Solutions of nonlinear equations to describe physical models in plasma

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Abstract. In this article, the solutions of the models that describe the behavior of one of the physical generators in plasma were obtained using soliton solutions. The behavior of ions and electrons in the plasma, and their relationship to the magnetic field, was demonstrated using the Korteweg–de Vries (KDV) equation and analyzed for its dispersion of the solitons obtained from the KDV equation solutions.

Keywords: degenerate pressure, degenerate ions, electrons fluid, reductive perturbation method, soliton solution.

1. Introduction

Plasma physics is the key to modern-day technology in applications such as electronics, plasma ions, and plasma lasers. The returns of plasma science have a great direct effect on industry and on the economy, with ample studies and attention dedicated to the field [1]-[12].

It is also very clear from the existing research that the acquisition of energy from thermal fusion reactions in plasma is not far-fetched. This will lead to increased interest and active participation in this area. From the recent results, it has been proven that there is an urgent need to build nuclear fusion reactors because their capabilities far exceed traditional sources. They even surpass the capacity of nuclear fission, all while maintaining a pure, safe, and clean source of energy [7, 8, 9, 10, 11, 12].

There are also many applications and uses, such as access to ozone gas, separation of isotopes, printing techniques, manufacture of generators, manufacture of thin films, development of communication devices and controls, and television screens, among others [1, 2, 3, 4, 5, 6].

Crookes released a fourth meaning of the term “plasma” to describe the confirmed part of gas permeability. Solids are transformed into plasma by heating through the liquid and gas states. When the atoms forming the gas ionize at temperatures greater than 100,000 Galvin, the material becomes uniform/homogenous; this is plasma. Langmuir and Tonks described plasma as a semi-electrolytic fluid composed of charged particles and other moderates that behave collectively. If the force produced by a loudspeaker generates sound waves, the charged particles in their motion form local clusters of positive or negative charges, which in turn generate electric fields and thus generate magnetic fields from the movement of molecules. Nuclear fission does not produce any radioactive products that pollute the environment of nuclear fusion.

Interactions through the magnetic and electric fields generate waves that represent waves of sound in a low-pressure ionized liquid. To avoid the dispersion of waves caused by the presence of ions and the fluctuation of electrons, we the behavior of the movements of ions and electrons in the absence of a magnetic field was investigated through equations [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12].

2. Physical models in plasma

Conservation mass and momentum to ion fluid within a one-dimensional space system [1, 2, 3, 4, 5], which is given by

$$\frac{\partial \mathrm{n_i}}{\partial \mathrm{t}}+\frac{\partial}{\partial \mathrm{x}}\left(\mathrm{n_i v_i}\right)=0, \tag{1}$$

$$\mathrm{m_i n_i}\left(\frac{\partial \mathrm{v_i}}{\partial \mathrm{t}}+\mathrm{v_i}\frac{\partial \mathrm{v_i}}{\partial \mathrm{x}}\right)+\frac{\partial \mathrm{p_i}}{\partial \mathrm{x}}=\mathrm{en_i E} \tag{2}$$

where $\mathrm{n_i}$ and $\mathrm{n_e}$ are the density number of ions and electrons, $\mathrm{p_i}$ and $\mathrm{p_e}$ are their pressures, $\mathrm{v_i}$ and $\mathrm{v_e}$ are their velocities, $\mathrm{m_i}$ is the mass of an ions, E is the electric field, and e is its charge. If we neglect its pressure, i.e. $\mathrm{p_i}=0$, then one has [4, 6]

$$\frac{\partial \mathrm{n_i}}{\partial \mathrm{t}}+\frac{\partial}{\partial \mathrm{x}}\left(\mathrm{n_i v_i}\right)=0, \tag{3}$$

$$\mathrm{m_i}\left(\frac{\partial \mathrm{v_i}}{\partial \mathrm{t}}+\mathrm{v_i}\frac{\partial \mathrm{v_i}}{\partial \mathrm{x}}\right)=\mathrm{eE}. \tag{4}$$

While conservation of mass and momentum to the electron fluid system is given by

$$\frac{\partial \mathrm{n_e}}{\partial \mathrm{t}}+\frac{\partial}{\partial \mathrm{x}}\left(\mathrm{n_e v_e}\right)=0, \tag{5}$$

$$\mathrm{m_e n_e}\left(\frac{\partial \mathrm{v_e}}{\partial \mathrm{t}}+\mathrm{v_e}\frac{\partial \mathrm{v_e}}{\partial \mathrm{x}}\right)+\frac{\partial \mathrm{p_e}}{\partial \mathrm{x}}=-\mathrm{en_e E} \tag{6}$$

where $--e$ is the charge of an electron and we denote its mass by $\mathrm{m_e}$. Noting that we can neglect the inertia of electrons (i.e. $\mathrm{m_e}=0$) and the reason for this is that the electrons are much lighter than ions. Hence, Eq. (6) can be reduced to

$$\frac{\partial p_e}{\partial x} = -en_eE. \tag{7}$$

Assuming an isothermal state of the electron fluid, the integration of Eq. (7) yields to following equation

$$p_e = kTn_e \tag{8}$$

where k and T represent the Boltzmann's constant and the temperature, respectively. Inserting Eq. (8) into Eq. (7) we get

$$kT\frac{\partial n_e}{\partial x} = en_e\frac{\partial \emptyset}{\partial x} \tag{9}$$

where $E = -\frac{\partial \emptyset}{\partial x}$. Using Eq. (9), one observe that n_e can be represent in terms of $\emptyset$ as

$$n_e = n_0 \exp\left(\frac{e\,\emptyset}{k\,T}\right) \tag{10}$$

in which the constant n_0 denoted to the density number of electrons at $\emptyset = 0$.

The Maxwell model to electrostatic field E can be generated by a charge density σ, which can be given by

$$\epsilon_0 \nabla.E \ = \sigma,$$

where ϵ_0 is a dielectric constant so that

$$\in_0 \frac{\partial E}{\partial x} = e\left(n_i - n_e\right). \tag{11}$$

Eq. (11) can be represent in terms of ϕ as

$$-\frac{\partial^2 \emptyset}{\partial x^2} = \frac{e}{\in_0}\left(n_i - n_e\right). \tag{12}$$

Note that, the number density of the electrons n_e can be eliminate from Eq. (12) using Eq. (10). By dropping the ion-superscript from the ion-variables (n_i, n_e) , it follows the improved system of equations in term of $(n, v, \emptyset)$ such that

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nv) \ = 0, \tag{13}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{e}{m}\frac{\partial \emptyset}{\partial x} \ = 0, \tag{14}$$

$$-\frac{\partial^2 \emptyset}{\partial \mathrm{x}^2}+\frac{\mathrm{e\ n_0}}{\in_0}\exp\left(\frac{\mathrm{e}\ \emptyset}{\mathrm{k\ T}}\right)=\frac{\mathrm{e}}{\in_0}\mathrm{n}. \tag{15}$$

By defining the Debye length, λ_0, and the ion-acoustic sound speed, c_0, respectively as follows:

$$\lambda_0^2=\frac{\in_0 \mathrm{k\ T}}{\mathrm{n_0 e^2}}; \qquad \mathrm{c}_0^2=\frac{\mathrm{k\ T}}{\mathrm{m}}. \tag{16}$$

Then, it can be non-dimensionalize the Eqs. (13)-(15). According to the conditions, such parameters λ_0 and c_0 will be different in plasma. For instance, in a dense laboratory plasma, one can get that $\mathrm{n_0} \approx 10^{20}m^{-3}$,$\mathrm{T} \approx 60.000K$, and $\lambda_0 \approx 10^{-6}\ m$, while in the solar wind near the earth, on can get that $\mathrm{n_0} \approx 10^7 m^{-3}$, $\mathrm{T} \approx 120.000K$, and $\lambda_0 \approx 10\ m$.

Define the dimensionless variables

$$\overline{\mathrm{x}}=\lambda_0^{-1}\mathrm{x},\ \overline{\mathrm{t}}=\lambda_0^{-1}\mathrm{c_0 t},\ \overline{\mathrm{n}}=\frac{\mathrm{n}}{\mathrm{n_0}},\ \overline{\mathrm{v}}=\frac{\mathrm{v}}{\mathrm{n_0}},\ \overline{\emptyset}=\frac{\mathrm{e}\ \emptyset}{\mathrm{k\ T}},\ \rho=\exp(\emptyset)\ . \tag{17}$$

then drop the "bars", we have the following non-dimensional equations

$$\frac{\partial \mathrm{n}}{\partial \mathrm{t}}+\frac{\partial}{\partial \mathrm{x}}(\mathrm{nv})=0, \tag{18}$$

$$\frac{\partial \mathrm{v}}{\partial \mathrm{t}}+\mathrm{v}\frac{\partial \mathrm{v}}{\partial \mathrm{x}}+\frac{\partial \emptyset}{\partial \mathrm{x}}=0, \tag{19}$$

$$\frac{\partial^2 \emptyset}{\partial \mathrm{x}^2}+\mathrm{n}=\mathrm{p}. \tag{20}$$

3. Solution

We studied the properties of soliton ion acoustic waves in plasma. According to Washimi and Taniuti [9], the reductive perturbation approach can be applied to the main equations. Consequently, the coordinates of space and time are stretched in the sense of the formed relations as follows:

$$\xi=\sqrt{\epsilon}\,(x-\lambda t)\,, \qquad \tau=\sqrt{\epsilon^3}\ t, \tag{21}$$

where ϵ is a small positive parameter and λ is the wave velocity.

Using Eq. (21), Eqs. (18)–(20) become

$$-\lambda\ \frac{\partial n}{\partial \xi}\ +\frac{\partial}{\partial \xi}(nv)+\epsilon\frac{\partial n}{\partial \tau}=0, \tag{22}$$

$$-\lambda\ \frac{\partial v}{\partial \xi}\ +\frac{\partial \emptyset}{\partial \xi}+v\frac{\partial v}{\partial \xi}+\epsilon\frac{\partial v}{\partial \tau}=0, \tag{23}$$

$$\text{p}-\epsilon\frac{\partial^2\emptyset}{\partial\xi^2}=\text{n}. \tag{24}$$

Thus, we need the asymptotic solution of Eqs. (22)–(24) of the form

$$\text{n} = 1+\epsilon\text{n}_1+\epsilon^2\text{n}_2+\ldots, \tag{25}$$

$$\text{v} = \epsilon\text{v}_1+\epsilon^2\text{v}_2+\ldots, \tag{26}$$

$$= 1+\epsilon\emptyset_1+\epsilon^2\emptyset_2+\ldots, \tag{27}$$

$$= 1+\epsilon\emptyset_1+\epsilon^2(\emptyset_2+\frac{{\emptyset_1}^2}{2})+\ldots. \tag{28}$$

Equating the coefficients of ϵ and ϵ^2 of the Taylor expansions in Eqs.(25)–(28)we find that

$$\frac{\partial v_1}{\partial\xi}-\lambda\frac{\partial n_1}{\partial\xi}=0, \tag{29}$$

$$-\lambda\frac{\partial v_1}{\partial\xi}+\frac{\partial\emptyset_1}{\partial\xi}=0, \tag{30}$$

$$\emptyset_1-\text{n}_1=0. \tag{31}$$

Equating the coefficients of ϵ^2, we find that

$$\frac{\partial v_2}{\partial\xi}-\lambda\frac{\partial n_2}{\partial\xi}+\frac{\partial}{\partial\xi}(n_1v_1)+\frac{\partial n_1}{\partial\tau}=0, \tag{32}$$

$$-\lambda\frac{\partial v_2}{\partial\xi}+\frac{\partial\emptyset_2}{\partial\xi}+v_1\frac{\partial v_1}{\partial\xi}+\frac{\partial v_1}{\partial\tau}=0, \tag{33}$$

$$\emptyset_2-\text{n}_2+\frac{1}{2}{\emptyset_1}^2+\frac{\partial^2\emptyset_1}{\partial\xi^2}=0. \tag{34}$$

Taking Φ_1 from Eqs.(29)–(31), we have the following homogeneous linear system for (n_1,v_1):

$$\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix}\begin{pmatrix} \frac{\partial n_1}{\partial\xi} \\ \frac{\partial v_1}{\partial\xi} \end{pmatrix}=0. \tag{35}$$

The system (35) has a nontrivial solution when $\lambda^2 = 1$. From definiteness and accordingly to a right-moving wave, we suppose that $\lambda = 1$. Then, one get

$$\begin{pmatrix} n_1 \\ v_1 \end{pmatrix} = \varphi(\xi,\tau)\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \emptyset_1 = \varphi(\xi,\tau), \tag{36}$$

where $\varphi(\xi,\tau)$ is an arbitrary scalar-valued function. Setting $\lambda = 1$ and eliminating Φ_2 from Eqs. (32)–(34), we get the following non-homogeneous linear system for (n_2,v_2):

$$\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix}\begin{pmatrix} \frac{\partial n_2}{\partial \xi} \\ \frac{\partial v_2}{\partial \xi} \end{pmatrix} + \begin{pmatrix} \frac{\partial}{\partial \xi}(n_1 v_1) + \frac{\partial n_1}{\partial \tau} \\ -\emptyset_1 \frac{\partial \emptyset_1}{\partial \xi} + \frac{\partial^3 \emptyset_1}{\partial \xi^3} + v_1 \frac{\partial v_1}{\partial \xi} + \frac{\partial v_1}{\partial \tau} \end{pmatrix} = 0. \tag{37}$$

The system (37) can be solved in term of (n_2,v_2) if and only if the non-homogeneous terms are orthogonal to the null vector $(1,1)$. It can be deduced from this condition using Eq. (6) that $\varphi(\xi\ ,\tau)$ satisfies a KDV equation:

$$\varphi_\tau + \varphi\varphi_\xi + \frac{1}{2}\varphi_{\xi\xi\xi} = 0. \tag{38}$$

It's worth to mention that the linearized dispersion relation of Eq. (38) together with original system agrees with a long-wave expansion. If it satisfies Eq. (38), then we can solve Eq. (37) in term of (n_2,v_2). However, the solution will be the sum of non-homogeneous solution with arbitrary multiple $\varphi_2(\xi,\tau)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ of the homogeneous. Hence, the asymptotic solution can be found in the same manner. Further, for the order $\in^k$, the non-homogeneous solution of a linear equation can be obtained in term of (n_k, v_k) as the following form

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \frac{\partial n_k}{\partial \xi} \\ \frac{\partial v_k}{\partial \xi} \end{pmatrix} + \begin{pmatrix} f_{k-1} \\ g_{k-1} \end{pmatrix} = 0, \tag{39}$$

where f_{k-1} and g_{k-1} are depending on (n_1, v_1), $\ldots$, (n_{k-1}, v_{k-1}), and Φ_k can be expressed explicitly in terms of $n_1, \ \ldots \ , n_k$. The condition $f_{k-1} + g_{k-1} = 0$ ensures that this equation can be solved in term of (n_k, v_k) that satisfied whenever φ_{k-1} satisfies a suitable equation. Therefore, the solution in term of (n_k, v_k) involves an arbitrary function of integration φ_k. An equation for φ_k follows formula of solvability condition at the $(k+1)$-order equations. Using the sine–cosine method, it can obtain a traveling wave solution for the KDV equation (38) by setting the wave variable $\theta = \xi - b\tau$, where b is a constant, which is reduced Eq. (38) into the following ordinary differential equation:

$$-b\varphi^{\backslash} + \varphi\varphi^{\backslash} + \frac{1}{2}\varphi^{\backslash\backslash\backslash} = 0, \tag{40}$$

where e$\backslash = \frac{d}{d\theta}$. Integrating Eq. (40) once, and considering the integration constant as zero, we can find

$$-b\varphi + \frac{1}{2}\varphi^2 + \frac{1}{2}\varphi^{\backslash\backslash} = 0. \tag{41}$$

We take [10, 11, 12]

$$\varphi(\theta) = a\ \cos^m(\mu\theta), \tag{42}$$

$$\varphi^{\backslash}(\theta) = -a\mu m\ \cos^{m-1}(\mu\theta)\sin(\mu\theta), \tag{43}$$

$$\varphi^{\backslash\backslash}(\theta) = -a\mu^2 m^2 \cos^m(\mu\theta) + a\mu^2 m(m-1)\cos^{m-2}(\mu\theta). \tag{44}$$

Substituting Eqs. (42)–(44) into Eq. (41) yields

$$\begin{aligned} &-\left(ab + \frac{1}{2}a\mu^2 m^2\right)\cos^m(\mu\theta) + \frac{1}{2}a\mu^2 m(m-1)\cos^{m-2}(\mu\theta) \\ &+ \frac{1}{2}a^2\cos^{2m}(\mu\theta) = 0. \end{aligned} \tag{45}$$

The following algebraic system can be obtained by equating the exponents and coefficients of each pair of cosine functions:

$$m \neq 0, 2m = m-2, ab + \frac{1}{2}a\mu^2 m^2 = 0, \frac{1}{2}a^2 + \frac{1}{2}a\mu^2 m(m-1) = 0. \tag{46}$$

From system (46), we obtain

$$m = -2, \qquad \mu = \sqrt{\frac{-b}{2}}, \qquad a = 3b. \tag{47}$$

Then, we obtain the formal solitary wave solutions for Eq. (38)

$$\varphi(\xi,\tau) = 3b\ \ sech^2\frac{1}{2}\left(\sqrt{2b}\,(\xi - b\tau)\right), \quad b > 0 \tag{48}$$

or

$$\varphi(\xi,\tau) = 3b\ \ sec^2\frac{1}{2}\left(\sqrt{-2b}\,(\xi - b\tau)\right), \quad b < 0. \tag{49}$$

Then, we can write the solution of the system (18)–(20) as

$$\begin{pmatrix} n \\ v \\ \emptyset \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 3b\ \in\ sech^2\frac{1}{2}\left(\sqrt{2b}\,(\xi - b\tau)\right)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + O\left(\in^2\right) \tag{50}$$

or

$$\begin{pmatrix} n \\ v \\ \emptyset \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 3b\ \in\ sec^2\frac{1}{2}\left(\sqrt{-2b}\,(\xi - b\tau)\right)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + O\left(\in^2\right). \tag{51}$$

In summary, the leading-order asymptotic solution of the system (18)–(20) as $\in \to 0^+$ is

$$\begin{pmatrix} n \\ v \\ \emptyset \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \ \in\ \varphi\left(\sqrt{\in}\,(x-t), \sqrt{\in^3}t\right)\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + O\left(\in^2\right). \tag{52}$$

We expect this asymptotic solution to be valid for long times, in the order

$$\tau = O(1) \quad \text{or} \quad t = O(\in^{\frac{-3}{2}}).$$

4. Conclusion

Recently, the new modes of waves associated with plasma have become an attractive topic for many researchers. In this effort, an analytical plasma model was applied showing that the nonlinear evolution of charge fluctuations has a significant effect on the instability of charged ions. Moreover, a new simulation model that can be used for studying the unique physical properties of waves in plasmas, caused by charging effects, was established.

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