft UP-ideal* of A.

From [12], we know that every anti-hesitant fuzzy UP-ideal of A is an anti-hesitant fuzzy UP-filter. Then we have the following Theorem:

Theorem 6.2. *Every p-anti-hesitant fuzzy soft UP-ideal of A is a p-anti-hesitant fuzzy soft UP-filter.*

The following example shows that the converse of Theorem 6.2 is not true in general.

Example 6.3. Let $A = \{0, 1, 2, 3, 4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3	4
0	0	1	2	3	4
1	0	0	2	3	4
2	0	0	0	3	3
3	0	1	2	0	3
4	0	1	2	0	0

Then $(A, \cdot, 0)$ is a UP-algebra. Let $Y = \{p_1, p_2, p_3, p_4\}$ be a parameter set. We define a hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over A by the following table:

$\widetilde{\mathrm{H}}$	0	1	2	3	4
p_1	$\{0.7\}$	$[0.7, 0.8)$	$[0.7, 0.9]$	$[0.6, 1]$	$[0.6, 1]$
p_2	$\emptyset$	$\{0.6\}$	$\{0.6\}$	$\{0.6, 0.9\}$	$\{0.6, 0.9\}$
p_3	$\{0.2\}$	$\{0.2, 0.3\}$	$\{0.1, 0.2, 0.3\}$	$\{0.1, 0.2, 0.3\}$	$\{0.1, 0.2, 0.3\}$
p_4	$\{0.3\}$	$\{0.3\}$	$[0.2, 0.4]$	$\{0.2, 0.3\}$	$\{0.2, 0.3\}$

Then $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft UP-filter of A, but not an anti-hesitant fuzzy soft UP-ideal of A based on parameters p_1 and p_2. Indeed,

$$\begin{aligned}\mathrm{h}_{\widetilde{\mathrm{H}}[p_1]}(3 \cdot 4) = \mathrm{h}_{\widetilde{\mathrm{H}}[p_1]}(3) = [0.7, 0.9] &\nsubseteq [0.7, 0.8) \\ &= \{0.7\} \cup [0.7, 0.8) \\ &= \mathrm{h}_{\widetilde{\mathrm{H}}[p_1]}(0) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p_1]}(2) \\ &= \mathrm{h}_{\widetilde{\mathrm{H}}[p_1]}(3 \cdot (2 \cdot 4)) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p_1]}(2)\end{aligned}$$

and

$$\begin{aligned}\mathrm{h}_{\widetilde{\mathrm{H}}[p_2]}(3 \cdot 4) = \mathrm{h}_{\widetilde{\mathrm{H}}[p_2]}(3) = \{0.6, 0.9\} &\nsubseteq \{0.6\} \\ &= \emptyset \cup \{0.6\} \\ &= \mathrm{h}_{\widetilde{\mathrm{H}}[p_2]}(0) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p_2]}(2) \\ &= \mathrm{h}_{\widetilde{\mathrm{H}}[p_2]}(3 \cdot (2 \cdot 4)) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p_2]}(2).\end{aligned}$$

Theorem 6.4. *If $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft UP-ideal of A, then it satisfies the condition* (5.1).

Proof. Assume that $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft UP-ideal of A. Let $p \in Y$ and let $w, x, y, z \in A$ be such that $x \leq w \cdot (y \cdot z)$. Then $\widetilde{\mathrm{H}}[p]$ is an anti-hesitant fuzzy UP-ideal of A and $x \cdot (w \cdot (y \cdot z)) = 0$. Thus $\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot (y \cdot z)) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot (w \cdot (y \cdot z))) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w) = \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(0) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w) = \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w)$. Therefore, $\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot z) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot (y \cdot z)) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y)$. □

Theorem 6.5. *A hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over A is an anti-hesitant fuzzy soft UP-ideal of A if and only if it satisfies the condition* (5.2).

Proof. Assume that $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft UP-ideal of A. Let $p \in Y$ and $w, x, y, z \in A$ be such that $w \leq x \cdot (y \cdot z)$. Then $\widetilde{\mathrm{H}}[p]$ is an anti-hesitant fuzzy UP-ideal of A. By Proposition 3.4, we have $\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot (y \cdot z))$. Therefore,

$$\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot z) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot (y \cdot z)) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y).$$

Conversely, assume that $(\widetilde{\mathrm{H}}, Y)$ satisfies the condition (5.2). Let $p \in Y$ and let $x \in A$. By Corollaries 5.9 and 4.6, we have $\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(0) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x)$. Let $x, y, z \in A$. By Proposition 2.5 (1), we have $(x \cdot (y \cdot z)) \cdot (x \cdot (y \cdot z)) = 0$ and thus $x \cdot (y \cdot z) \leq x \cdot (y \cdot z)$. It follows from (5.2) that

$$\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot z) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot (y \cdot z)) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y).$$

Therefore, $\widetilde{\mathrm{H}}[p]$ is an anti-hesitant fuzzy UP-ideal of A. Hence, $(\widetilde{\mathrm{H}}, Y)$ is a p-anti-hesitant fuzzy soft UP-ideal of A. Since p is arbitrary, we know that $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft UP-ideal of A. □

Theorem 6.6. *Let* $(\widetilde{\mathrm{H}}, Y)$ *be a hesitant fuzzy soft set over* A *which satisfies the condition:*

(6.1) $(\forall p \in Y \forall w, x, y, z \in A)(w \leq (z \cdot y) \cdot (z \cdot x) \Rightarrow \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y))$.

Then it is an anti-hesitant fuzzy soft UP-ideal of A*.*

Proof. Assume that $(\widetilde{\mathrm{H}}, Y)$ is a hesitant fuzzy soft set over A which satisfies the condition (6.1). Let $p \in Y$ and let $x, y, z \in A$. By Proposition 2.5 (1) and (UP-3), we have $(x \cdot (y \cdot z)) \cdot (((x \cdot z) \cdot y) \cdot ((x \cdot z) \cdot (x \cdot z))) = (x \cdot (y \cdot z)) \cdot (((x \cdot z) \cdot y) \cdot 0) = (x \cdot (y \cdot z)) \cdot 0 = 0$ and thus $x \cdot (y \cdot z) \leq ((x \cdot z) \cdot y) \cdot ((x \cdot z) \cdot (x \cdot z))$. It follows from (6.1) that

$$\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot z) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x \cdot (y \cdot z)) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y).$$

Therefore, $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft UP-ideal of A. □

Corollary 6.7. *If* $(\widetilde{\mathrm{H}}, Y)$ *is a hesitant fuzzy soft set over* A *which satisfies the condition* (6.1)*, then it satisfies the conditions* (5.1) *and* (5.2).

Proof. It is straightforward from Theorems 6.6, 6.4 and 6.5. □

7. Anti-hesitant fuzzy soft strongly UP-ideals

Definition 7.1. Let Y be a nonempty subset of P. A hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over A is called a *anti-hesitant fuzzy soft strongly UP-ideal* based on $p \in Y$ (we shortly call a *p-anti-hesitant fuzzy soft strongly UP-ideal*) of A if the hesitant fuzzy set

$$\widetilde{\mathrm{H}}[p] := \{(a, \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(a)) \mid a \in A\}$$

on A is an anti-hesitant fuzzy strongly UP-ideal of A. If $(\widetilde{\mathrm{H}}, Y)$ is a p-anti-hesitant fuzzy soft strongly UP-ideal of A for all $p \in Y$, we state that $(\widetilde{\mathrm{H}}, Y)$ is an *anti-hesitant fuzzy soft strongly UP-ideal* of A.

From [12], we know that every anti-hesitant fuzzy strongly UP-ideal of A is an anti-hesitant fuzzy UP-ideal. Then we have the following Theorem:

Theorem 7.2. *Every p-anti-hesitant fuzzy soft strongly UP-ideal of A is a p-anti-hesitant fuzzy soft UP-ideal.*

The following example shows that the converse of Theorem 7.2 is not true in general.

Example 7.3. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	3	3
2	0	1	0	0
3	0	1	2	0

Then $(A, \cdot, 0)$ is a UP-algebra. Let $Y = \{p_1, p_2, p_3, p_4\}$ be a parameter set. We define a hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over A by the following table:

$\widetilde{\mathrm{H}}$	0	1	2	3
p_1	$[0.3, 0.8)$	$[0.3, 0.8)$	$[0.3, 0.8)$	$[0.3, 0.8]$
p_2	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
p_3	$[0, 0.2]$	$[0, 0.2]$	$[0, 0.2]$	$[0, 0.2]$
p_4	$(0, 0.1]$	$[0, 0.1]$	$[0, 0.1]$	$[0, 0.1]$

Then $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft UP-ideal of A, but not an anti-hesitant fuzzy soft strongly UP-ideal of A based on parameters p_1 and p_4. Indeed,

$$\begin{aligned} \mathrm{h}_{\widetilde{\mathrm{H}}[p_1]}(3) &= [0.3, 0.8] \\ &\not\subseteq [0.3, 0.8) \\ &= [0.3, 0.8) \cup [0.3, 0.8) \\ &= \mathrm{h}_{\widetilde{\mathrm{H}}[p_1]}(0) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p_1]}(1) \\ &= \mathrm{h}_{\widetilde{\mathrm{H}}[p_1]}((3 \cdot 1) \cdot (3 \cdot 3)) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p_1]}(1) \end{aligned}$$

and

$$\begin{aligned} \mathrm{h}_{\widetilde{\mathrm{H}}[p_4]}(2) &= [0, 0.1] \\ &\not\subseteq (0, 0.1] \\ &= (0, 0.1] \cup (0, 0.1] \\ &= \mathrm{h}_{\widetilde{\mathrm{H}}[p_4]}(0) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p_4]}(0) \\ &= \mathrm{h}_{\widetilde{\mathrm{H}}[p_4]}((2 \cdot 0) \cdot (2 \cdot 2)) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p_4]}(0). \end{aligned}$$

By Theorems 5.2, 6.2, and 7.2 and Examples 5.3, 6.3, and 7.3, we have that the notion of p-anti-hesitant fuzzy soft UP-subalgebras is a generalization of

p-anti-hesitant fuzzy soft UP-filters, the notion of p-anti-hesitant fuzzy soft UP-filters is a generalization of p-anti-hesitant fuzzy soft UP-ideals, and the notion of p-anti-hesitant fuzzy soft UP-ideals is a generalization of p-anti-hesitant fuzzy soft strongly UP-ideals.

Theorem 7.4. *A hesitant fuzzy soft set* $(\widetilde{\mathrm{H}}, Y)$ *over* A *is an anti-hesitant fuzzy soft strongly UP-ideal of* A *if and only if it satisfies the condition* (6.1).

Proof. Assume that $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft strongly UP-ideal of A. Let $p \in Y$ and let $w, x, y, z \in A$ be such that $w \leq (z \cdot y) \cdot (z \cdot x)$. Then $\widetilde{\mathrm{H}}[p]$ is an anti-hesitant fuzzy strongly UP-ideal of A. By Proposition 3.4, we have $\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w) \supseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}((z \cdot y) \cdot (z \cdot x))$. Therefore,

$$\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}((z \cdot y) \cdot (z \cdot x)) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(w) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y).$$

Conversely, assume that $(\widetilde{\mathrm{H}}, Y)$ satisfies the condition (6.1). Let $p \in Y$ and let $x \in A$. By Corollaries 6.7, 5.9 and 4.6, respectively, we have $\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(0) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x)$. Let $x, y, z \in A$. Since $((z \cdot y) \cdot (z \cdot x)) \cdot ((z \cdot y) \cdot (z \cdot x)) = 0$, we have $(z \cdot y) \cdot (z \cdot x) \leq (z \cdot y) \cdot (z \cdot x)$. It follows from (6.1) that

$$\mathrm{h}_{\widetilde{\mathrm{H}}[p]}(x) \subseteq \mathrm{h}_{\widetilde{\mathrm{H}}[p]}((z \cdot y) \cdot (z \cdot x)) \cup \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(y).$$

Therefore, $\widetilde{\mathrm{H}}[p]$ is an anti-hesitant fuzzy strongly UP-ideal of A. Hence, $(\widetilde{\mathrm{H}}, Y)$ is a p-anti-hesitant fuzzy soft strongly UP-ideal of A. Since p is arbitrary, we know that $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft strongly UP-ideal of A. □

Theorem 7.5. *Let* $(\widetilde{\mathrm{H}}, Y)$ *be a hesitant fuzzy soft set over* A *and* N *be a nonempty subset of* Y. *Then the following statements are hold:*

(1) *If* $(\widetilde{\mathrm{H}}, Y)$ *is an anti-hesitant fuzzy soft strongly UP-ideal (resp., anti-hesitant fuzzy soft UP-ideal, anti-hesitant fuzzy soft UP-filter) of* A*, then* $(\widetilde{\mathrm{H}}|_N, N)$ *is an anti-hesitant fuzzy soft strongly UP-ideal (resp., anti-hesitant fuzzy soft UP-ideal, anti-hesitant fuzzy soft UP-filter) of* A*, and*

(2) *there exists* $(\widetilde{\mathrm{H}}|_N, N)$ *is an anti-hesitant fuzzy soft strongly UP-ideal (resp., anti-hesitant fuzzy soft UP-ideal, anti-hesitant fuzzy soft UP-filter) of* A*, but* $(\widetilde{\mathrm{H}}, Y)$ *is not an anti-hesitant fuzzy soft strongly UP-ideal (resp., anti-hesitant fuzzy soft UP-ideal, anti-hesitant fuzzy soft UP-filter) of* A.

Proof. (1) Assume that $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft strongly UP-ideal (resp., anti-hesitant fuzzy soft UP-ideal, anti-hesitant fuzzy soft UP-filter) of A. In the same way as Theorem 4.7, we can show that $(\widetilde{\mathrm{H}}|_N, N)$ is an anti-hesitant fuzzy soft strongly UP-ideal (resp., anti-hesitant fuzzy soft UP-ideal, anti-hesitant fuzzy soft UP-filter) of A.

(2) Example 7.3 (resp., Example 6.3, Example 5.3), if we choose $N = \{p_2, p_3\}$ (resp., $\{p_3, p_4\}$, $\{p_1, p_2\}$) , then $(\widetilde{\mathrm{H}}|_N, N)$ is an anti-hesitant fuzzy soft strongly

UP-ideal (resp., anti-hesitant fuzzy soft UP-ideal, anti-hesitant fuzzy soft UP-filter) of A, but $(\widetilde{\mathrm{H}}, Y)$ is not an anti-hesitant fuzzy soft strongly UP-ideal (resp., anti-hesitant fuzzy soft UP-ideal, anti-hesitant fuzzy soft UP-filter) of A. □

Definition 7.6 ([13]). Let Y be a nonempty subset of P. A hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over A is called a *constant hesitant fuzzy soft set* based on $p \in Y$ (we shortly call a *p-constant hesitant fuzzy soft set*) over A if the hesitant fuzzy set

$$\widetilde{\mathrm{H}}[p] := \{(a, \mathrm{h}_{\widetilde{\mathrm{H}}[p]}(a)) \mid a \in A\}$$

on A is a constant hesitant fuzzy set on A. If $(\widetilde{\mathrm{H}}, Y)$ is a p-constant hesitant fuzzy soft set over A for all $p \in Y$, we state that $(\widetilde{\mathrm{H}}, Y)$ is a *constant hesitant fuzzy soft set* over A.

Theorem 7.7 ([13]). *A hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over A is a hesitant fuzzy soft strongly UP-ideal of A if and only if is a constant hesitant fuzzy soft set over A.*

Theorem 7.8 ([12]). *A hesitant fuzzy set* H *on A is an anti-hesitant fuzzy strongly UP-ideal of A if and only if it is a constant hesitant fuzzy set on A.*

Theorem 7.9. *A hesitant fuzzy soft set $(\widetilde{\mathrm{H}}, Y)$ over A is an anti-hesitant fuzzy soft strongly UP-ideal of A if and only if is a constant hesitant fuzzy soft set over A.*

Proof. Assume that $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft strongly UP-ideal of A and let $p \in Y$. Then $\widetilde{\mathrm{H}}[p]$ is an anti-hesitant fuzzy strongly UP-ideal of A. By Theorem 7.8, we obtain $\widetilde{\mathrm{H}}[p]$ is a constant hesitant fuzzy set on A. Thus $(\widetilde{\mathrm{H}}, Y)$ is a p-constant hesitant fuzzy soft set over A. Since p is arbitrary, we know that $(\widetilde{\mathrm{H}}, Y)$ is a constant hesitant fuzzy soft set over A.

Conversely, let $p \in Y$. Assume that $(\widetilde{\mathrm{H}}, Y)$ is a constant hesitant fuzzy soft set over A. Then $\widetilde{\mathrm{H}}[p]$ is a constant hesitant fuzzy set on A. By Theorem 7.8, we have $\widetilde{\mathrm{H}}[p]$ is an anti-hesitant fuzzy strongly UP-ideal of A. Since p is arbitrary, we state that $(\widetilde{\mathrm{H}}, Y)$ is an anti-hesitant fuzzy soft strongly UP-ideal of A. □

Corollary 7.10. *For UP-algebras, we can conclude that the notions of anti-hesitant fuzzy soft strongly UP-ideals and hesitant fuzzy soft strongly UP-ideals coincide.*

Proof. It is straightforward by Theorems 7.7 and 7.9. □

8. Conclusions and future work

In this paper, we have introduced the notion of anti-hesitant fuzzy soft sets over UP-algebras which is a new extension of anti-type of hesitant fuzzy sets on UP-algebras and the notions of anti-hesitant fuzzy soft UP-subalgebras, anti-hesitant fuzzy soft UP-filters, anti-hesitant fuzzy soft UP-ideals and anti-hesitant fuzzy soft strongly UP-ideals of UP-algebras and investigated some of its important properties. Then we have the diagram of anti-type of hesitant fuzzy soft sets over UP-algebras below.

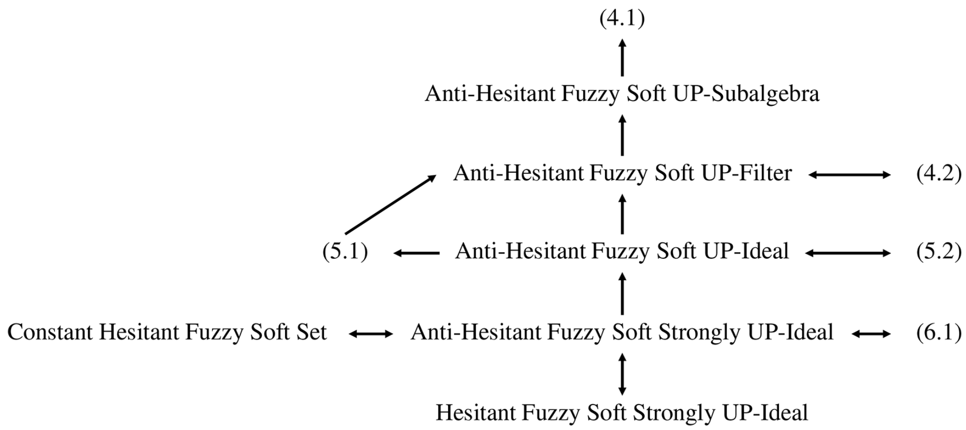

Figure 1: Anti-type of hesitant fuzzy soft sets over UP-algebras

In our future study of UP-algebras, the following objectives considered:

- To get more results in anti-type of hesitant fuzzy soft sets over UP-algebras.
- To define operations of anti-type of hesitant fuzzy soft sets over UP-algebras.
- To study anti-type of hesitant fuzzy soft sets over UP-algebras used in decision making.

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