Convolution conditions for q-Sakaguchi-Janowski type functions

H. Shamsan
Department of Mathematics
Yuvaraja's College
University of Mysore

Mysore 570 005
India
hmas19771@gmail.com

S. Latha
Department of Mathematics
Yuvaraja's College
University of Mysore
Mysore, 570 005
India
drlatha@gmail.com

B.A. Frasin*
Faculty of Science
Department of Mathematics
Al al-Bayt University
Mafraq
Jordan
bafrasin@yahoo.com

Abstract. In this paper, we derived convolution conditions for Sakaguchi-Janowski type functions. Those results contains some interesting corollaries as special cases.

Keywords: Janowski functions, Sakaguchi functions, convolution, q-derivative.

1. Introduction and definitions

Bieberbach conjecture, Milin conjecture, Robertson conjecture, Sheil-small conjecture, Rogosinski conjecture, Littlewood conjecture, etc, attracted eminent mathematicians to work in the theory of univalent functions. Attempts to prove or disprove these conjectures inspired research not only to develop elegant and useful techniques in complex analysis but also led to the introduce and study of various subclasses of univalent functions. Functions with positive real part play a crucial role in Geometric Function Theory as its significance can be seen from that all the simple subclasses of the class of univalent functions have been

*. Corresponding author

defined by using this concept. Motivated by this class Janowski [6] defined the class $P(A, B)$.

Let $\mathcal{A}$ denote the class of functions of form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk

$$\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\},$$

and normalized by $f(0) = f'(0) - 1 = 0$. Let Ω denote the class of analytic functions ω in $\mathcal{U}$ with $\omega(0) = 0$, $|\omega(z)| < 1$. We denote by $P(A, B)$ the Janowski class containing functions p of the form

$$p(z) = \frac{1 + A\omega(z)}{1 + B\omega(z)}, \quad -1 \le B < A \le 1, \quad \omega \in \Omega.$$

Definition 1.1. The convolution or Hadamard product, of two analytic functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n (|z| < R_1) \text{ and } g(z) = z + \sum_{n=2}^{\infty} b_n z^n (|z| < R_2),$$

is defined as the power series

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, |z| < R_1 R_2.$$

The fractional q-calculus is the extension of the ordinary fractional calculus in the q-theory. The theory of q-calculus operators in recent past have been applied in the areas of ordinary fractional calculus, optimal control problems and in finding solutions of the q-difference and q-integral equations, and in q-transform analysis and also in the geometric function theory of complex analysis. For more details on the subject, one may refer to [1, 2, 3, 9], and [15].

In early twentieth century, Jackson [4] initiated q-calculus and developed the concept of the q-derivative and q-integral.

Definition 1.2. For a function $f \in \mathcal{S}$ given by (1) and $0 < q < 1$, the q-derivative of f is defined by

$$D_q f(z) = \begin{cases} \frac{f(z) - f(qz)}{z(1-q)}, & z \neq 0, \\ f'(0), & z = 0. \end{cases} \quad , \quad \text{where } (0 < q < 1) \tag{2}$$

Equivalently (2), may be written as

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, \quad z \neq 0$$

where

$$[n]_q = \begin{cases} \frac{1-q^n}{1-q}, & q \neq 1 \\ n, & q = 1. \end{cases}$$

Note that as $q \to 1^-$, $[n]_q \to n$. For a function $f(z) = z^n$, we can observe that

$$D_q f(z) = D_q(z^n) = \frac{1-q^n}{1-q} z^{n-1} = [n]_q z^{n-1}.$$

Then

$$\lim_{q \to 1^-} D_q f(z) = \lim_{q \to 1^-} [n]_q z^{n-1} = n z^{n-1} = f'(z),$$

where $f'(z)$ is the ordinary derivative. As a right inverse, Jackson [5] presented the q-integral of a function f as

$$\int_0^z f(t) d_q t = z(1-q) \sum_{n=0}^{\infty} q^n f(zq^n),$$

provided that the series converges.

Under the hypothesis of the definition of q-difference operator, we have the following rules:

(i) $D_q(af(z) \pm bg(z)) = aD_q f(z) \pm bD_q g(z)$, where a and b any real (or complex) constants;

(ii) $D_q(f(z)g(z)) = g(qz)D_q f(z) + f(z)D_q g(z) = f(z)D_q g(z) + D_q f(z)g(qz)$;

(iii) $D_q\left(\frac{f(z)}{g(z)}\right) = \frac{g(z)D_q f(z) - f(z)D_q g(z)}{g(qz)g(z)}$.

Now, we introduce the following classes of analytic functions

(i) $f \in S_q^*(A, B, t)$ if $\frac{(1-t)zD_q f(z)}{f(z)-f(tz)} \in P(A, B)$;

(ii) $f \in K_q(A, B, t)$ if and only if $zD_q f \in S_q^*(A, B, t)$;

(iii) $f \in S_q^{*\lambda}(A, B, t)$ if $\frac{e^{i\lambda}\frac{(1-t)zD_q f(z)}{f(z)-f(tz)} - i\sin\lambda}{\cos\lambda} \in P(A, B)$;

(iv) $f \in K_q^{\lambda}(A, B, t)$ if and only if $zD_q f \in S_q^{*\lambda}(A, B, t)$;

where in (iii) and (iv) λ is real and $|\lambda| < \frac{\pi}{2}$.

Remark 1.3. (i) As $q \to 1^-$, $t = o$, $A = 1 - 2\alpha$ and $B = -1$, we get the class $S^*(\alpha, 1, t)$ introduced by Owa [8].

(ii) As $q \to 1^-$, $t = -1$, $A = 1 - 2\alpha$ and $B = -1$, we get the class $S^*(\alpha, 1, -1)$ introduced by Sakaguchi [12].

2. Main results

Theorem 2.1. *The function $f \in K_q(A, B, t)$ in $|z| < R \leq 1$ if and only if*

$$\frac{1}{z}\left[f * \frac{z + \frac{(1+Bx)+([2]_q-1)u_n(t)(1+Ax)}{(1+Bx)-u_n(t)(1+Ax)}qz^2 + \frac{(1+q-[2]_q)u_n(t)(1+Ax)}{(1+Bx)-u_n(t)(1+Ax)}qz^3}{(1-z)(1-qz)(1-q^2z)}\right] \neq 0,$$

$|z| < R$, $|x| = 1$, *where* $u_n(t) = \sum_{j=1}^{n} t^{j-1}$ *and* $|x| = 1$.

Proof. The function $f \in K_q(A,B,t)$ if and only if

$$\frac{(1-t)D_q(zD_qf(z))}{D_qf(z)-D_qf(tz)} \neq \frac{1+Ax}{1+Bx}, \quad (|z|<R,\ |x|=1,\ Bx \neq -1)$$

which implies

$$(1-t)D_q(zD_qf(z))(1+Bx)-(1+Ax)\left(D_qf(z)-D_qf(tz)\right) \neq 0. \tag{3}$$

Setting $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$, we have

$$D_qf = 1+\sum_{n=2}^{\infty}[n]_q a_n z^{n-1}$$
$$D_q\left(zD_qf\right) = 1+\sum_{n=2}^{\infty}[n]_q^2 a_n z^{n-1} = D_qf * \frac{1}{(1-z)(1-qz)}.$$

The left hand side of (3) is equivalent to

$$\begin{aligned}
&(1+Bx)\left[D_qf * \sum_{n=1}^{\infty}[n]_q a_n z^{n-1}\right] - D_qf * \sum_{n=1}^{\infty} u_n(t)(1+Ax)z^{n-1} \\
&= D_qf * \sum_{n=1}^{\infty}\left[(1+Bx)[n]_q - u_n(t)(1+Ax)\right]z^{n-1} \\
&= D_qf * \left(\frac{-u_n(t)(1+Ax)}{1-z} + \frac{1+Bx}{(1-z)(1-qz)}\right) \\
&= D_qf * \left(\frac{(1+Bx)-(1-qz)u_n(t)(1+Ax)}{(1-z)(1-qz)}\right).
\end{aligned}$$

Thus

$$\frac{1}{z}\left[zD_qf * \frac{z+\frac{u_n(t)(1+Ax)}{(1+Bx)-u_n(t)(1+Ax)}qz^2}{(1-z)(1-qz)}\right] \neq 0. \tag{4}$$

Since $zD_qf * g = f * zD_qg$, we can write (4) as

$$\frac{1}{z}\left[f * \frac{z+\frac{(1+Bx)+([2]_q-1)u_n(t)(1+Ax)}{(1+Bx)-u_n(t)(1+Ax)}qz^2+\frac{(1+q-[2]_q)u_n(t)(1+Ax)}{(1+Bx)-u_n(t)(1+Ax)}qz^3}{(1-z)(1-qz)(1-q^2z)}\right] \neq 0,$$

$|z|<R,\ |\rho|=1$, which completes the proof. □

As $q \to 1^-$ and $t=0$, we have following result proved by Ganesan and Padmanabhan in [7].

Corollary 2.2. *The function $f \in K(A,B)$ in $|z|<R\leq 1$ if and only if*

$$\frac{1}{z}\left[f * \frac{xz+\frac{(Ax+Bx+2)}{B-A}z^2}{(1-z)^3}\right] \neq 0.$$

Remark 2.3. As $q \to 1^-$, $t = 0$ and $A = 1$, $B = -1$, we get convolution condition characterizing convex functions as in Silverman et al. in [13] with a suitable modification.

Theorem 2.4. *The function $f \in S_q^*(A,B,t)$ in $|z| < R \le 1$ if and only if*

$$\frac{1}{z}\left[f * \frac{z + \frac{u_n(t)(1+Ax)}{(1+Bx)-u_n(t)(1+Ax)}qz^2}{(1-z)(1-qz)}\right] \neq 0, \quad (|z| < R,\ |x| = 1),$$

where $u_n(t) = \sum_{j=1}^{n} t^{j-1}$.

Proof. Since $f \in S_q^*(A,B,t)$ if and only if $g(z) = \int_0^z \frac{f(\zeta)}{\zeta} d_q\zeta \in K_q(A,B,t)$, we have

$$\frac{1}{z}\left[g * \frac{z + \frac{(1+Bx)+([2]_q-1)u_n(t)(1+Ax)}{(1+Bx)-u_n(t)(1+Ax)}qz^2 + \frac{(1+q-[2]_q)u_n(t)(1+Ax)}{(1+Bx)-u_n(t)(1+Ax)}qz^3}{(1-z)(1-qz)(1-q^2z)}\right]$$
$$= \frac{1}{z}\left[f * \frac{z + \frac{u_n(t)(1+Ax)}{(1+Bx)-u_n(t)(1+Ax)}qz^2}{(1-z)(1-qz)}\right].$$

Thus the result follows from Theorem 2.1. □

As $q \to 1^-$, $t = 0$, we have following result proved by Ganesan and Padmanabhan in [7].

Corollary 2.5. *The function $f \in S^*(A,B)$ in $|z| < R \le 1$ if and only if*

$$\frac{1}{z}\left[f * \frac{xz + \frac{1+Ax}{B-A}z^2}{(1-z)^2}\right] \neq 0, \quad (|z| < R,\ |x| = 1).$$

As a corollary we can derive coefficient inequalities for the class $S_q^*(A,B,t)$.

Corollary 2.6. *A function $f \in \mathcal{A}$ is in the class $S_q^*(A,B,t)$ if and only if*

$$f(z) = 1 + \sum_{n=2}^{\infty} A_n z^{n-1} \neq 0,$$

where $A_n = \frac{([n]_q - u_n(t)) + ([n]_q B - A u_n(t))\rho}{\rho(B-A)} a_n$ and $u_n(t) = \sum_{j=1}^{n} t^{j-1}$.

Proof. A function $f \in S_q^*(A,B,t)$ if and only if

$$\frac{(1-t)zD_q f(z)}{f(z) - f(tz)} \neq \frac{1 + A\rho}{1 + B\rho}.$$

That is

$$(1 + B\rho)(1 - t)(zD_q f(z)) - (1 + A\rho)\left[f(z) - f(tz)\right] \neq 0$$

which implies

$$(B-A)\rho z\left[1+\sum_{n=2}^{\infty}[n]_q(1+B\rho)-u_n(t)(1+A\rho)\right]a_nz^n\neq 0.$$

This simplifies into

$$1+\sum_{n=2}^{\infty}\frac{([n]_q-u_n(t))([n]_qB-Au_n(t))\rho}{\rho(B-A)}a_nz^{n-1}\neq 0,$$

which completes the proof. □

Remark 2.7. As $q\to 1^-$, $t=0$ and $A=1$, $B=-1$, we get convolution condition characterizing starlike functions as in Silverman et al. in [13] with a suitable modification.

Theorem 2.8. *For $|z|<R\leq 1$, λ real with $|\lambda|<\frac{\pi}{2}$ and $|x|=1$, we have $f\in K_q^{\lambda}(A,B,t)$ if and only if*

$$\frac{1}{z}\left[f*\frac{z+\frac{(1+Bx)+([2]_q-1)u_n(t)(1+\phi x)}{(1+Bx)-u_n(t)(1+\phi x)}qz^2+\frac{(1+q-[2]_q)u_n(t)(1+\phi x)}{(1+Bx)-u_n(t)(1+\phi x)}qz^3}{(1-z)(1-qz)(1-q^2z)}\right]\neq 0,$$

$|z|<R$, $|x|=1$, where $\phi=(A\cos\lambda+iB\sin\lambda)\,e^{-i\lambda}$ and $u_n(t)=\sum_{j=1}^{n}t^{j-1}$.

Proof. We have, $f\in K_q^{\lambda}(A,B,t)$ if and only if

$$\frac{e^{i\lambda}\frac{(1-t)D_q(zD_qf(z))}{D_qf(z)-D_qf(tz)}-i\sin\lambda}{\cos\lambda}\neq\frac{1+Ax}{1+Bx},\quad (|z|<R,\ |x|=1,\ x\neq -1)$$

which implies

$$(1-t)(D_q(zD_qf(z))(1+Bx)-(1+\phi x)\left(D_qf(z)-D_qf(tz)\right)\neq 0. \tag{5}$$

Proceeding exactly as in Theorem 2.1 we arrive at the result by replacing A by ϕ. □

As $q\to 1^-$, $t=0$, we have following result proved by Ganesan and Padmanabhan in [7].

Corollary 2.9. *For $|z|<R\leq 1$, λ real with $|\lambda|<\frac{\pi}{2}$ and $|x|=1$, we have*

$$e^{i\lambda}\frac{(zf'(z))'}{f'(z)}\in P(A,B)$$

if and only if

$$\frac{1}{z}\left[f*\frac{xz+\left(x+\frac{2(1+\rho e^{i\lambda}(A\cos\lambda+iB\sin\lambda))}{B-e^{-i\lambda}(A\cos\lambda+iB\sin\lambda)}\right)z^2+}{(1-z)^3}\right]\neq 0.$$

Theorem 2.10. *For* $|z| < R \leq 1$, λ *real with* $|\lambda| < \frac{\pi}{2}$ *and* $|x| = 1$, *we have* $f \in S_q^{*\lambda}(A, B, t)$ *if and only if*

$$\frac{1}{z}\left[f * \frac{z + \frac{([2]_q - 1)u_n(t)(1+\phi x)}{(1+Bx) - u_n(t)(1+\phi x)} qz^2 + \frac{(1+q-[2]_q)u_n(t)(1+\phi x)}{(1+Bx) - u_n(t)(1+\phi x)} qz^3}{(1-z)(1-qz)(1-q^2 z)}\right] \neq 0, |z| < R,\ |x| = 1,$$

where $\phi = (A\cos\lambda + iB\sin\lambda)\, e^{-i\lambda}$ *and* $u_n(t) = \sum_{j=1}^{n} t^{j-1}$.

Proof. The result follows from Theorem 2.8 in the same manner that Theorem 2.4 followed from Theorem 2.1. □

For $t = 0$, we have following result proved by Ganesan and Padmanabhan in [7].

Corollary 2.11. *For* $|z| < R \leq 1$, λ *real with* $|\lambda| < \frac{\pi}{2}$ *and* $|x| = 1$, *we have*

$$e^{i\lambda}\frac{zf'(z)}{f(z)} \in P(A, B)$$

if and only if

$$\frac{1}{z}\left[f * \left(\frac{\left[B - e^{i\lambda}(A\cos\lambda + iB\sin\lambda)\right] xz + \left[1 + xe^{i\lambda}(A\cos\lambda + iB\sin\lambda)\right] z^2}{(1-z)^2}\right)\right] \neq 0.$$

As a corollary we can derive coefficient inequalities for the class $S_q^{*\lambda}(A, B, t)$.

Corollary 2.12. *A function* $f \in \mathcal{A}$ *is in the class* $S_q^{*\lambda}(A, B, t)$ *if and only if*

$$f(z) = 1 + \sum_{n=2}^{\infty} d_n z^{n-1} \neq 0,$$

where $d_n = \frac{([n]_q - u_n(t)) + ([n]_q B - \gamma u_n(t))\rho}{\rho(B-A)} a_n$ *and* $\gamma = (A\cos\lambda + iB\sin\lambda)\, e^{-i\lambda}$.

Proof. A function $f \in S_\lambda^*(A, B, t)$ if and only if

$$\frac{e^{i\lambda}\frac{(1-t)zD_q f(z)}{f(z) - f(tz)} - i\sin\lambda}{\cos\lambda} \neq \frac{1 + A\rho}{1 + B\rho}.$$

That is

$$(1 + B\rho)(1 - t)(zD_q f(z)) - (1 + \gamma\rho)\left[f(z) - f(tz)\right] \neq 0.$$

The rest of the proof follows as in Corollary 2.6. □

Remark 2.13. As $t = 0$ and $A = 1$, $B = -1$, we get convolution condition characterizing spirallikeness of functions as in Silverman et al. in [13].

Acknowledgements

The authors would like to thank the referee for his helpful comments and suggestions.

References

[1] M.H. Abu-Risha, M.H. Annaby, M.E.H. Isail and Z.S. Mansour, *Linear q-difference equations*, Z. Anal. Anwend., 26 (2007), 481-494.

[2] T. Al-Hawary, F. Yousef, and B. A. Frasin, *Subclasses of analytic functions of complex order involving Jackson's (p, q)-derivative,* Available at SSRN 3289803 (2018).

[3] G. Bangerzako, *Variational calculus on q-nonuniform lattices*, J. Math. Anal. Appl., 306 (2005), 161-179.

[4] F. H. Jackson, *On q-functions and a certain difference operator,* Trans. Royal Soc. Edinburgh., 46 (1909), 253-281.

[5] F. H. Jackson, *On q-definite integrals,* Quarterly J. Pure Appl. Math., 41 (1910), 193-203.

[6] W. Janowski, *Extremal problems for a family of functions with positive real part and for some related families,* Ann. Polon. Math., 23 (1970), 159-177.

[7] M. S. Ganesan, and K. S. Padmanabhan, *Convolution conditions for certain classes of analytic functions,* Int. J. Pure and Appl. Math., 15 (1984), 777-780.

[8] S. Owa, T. Sekine, Rikuo Yamakawa, *On Sakaguchi type functions* Applied Mathematics and Computation, 187 (2007), 356-361.

[9] C. Ramachandran, T. Soupramanien and B.A. Frasin, *New subclasses of analytic function associated with q-difference operator*, European journal of pure and applied mathematics, 10 (2017), 348-362.

[10] S. Ruscheweyh and T. Shell-Small, *Hadamard products of schlicht functions and the Pólya-Schoenberg conjecture,* Comment. Math. Helv., 48 (1973), 119-135.

[11] S. Ruscheweyh, *Linear operators between classes of prestarlike functions,* Comment. Math. Helv., 52 (1977), 497-509.

[12] K. Sakaguchi, *On a certain univalent mapping,* J. Math. Soc. Japan., 11 (1959), 72-75.

[13] H. Silverman, E. M. Silvia and D. Telage, *Convolution conditions for convexity, starlikeness and spiral-likeness,* Mathe. Zeitschrift, 162 (1978), 125-130.

[14] T. J. Suffridge, *Starlike functions as limits of polynomials,* In: Advances in Complex Function Theory. Lecture Notes in Mathematics, Berlin-Heidelberg-New York: Springer, 505 (1976), 164-202.

[15] F. Yousef, A. Al-Shible, and S. Yalçın, *An application of Jackson's* (p, q)*-derivative to a subclass of starlike functions with negative coefficients,* arXiv preprint arXiv:1903.01336 (2019).

Accepted: 25.04.2019