Integral inequalities of Hermite-Hadamard type for extended (s,m)-GA-ε-convex functions

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Abstract. In the paper, by introducing the concept of (s,m)-GA-ε-functions and using Hölder's integral inequality, the authors develop several new integral inequalities of the Hermite-Hadamard type for extended (s,m)-ε-GA-convex functions.
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1. Introduction

Throughout this paper, we use the following notations:

$$\mathbb{R} = (-\infty, \infty), \quad \mathbb{R}_0 = [0, \infty), \quad \text{and} \quad \mathbb{R}_+ = (0, \infty).$$

Let us recall and review some concepts of convex functions.

Definition 1.1 ([13]). *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is called to be convex on an interval I if*

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$$

for all $x, y \in I$ and $\lambda \in [0,1]$.

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Definition 1.2 ([2])**.** *For some $s \in (0,1]$, a function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is called to be s-convex in the second sense on an interval I if*

$$f(\lambda x + (1-\lambda)y) \le \lambda^s f(x) + (1-\lambda)^s f(y) \tag{1.1}$$

for all $x, y \in I$ and $\lambda \in [0,1]$.

Definition 1.3 ([23])**.** *For some $s \in [-1,1]$, a function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is called to be extended s-convex on a interval I if the inequality* (1.1) *holds for all $x, y \in I$ and $\lambda \in (0,1)$.*

Definition 1.4 ([19])**.** *For some $m \in (0,1]$, a function $f : [0,b] \to \mathbb{R}$ is called to be m-convex on $[0,b]$ if*

$$f(\lambda x + m(1-\lambda)y) \le \lambda f(x) + m(1-\lambda) f(y)$$

for all $x, y \in [0,b]$ and $\lambda \in [0,1]$.

Definition 1.5 ([5, 15])**.** *For some $(s,m) \in (0,1] \times (0,1]$, a function $f : [0,b] \to \mathbb{R}$ is called to be (s,m)-convex on $[0,b]$ if*

$$f(\lambda x + m(1-\lambda)y) \le \lambda^s f(x) + m(1-\lambda)^s f(y) \tag{1.2}$$

for all $x, y \in [0,b]$ and $\lambda \in [0,1]$.

Definition 1.6 ([27])**.** *For some $(s,m) \in [-1,1] \times (0,1]$, a function $f : [0,b] \to \mathbb{R}$ is called to be extended (s,m)-convex on $[0,b]$ if the inequality* (1.2) *is valid for all $x, y \in [0,b]$ and $\lambda \in (0,1)$.*

Definition 1.7 ([14])**.** *A function $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ is called to be geometrically-arithmetically-convex (or say, GA-convex) on I if*

$$f\big(x^\lambda y^{1-\lambda}\big) \le \lambda f(x) + (1-\lambda) f(y)$$

for all $x, y \in I$ and $\lambda \in [0,1]$.

Definition 1.8 ([16])**.** *For $\varepsilon \ge 0$, a function $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ is called to be ε-GA-convex on I if*

$$f\big(x^\lambda y^{1-\lambda}\big) \le \lambda f(x) + (1-\lambda) f(y) + \varepsilon$$

for all $x, y \in I$ and $\lambda \in [0,1]$.

Definition 1.9 ([11, 12])**.** *For some $(s,m) \in [-1,1] \times (0,1]$, a function $f : (0,b] \to \mathbb{R}$ is called to be extended (s,m)-GA-convex on $(0,b]$ if*

$$f\big(x^\lambda y^{m(1-\lambda)}\big) \le \lambda^s f(x) + m(1-\lambda)^s f(y)$$

for all $x, y \in (0,b]$ and $\lambda \in (0,1)$.

The well-known integral Hermite-Hadamard inequality for convex functions reads that, if f is a convex function on $[a, b]$, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x \le \frac{f(a)+f(b)}{2}.$$

In recent decades, plenty of integral inequalities of the Hermite-Hadamard type for various kinds of convex functions have been established in, for example, [1, 3, 4, 6, 7, 8, 17, 20, 22, 25, 26]. In particular, some integral inequalities of the Hermite-Hadamard type for geometrically convex functions and for s-GA-convex functions were studied in [9, 10, 18, 21, 24, 28].

2. A new definition and two lemmas

Now we extend Definition 1.9 by introducing the definition of extended (s, m)-ε-GA-convex functions.

Definition 2.1. *For some $(s, m) \in [-1, 1] \times (0, 1]$ and $\varepsilon \ge 0$, a function $f : (0, b] \to \mathbb{R}_0$ is called to be extended (s, m)-ε-GA-convex on $(0, b]$ if*

$$f\left(x^\lambda y^{m(1-\lambda)}\right) \le \lambda^s f(x) + m(1-\lambda)^s f(y) + \varepsilon$$

for all $x, y \in (0, b]$ and $\lambda \in (0, 1)$.

It is clear that, when $\varepsilon = 0$ in Definition 2.1, one will derive Definition 1.9.

In order to prove our main results, we need the following lemmas.

Lemma 1. *Let $f : I \subseteq \mathbb{R}_+ \to \mathbb{R}$ be differentiable on the interior I° and $a, b \in I$ with $a < b$. If $f' \in L_1([a, b])$ and $\lambda \in (0, 1)$, then*

$$\begin{aligned} f\left(a^{1-\lambda}b^\lambda\right) &- \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}\,x \\ &= (\ln b - \ln a)\left[\int_0^\lambda t a^{1-t}b^t f'\left(a^{1-t}b^t\right)\mathrm{d}\,t - \int_\lambda^1 (1-t)a^{1-t}b^t f'\left(a^{1-t}b^t\right)\mathrm{d}\,t\right]. \end{aligned}$$

Proof. Integrating by part and changing variables of integration yield

$$\begin{aligned} &(\ln b - \ln a)\left[\int_0^\lambda t a^{1-t}b^t f'\left(a^{1-t}b^t\right)\mathrm{d}\,t - \int_\lambda^1 (1-t)a^{1-t}b^t f'\left(a^{1-t}b^t\right)\mathrm{d}\,t\right] \\ &= \int_0^\lambda t f'\left(a^{1-t}b^t\right)\mathrm{d}\left(a^{1-t}b^t\right) - \int_\lambda^1 (1-t) f'\left(a^{1-t}b^t\right)\mathrm{d}\left(a^{1-t}b^t\right) \\ &= \lambda f\left(a^{1-\lambda}b^\lambda\right) + (1-\lambda) f\left(a^{1-\lambda}b^\lambda\right) - \int_0^1 f\left(a^{1-t}b^t\right)\mathrm{d}\,t \\ &= f\left(a^{1-\lambda}b^\lambda\right) - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}\,x. \end{aligned}$$

This completes the proof of Lemma 1. □

Lemma 2. *Let $x, y > 0$, $\lambda \in (0,1)$, and $s \in [-1,1]$. Then*

$$F(x,y;\lambda) = \int_0^\lambda t x^t y^{1-t} \,\mathrm{d}\, t = \begin{cases} \dfrac{\lambda y^{1-\lambda}[L(x^\lambda, y^\lambda) - x^\lambda]}{\ln y - \ln x}, & x \neq y; \\ \dfrac{\lambda^2 x}{2}, & x = y, \end{cases}$$

$$\begin{aligned} G(x,y;\lambda,s) &= \int_0^\lambda t[tx + (1-t)y]t^s \,\mathrm{d}\, t \\ &= \frac{(s+2)\lambda x + (s+3-(s+2)\lambda)y}{(s+2)(s+3)}\lambda^{s+2}, \end{aligned}$$

and

$$\begin{aligned} H(x,y;\lambda,s) &= \int_0^\lambda t[tx + (1-t)y](1-t)^s \,\mathrm{d}\, t \\ &= \begin{cases} \dfrac{\begin{array}{c} 2x + (s+1)y - [(1-\lambda)(s+1)(1+2\lambda+\lambda s)y \\ +(2+\lambda(s+1)(2+2\lambda+\lambda s)x)](1-\lambda)^{s+1} \end{array}}{(s+1)(s+2)(s+3)}, & -1 < s \le 1; \\ \dfrac{\lambda^2(y-x) - 2(\ln(1-\lambda)+\lambda)x}{2}, & s = -1, \end{cases} \end{aligned}$$

where the logarithmic mean $L(x,y)$ is defined as

$$L(x,y) = \begin{cases} \dfrac{y-x}{\ln y - \ln x}, & x \neq y; \\ x, & x = y. \end{cases}$$

3. Main results

We are now in a position to establish some integral inequalities of the Hermite-Hadamrd type for functions of the extended (s,m)-ε-GA-convexity.

Theorem 1. *Let $(s,m) \in [-1,1] \times (0,1]$ and $\lambda \in (0,1)$ and let $f : (0,b^*] \to \mathbb{R}$ be a differentiable mapping on $(0,b^*]$, where $a, b \in (0,b^*]$, $a < b$, $b^{1/m} \le b^*$, and $f' \in L_1([a,b])$. If $|f'|^q$ is extended (s,m)-ε-GA-convex on $(0, \max\{b^{1/m}, b\}]$ for $q \ge 1$, then*

$$\begin{aligned} &\left| f\big(a^\lambda b^{1-\lambda}\big) - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}\, x \right| \\ &\le \frac{\ln b - \ln a}{6^{1/q}}\big[F^{1-1/q}(a,b;\lambda)\big(6G(a,b;\lambda,s)|f'(a)|^q \\ &+ 6mH(a,b;\lambda,s)\big|f'\big(b^{1/m}\big)\big|^q + \varepsilon\lambda^2[2\lambda a + (3-2\lambda)b]\big)^{1/q} \\ &+ F^{1-1/q}(b,a;1-\lambda,s)\big(6H(b,a;1-\lambda,s)|f'(a)|^q \\ &+ 6mG(b,a;1-\lambda,s)\big|f'\big(b^{1/m}\big)\big|^q + \varepsilon(1-\lambda)^2[(1+2\lambda)a + 2(1-\lambda)b]\big)^{1/q}\big], \end{aligned} \tag{3.1}$$

where $F(a,b;\lambda)$, $G(a,b;\lambda,s)$, and $H(a,b;\lambda,s)$ are defined as in Lemma 2.

Proof. By Lemma 1 and Hölder's integral inequality, we have

$$\begin{aligned}
&\left|f\left(a^\lambda b^{1-\lambda}\right)-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\,\mathrm{d}\,x\right| \\
&\le(\ln b-\ln a)\left[\int_0^\lambda ta^tb^{1-t}\left|f'\left(a^tb^{1-t}\right)\right|\mathrm{d}\,t+\int_\lambda^1(1-t)a^tb^{1-t}\left|f'\left(a^tb^{1-t}\right)\right|\mathrm{d}\,t\right] \\
&\le(\ln b-\ln a)\left[\left(\int_0^\lambda ta^tb^{1-t}\,\mathrm{d}\,t\right)^{1-1/q}\left(\int_0^\lambda ta^tb^{1-t}\left|f'\left(a^tb^{1-t}\right)\right|^q\mathrm{d}\,t\right)^{1/q}\right. \qquad (3.2)\\
&\left.+\left(\int_\lambda^1(1-t)a^tb^{1-t}\,\mathrm{d}\,t\right)^{1-1/q}\left(\int_\lambda^1(1-t)a^tb^{1-t}\left|f'\left(a^tb^{1-t}\right)\right|^q\mathrm{d}\,t\right)^{1/q}\right].
\end{aligned}$$

By Lemma 2, we can obtain

$$\int_0^\lambda(1-t)a^tb^{1-t}\,\mathrm{d}\,t=F(a,b;\lambda,s) \tag{3.3}$$

and

$$\int_\lambda^1(1-t)a^tb^{1-t}\,\mathrm{d}\,t=F(a,b;1-\lambda,s). \tag{3.4}$$

Using the extended (s,m)-ε-GA-convexity of $|f'|^q$, the geometric-arithmetic mean inequality, and Lemma 2 gives

$$\begin{aligned}
&\int_0^\lambda ta^tb^{1-t}\left|f'\left(a^tb^{1-t}\right)\right|^q\mathrm{d}\,t \\
&\le\int_0^\lambda t[ta+(1-t)b]\left[t^s|f'(a)|^q+m(1-t)^s\left|f'\left(b^{1/m}\right)\right|^q+\varepsilon\right]\mathrm{d}\,t \qquad (3.5)\\
&=G(a,b;\lambda,s)|f'(a)|^q+mH(a,b;\lambda,s)\left|f'\left(b^{1/m}\right)\right|^q+\frac{1}{6}\varepsilon\lambda^2[2\lambda a+(3-2\lambda)b]
\end{aligned}$$

and

$$\begin{aligned}
&\int_\lambda^1(1-t)a^tb^{1-t}\left|f'\left(a^tb^{1-t}\right)\right|^q\mathrm{d}\,t\le\int_\lambda^1(1-t)[ta+(1-t)b]\left[t^s|f'(a)|^q\right. \\
&\left.+m(1-t)^s\left|f'\left(b^{1/m}\right)\right|^q+\varepsilon\right]\mathrm{d}\,t=H(b,a;1-\lambda,s)|f'(a)|^q \qquad (3.6)\\
&+mG(b,a;1-\lambda,s)\left|f'\left(b^{1/m}\right)\right|^q+\frac{\varepsilon(1-\lambda)^2[(1+2\lambda)a+2(1-\lambda)b]}{6}.
\end{aligned}$$

Substituting inequalities (3.3), (3.4), (3.5), and (3.6) into the inequality (3.2) results in the inequality (3.1). The proof of Theorem 1 is complete. □

Corollary 1. *Under conditions of Theorem 1, if $s=-1$, we have*

$$\begin{aligned}&\left|f\big(a^\lambda b^{1-\lambda}\big)-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\,\mathrm{d}\,x\right|\\&\le\frac{\ln b-\ln a}{6^{1/q}}\big[F^{1-1/q}(a,b;\lambda,-1)\big(3(\lambda^2(a-b)+2\lambda b)|f'(a)|^q\\&+3m[\lambda^2(b-a)-2a(\ln(1-\lambda)+\lambda)]\big|f'\big(b^{1/m}\big)\big|^q+\varepsilon\lambda^2[2\lambda a+(3-2\lambda)b]\big)^{1/q}\\&+F^{1-1/q}(b,a;1-\lambda,-1)\big(3\big[(1-\lambda)^2(a-b)-2b(1-\lambda+\ln\lambda)\big]|f'(a)|^q\\&+3m[(b-a)(1-\lambda)^2+2a(1-\lambda)]\big|f'\big(b^{1/m}\big)\big|^q\\&+\varepsilon(1-\lambda)^2[(1+2\lambda)a+2(1-\lambda)b]\big)^{1/q}\big].\end{aligned}$$

Theorem 2. *Let $(s,m)\in[-1,1]\times(0,1]$ and $\lambda\in(0,1)$ and let $f:(0,b^*]\to\mathbb{R}$ be a differentiable mapping on $(0,b^*]$, where $a,b\in(0,b^*]$, $a<b$, $b^{1/m}\le b^*$, and $f'\in L_1([a,b])$. If $|f'|^q$ is extended (s,m)-ε-GA-convex on $(0,\max\{b^{1/m},b\}]$ for $q>1$, then*

$$\begin{aligned}&\left|f\big(a^\lambda b^{1-\lambda}\big)-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\,\mathrm{d}\,x\right|\le\frac{\ln b-\ln a}{6^{1/q}}\Bigg[F^{1-1/q}(a,b;\lambda)\\&\times\left(\frac{\lambda^{s+2}}{s+2}|f'(a)|^q+mT(\lambda,s)\big|f'(b^{1/m})\big|^q+\frac12\varepsilon\lambda^2\right)^{1/q}+F^{1-1/q}(b,a;\\&1-\lambda,s)T(1-\lambda,s)|f'(a)|^q+\frac{m(1-\lambda)^{s+2}}{s+2}\big|f'\big(b^{1/m}\big)\big|^q+\frac{\varepsilon(1-\lambda)^2}{2}\Bigg],\end{aligned}\tag{3.7}$$

where

$$T(\lambda,s)=\int_0^\lambda t(1-t)^s\,\mathrm{d}\,t=\begin{cases}\dfrac{1-(1-\lambda)^{s+1}(1+\lambda+\lambda s)}{(s+1)(s+2)}, & -1<s\le1;\\ -\lambda-\ln(1-\lambda), & s=-1.\end{cases}$$

Proof. By Lemma 1 and Hölder's integral inequality, we have

$$\begin{aligned}&\left|f\big(a^\lambda b^{1-\lambda}\big)-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\,\mathrm{d}\,x\right|\\&\le(\ln b-\ln a)\left[\int_0^\lambda ta^tb^{1-t}\big|f'\big(a^tb^{1-t}\big)\big|\,\mathrm{d}\,t+\int_\lambda^1(1-t)a^tb^{1-t}\big|f'\big(a^tb^{1-t}\big)\big|\mathrm{d}\,t\right]\\&\le(\ln b-\ln a)\left[\left(\int_0^\lambda t\big(a^tb^{1-t}\big)^{q/(q-1)}\,\mathrm{d}\,t\right)^{1-1/q}\left(\int_0^\lambda t\big|f'\big(a^tb^{1-t}\big)\big|^q\,\mathrm{d}\,t\right)^{1/q}\right.\\&\left.+\left(\int_\lambda^1(1-t)\big(a^tb^{1-t}\big)^{q/(q-1)}\,\mathrm{d}\,t\right)^{1-1/q}\left(\int_\lambda^1(1-t)\big|f'\big(a^tb^{1-t}\big)\big|^q\,\mathrm{d}\,t\right)^{1/q}\right].\end{aligned}\tag{3.8}$$

By Lemma 2, we obtain

$$\int_0^\lambda t\big(a^tb^{1-t}\big)^{q/(q-1)}\,\mathrm{d}\,t=F\big(a^{q/(q-1)},b^{q/(q-1)};\lambda,s\big)\tag{3.9}$$

and

$$\int_{\lambda}^{1} (1-t)\big(a^t b^{1-t}\big)^{q/(q-1)}\,\mathrm{d}\,t = F\big(b^{q/(q-1)}, a^{q/(q-1)}; 1-\lambda, s\big). \tag{3.10}$$

From the extended (s,m)-ε-GA-convexity of $|f'|^q$, we have

$$\begin{aligned}\int_{0}^{\lambda} t a^t b^{1-t}\big|f'\big(a^t b^{1-t}\big)\big|^q\,\mathrm{d}\,t &\le \int_0^{\lambda} t\big[t^s|f'(a)|^q + m(1-t)^s\big|f'\big(b^{1/m}\big)\big|^q + \varepsilon\big]\,\mathrm{d}\,t \\ &= \frac{\lambda^{s+2}}{s+2}|f'(a)|^q + mT(\lambda,s)\big|f'\big(b^{1/m}\big)\big|^q + \frac{1}{2}\varepsilon\lambda^2\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}\int_{\lambda}^{1} (1-t)\big|f'\big(a^t b^{1-t}\big)\big|^q\,\mathrm{d}\,t &\le \int_{\lambda}^{1} (1-t)\big[t^s|f'(a)|^q + m(1-t)^s\big|f'\big(b^{1/m}\big)\big|^q + \varepsilon\big]\,\mathrm{d}\,t \\ &= T(1-\lambda)|f'(a)|^q + \frac{m(1-\lambda)^{s+2}}{s+2}\big|f'\big(b^{1/m}\big)\big|^q + \frac{1}{2}\varepsilon(1-\lambda)^2.\end{aligned} \tag{3.12}$$

Substituting (3.9), (3.10), (3.11), and (3.12) into the inequality (3.8) leads to (3.7). The proof of Theorem 2 is complete. □

Theorem 3. *Let $(s,m) \in [-1,1]\times(0,1]$ and $\lambda \in (0,1)$ and let $f : (0,b^*] \to \mathbb{R}$ be a differentiable mapping on $(0,b^*]$, where $a,b \in (0,b^*]$, $a<b$, $b^{1/m} \le b^*$, and $f' \in L_1([a,b])$. If $|f'|^q$ is extended (s,m)-ε-GA-convex on $(0,\max\{b^{1/m},b\}]$ for $q>1$, then*

$$\begin{aligned}&\left|f\big(a^{\lambda}b^{1-\lambda}\big) - \frac{1}{\ln b - \ln a}\int_a^b \frac{f(x)}{x}\,\mathrm{d}\,x\right| \\ &\le \frac{\ln b - \ln a}{6^{1/q}}\Bigg[\Big(\lambda L\big(a^{\lambda q/(q-1)}, b^{\lambda q/(q-1)}\big)b^{(1-\lambda)q/(q-1)}\Big)^{1-1/q} \\ &\times\left(\frac{\lambda^{q+s+1}}{q+s+1}|f'(a)|^q + mB(\lambda, q+1, s+1)\big|f'\big(b^{1/m}\big)\big|^q + \frac{1}{q+1}\varepsilon\lambda^{q+1}\right)^{1/q} \\ &+ \Big((1-\lambda)a^{\lambda q/(q-1)}L\big(a^{(1-\lambda)q/(q-1)}, b^{(1-\lambda)q/(q-1)}\big)\Big)^{1-1/q}\Bigg(B(1-\lambda, \\ &\times q+1, s+1)|f'(a)|^q + \frac{m(1-\lambda)^{q+s+1}}{q+s+1}\big|f'\big(b^{1/m}\big)\big|^q + \frac{\varepsilon(1-\lambda)^{q+1}}{q+1}\Bigg)^{1/q}\Bigg],\end{aligned} \tag{3.13}$$

where $B(\lambda,\alpha,\beta)$ is incomplete Beta function defined by

$$B(\lambda,\alpha,\beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^{\lambda} t^{\alpha-1}(1-t)^{\beta-1}\,\mathrm{d}\,t, \quad \alpha,\beta>0.$$

Proof. By Lemma 1 and Hölder's integral inequality, we have

$$
\begin{aligned}
&\left|f\big(a^\lambda b^{1-\lambda}\big)-\frac{1}{\ln b-\ln a}\int_a^b\frac{f(x)}{x}\,\mathrm{d}\,x\right|\\
&\le(\ln b-\ln a)\left[\int_0^\lambda ta^tb^{1-t}\big|f'\big(a^tb^{1-t}\big)\big|\,\mathrm{d}\,t+\int_\lambda^1(1-t)a^tb^{1-t}\big|f'\big(a^tb^{1-t}\big)\big|\,\mathrm{d}\,t\right]\\
&\le(\ln b-\ln a)\left[\left(\int_0^\lambda\big(a^tb^{1-t}\big)^{q/(q-1)}\,\mathrm{d}\,t\right)^{1-1/q}\left(\int_0^\lambda t^q\big|f'\big(a^tb^{1-t}\big)\big|^q\,\mathrm{d}\,t\right)^{1/q}\right.\\
&\left.+\left(\int_\lambda^1\big(a^tb^{1-t}\big)^{q/(q-1)}\,\mathrm{d}\,t\right)^{1-1/q}\left(\int_\lambda^1(1-t)^q\big|f'\big(a^tb^{1-t}\big)\big|^q\,\mathrm{d}\,t\right)^{1/q}\right],
\end{aligned}
\tag{3.14}
$$

where

$$
\int_0^\lambda\big(a^tb^{1-t}\big)^{q/(q-1)}\,\mathrm{d}\,t=\lambda b^{(1-\lambda)q/(q-1)}L\big(a^{\lambda q/(q-1)},b^{\lambda q/(q-1)}\big)
\tag{3.15}
$$

and

$$
\int_\lambda^1\big(a^tb^{1-t}\big)^{q/(q-1)}\,\mathrm{d}\,t=(1-\lambda)a^{\lambda q/(q-1)}L\big(a^{(1-\lambda)q/(q-1)},b^{(1-\lambda)q/(q-1)}\big).
\tag{3.16}
$$

From the extended (s,m)-ε-GA-convexity of $|f'|^q$, we have

$$
\begin{aligned}
\int_0^\lambda t^q\big|f'\big(a^tb^{1-t}\big)\big|^q\,\mathrm{d}\,t&\le\int_0^\lambda t^q\Big[t^s|f'(a)|^q+m(1-t)^s\big|f'\big(b^{1/m}\big)\big|^q+\varepsilon\Big]\,\mathrm{d}\,t\\
&=\frac{\lambda^{q+s+1}}{q+s+1}|f'(a)|^q+mB(\lambda,q+1,s+1)\big|f'\big(b^{1/m}\big)\big|^q+\frac{1}{q+1}\varepsilon\lambda^{q+1}.
\end{aligned}
\tag{3.17}
$$

and

$$
\begin{aligned}
&\int_\lambda^1(1-t)^q\Big|f'(a^tb^{1-t})\Big|^q\,\mathrm{d}\,t\le\int_\lambda^1(1-t)^q\Big[t^s|f'(a)|^q\\
&+m(1-t)^s\big|f'\big(b^{1/m}\big)\big|^q+\varepsilon\Big]\,\mathrm{d}\,t\\
&=B(1-\lambda,q+1,s+1)|f'(a)|^q\\
&+\frac{m(1-\lambda)^{q+s+1}}{q+s+1}\big|f'\big(b^{1/m}\big)\big|^q+\frac{1}{q+1}\varepsilon(1-\lambda)^{q+1}.
\end{aligned}
\tag{3.18}
$$

Substituting (3.15), (3.16), (3.17), and (3.18) into the inequality (3.14) derives the inequality (3.13). The proof of Theorem 3 is complete. □

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