Convolution conditions for $q$-Sakaguchi-Janowski type functions

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Abstract. In this paper, we derived convolution conditions for Sakaguchi-Janowski type functions. Those results contains some interesting corollaries as special cases.

Keywords: Janowski functions, Sakaguchi functions, convolution, $q$-derivative.

1. Introduction and definitions

Bieberbach conjecture, Milin conjecture, Robertson conjecture, Sheil-small conjecture, Rogosinski conjecture, Littlewood conjecture, etc, attracted eminent mathematicians to work in the theory of univalent functions. Attempts to prove or disprove these conjectures inspired research not only to develop elegant and useful techniques in complex analysis but also led to the introduce and study of various subclasses of univalent functions. Functions with positive real part play a crucial role in Geometric Function Theory as its significance can be seen from that all the simple subclasses of the class of univalent functions have been

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defined by using this concept. Motivated by this class Janowski [6] defined the class $P(A, B)$.

Let $\mathcal{A}$ denote the class of functions of form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk

$$\mathcal{U} = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \},$$

and normalized by $f(0) = f'(0) = 1$. Let $\Omega$ denote the class of analytic functions $\omega$ in $\mathcal{U}$ with $\omega(0) = 0$, $|\omega(z)| < 1$. We denote by $P(A, B)$ the Janowski class containing functions $p$ of the form

$$p(z) = \frac{1 + A\omega(z)}{1 + B\omega(z)}, \quad -1 \leq B < A \leq 1, \quad \omega \in \Omega.$$

**Definition 1.1.** The convolution or Hadamard product, of two analytic functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad \text{and} \quad g(z) = z + \sum_{n=2}^{\infty} b_n z^n \quad (|z| < R_1)$$

is defined as the power series

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad |z| < R_1 R_2.$$

The fractional $q$-calculus is the extension of the ordinary fractional calculus in the $q$-theory. The theory of $q$-calculus operators in recent past have been applied in the areas of ordinary fractional calculus, optimal control problems and in finding solutions of the $q$-difference and $q$-integral equations, and in $q$-transform analysis and also in the geometric function theory of complex analysis. For more details on the subject, one may refer to [1, 2, 3, 9], and [15].

In early twentieth century, Jackson [4] initiated $q$-calculus and developed the concept of the $q$-derivative and $q$-integral.

**Definition 1.2.** For a function $f \in \mathcal{S}$ given by (1) and $0 < q < 1$, the $q$-derivative of $f$ is defined by

$$D_q f(z) = \begin{cases} \frac{f(z) - f(qz)}{z(1-q)}, & z \neq 0, \\ f'(0), & z = 0. \end{cases}, \quad \text{where } (0 < q < 1)$$

Equivalently (2), may be written as

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, \quad z \neq 0,$$
where

\[ [n]_q = \begin{cases} 1 - q^n, & q \neq 1 \\ n, & q = 1. \end{cases} \]

Note that as \( q \to 1^- \), \( [n]_q \to n \). For a function \( f(z) = z^n \), we can observe that

\[ D_q f(z) = D_q (z^n) = \frac{1 - q^n}{1 - q} z^{n-1} = [n]_q z^{n-1}. \]

Then

\[ \lim_{q \to 1^-} D_q f(z) = \lim_{q \to 1^-} [n]_q z^{n-1} = n z^{n-1} = f'(z), \]

where \( f'(z) \) is the ordinary derivative. As a right inverse, Jackson [5] presented the \( q \)-integral of a function \( f \) as

\[ \int_0^z f(t) d_q t = z(1 - q) \sum_{n=0}^{\infty} q^n f(z q^n), \]

provided that the series converges.

Under the hypothesis of the definition of \( q \)-difference operator, we have the following rules:

(i) \( D_q (a f(z) \pm b g(z)) = a D_q f(z) \pm b D_q g(z) \), where \( a \) and \( b \) any real (or complex) constants;

(ii) \( D_q (f(z) g(z)) = g(qz) D_q f(z) + f(z) D_q g(z) = f(z) D_q g(z) + D_q f(z) g(qz) \);

(iii) \( D_q \left( \frac{f(z)}{g(z)} \right) = \frac{g(z) D_q f(z) - f(z) D_q g(z)}{g(qz) g(z)} \).

Now, we introduce the following classes of analytic functions

(i) \( f \in S_q^*(A, B, t) \) if \( \frac{(1-t) z D_q f(z)}{f(z) - f(tz)} \in P(A, B) \);

(ii) \( f \in K_q(A, B, t) \) if and only if \( z D_q f \in S_q^*(A, B, t) \);

(iii) \( f \in S_q^{*\lambda}(A, B, t) \) if and only if \( \frac{e^{i\lambda} (1-t) z D_q f(z)}{f(z) - f(tz)} - i \sin \lambda \cos \lambda \in P(A, B) \);

(iv) \( f \in K_q^{*\lambda}(A, B, t) \) if and only if \( z D_q f \in S_q^{*\lambda}(A, B, t) \);

where in (iii) and (iv) \( \lambda \) is real and \( |\lambda| < \frac{\pi}{2} \).

**Remark 1.3.** (i) As \( q \to 1^- \), \( t = 0 \), \( A = 1 - 2\alpha \) and \( B = -1 \), we get the class \( S^*(\alpha, 1, t) \) introduced by Owa [8].

(ii) As \( q \to 1^- \), \( t = 1 \), \( A = 1 - 2\alpha \) and \( B = -1 \), we get the class \( S^*(\alpha, 1, -1) \) introduced by Sakaguchi [12].

2. Main results

**Theorem 2.1.** The function \( f \in K_q(A, B, t) \) in \( |z| < R \leq 1 \) if and only if

\[
\frac{1}{z} \left[ f * \left( \frac{z + \frac{(1+Bx) + (1-Bx)([2z]_{q=-1} u_n(t)(1+Ax)}{(1+Bx)-u_n(t)(1+Ax)} q^2}{(1-z)(1-qz)(1-q^2z)} \right) \right] \neq 0,
\]

\[ |z| < R, \ |x| = 1, \text{ where } u_n(t) = \sum_{j=1}^{n} t^{j-1} \text{ and } |x| = 1. \]
**Proof.** The function $f \in K_q(A, B, t)$ if and only if 

$$
\frac{(1-t)D_q(zD_qf(z))}{D_qf(z) - D_qf(tz)} \neq \frac{1+Ax}{1+Bx}, \quad (|z| < R, \ |x| = 1, \ Bx \neq -1)
$$

which implies

(3) $$(1-t)D_q(zD_qf(z))(1+Bx) - (1+Ax)(D_qf(z) - D_qf(tz)) \neq 0.$$

Setting $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, we have

$$D_qf = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}$$

$$D_q(zD_qf) = 1 + \sum_{n=2}^{\infty} [n]_q^2 a_n z^{n-1} = D_qf * \frac{1}{(1-z)(1-qz)}.$$

The left hand side of (3) is equivalent to

$$(1+Bx) \left[ D_qf * \sum_{n=1}^{\infty} [n]_q a_n z^{n-1} \right] - D_qf \sum_{n=1}^{\infty} u_n(t)(1+Ax) z^{n-1}$$

$$= D_qf * \sum_{n=1}^{\infty} [(1+Bx)[n]_q - u_n(t)(1+Ax)] z^{n-1}$$

$$= D_qf * \left( \frac{-u_n(t)(1+Ax)}{1-z} + \frac{1+Bx}{(1-z)(1-qz)} \right)$$

$$= D_qf * \left( \frac{(1+Bx) - (1-qz)u_n(t)(1+Ax)}{(1-z)(1-qz)} \right).$$

Thus

(4) $$\frac{1}{z} \left[ zD_qf * \frac{z + \frac{u_n(t)(1+Ax)}{(1+Bx)-u_n(t)(1+Ax)} qz^2}{(1-z)(1-qz)} \right] \neq 0.$$

Since $zD_qf * g = f * zD_qg$, we can write (4) as

$$\frac{1}{z} \left[ \frac{z + \frac{(1+Bx)+(2q-1)u_n(t)(1+Ax)}{(1+Bx)-u_n(t)(1+Ax)} qz^2 + (1+q-2q)u_n(t)(1+Ax)}{(1-z)(1-qz)(1-q^2z)} \right] \neq 0,$$

$|z| < R, \ |\rho| = 1$, which completes the proof. \qed

As $q \to 1^-$ and $t = 0$, we have following result proved by Ganesan and Padmanabhan in [7].

**Corollary 2.2.** The function $f \in K(A, B)$ in $|z| < R \leq 1$ if and only if

$$\frac{1}{z} \left[ f * \frac{xz + \frac{(Ax+Bx+2)}{B-A} z^2}{(1-z)^3} \right] \neq 0.$$
Remark 2.3. As \( q \to 1^- \), \( t = 0 \) and \( A = 1, B = -1 \), we get convolution condition characterizing convex functions as in Silverman et al. in [13] with a suitable modification.

Theorem 2.4. The function \( f \in S_q(A, B, t) \) in \(|z| < R \leq 1 \) if and only if

\[
\frac{1}{z} \left[ f * \left( z + \frac{u_n(t)(1+Ax)}{(1+Bx)-u_n(t)(1+Ax)qz^2}(1-z)(1-qz) \right) \right] \neq 0, \quad (|z| < R, |x| = 1),
\]

where \( u_n(t) = \sum_{j=1}^{n} t^j \).

Proof. Since \( f \in S_q(A, B, t) \) if and only if \( g(z) = \int_0^z \frac{f(c)}{s} d_q \z \in K_q(A, B, t) \), we have

\[
1 \left[ g * \left( \frac{z + (1+Bx)+(2|q|-1)u_n(t)(1+Ax)}{(1+Bx)-u_n(t)(1+Ax)qz^2}(1-z)(1-qz)(1-q^2z) \right) \right]
= 1 \left[ f * \frac{z + u_n(t)(1+Ax)}{(1+Bx)-u_n(t)(1+Ax)qz^2}(1-z)(1-qz) \right].
\]

Thus the result follows from Theorem 2.1. \( \square \)

As \( q \to 1^- \), \( t = 0 \), we have following result proved by Ganesan and Padmanabhan in [7].

Corollary 2.5. The function \( f \in S_q(A, B, t) \) if and only if

\[
\frac{1}{z} \left[ f * \left( xz + \frac{1+Ax}{B-A} z^2 \right) \right] \neq 0, \quad (|z| < R, |x| = 1).
\]

As a corollary we can derive coefficient inequalities for the class \( S_q(A, B, t) \).

Corollary 2.6. A function \( f \in A \) is in the class \( S_q(A, B, t) \) if and only if

\[
f(z) = 1 + \sum_{n=2}^{\infty} A_n z^{n-1} \neq 0,
\]

where \( A_n = \frac{([n]_q-u_n(t))}{\rho(B-A)} \) and \( u_n(t) = \sum_{j=1}^{n} t^j \).

Proof. A function \( f \in S_q(A, B, t) \) if and only if

\[
\frac{(1-t)D_q f(z)}{f(z) - f(tz)} \neq \frac{1 + A\rho}{1 + B\rho}
\]

That is

\[
(1 + B\rho)(1-t)(D_q f(z)) - (1 + A\rho) [f(z) - f(tz)] \neq 0
\]
which implies

\[(B - A)\rho z \left[ 1 + \sum_{n=2}^{\infty} \left[ n \right] q(1 + B\rho) - u_n(t)(1 + A\rho) \right] a_n z^n \neq 0.\]

This simplifies into

\[1 + \sum_{n=2}^{\infty} \left( [n]_q - u_n(t) \right) \left( [n]_q B - A u_n(t) \right) \frac{\rho(B - A)}{\rho(B - A)} a_n z^{n-1} \neq 0,
\]

which completes the proof.

\[\square\]

**Remark 2.7.** As \( q \to 1^- \), \( t = 0 \) and \( A = 1, B = -1 \), we get convolution condition characterizing starlike functions as in Silverman et al. in [13] with a suitable modification.

**Theorem 2.8.** For \( \lvert z \rvert < R \leq 1, \lambda \) real with \( \lvert \lambda \rvert < \frac{T}{2} \) and \( \lvert x \rvert = 1 \), we have \( f \in K^\lambda_\lambda(A, B, t) \) if and only if

\[1 \frac{e^{i\lambda(1-t)D_q(zD_qf(z))} - i \sin \lambda}{\cos \lambda} \neq \frac{1 + Ax}{1 + Bx}, \quad (\lvert z \rvert < R, \lvert x \rvert = 1, x \neq -1)\]

which implies

\[(5) \quad (1 - t)(D_q(zD_qf(z))(1 + Bx) - (1 + \phi x)(D_qf(z) - D_qf(tz)) \neq 0.\]

Proceeding exactly as in Theorem 2.1 we arrive at the result by replacing \( A \) by \( \phi \). \[\square\]

As \( q \to 1^- \), \( t = 0 \), we have following result proved by Ganesan and Padmanabhan in [7].

**Corollary 2.9.** For \( \lvert z \rvert < R \leq 1, \lambda \) real with \( \lvert \lambda \rvert < \frac{T}{2} \) and \( \lvert x \rvert = 1 \), we have

\[e^{i\lambda(\frac{zf'(z)}{f'(z)})'} \in P(A, B)\]

if and only if

\[1 \frac{1 xz + \left( x + 2(1 + \rho e^{i\lambda(A \cos \lambda + iB \sin \lambda)}) (1 + z^2) \right) z^2}{(1 - z)^2} \neq 0.\]
Theorem 2.10. For $|z| < R \leq 1$, $\lambda$ real with $|\lambda| < \frac{\pi}{2}$ and $|x| = 1$, we have $f \in S^\lambda_q(A, B, t)$ if and only if
\[
\frac{1}{z} \left[ f^* \left( \frac{z + \left( [q] - 1 \right) u_n(t) \left( 1 + \phi x \right) q^2 z^2 + \left( \frac{1 + q - [q] u_n(t)}{1 + B x - u_n(t)} \right) q^3 z^3 \right)}{(1 - z)(1 - qz)(1 - q^2z)} \right] \neq 0, \quad |z| < R, \ |x| = 1,
\]
where $\phi = (A \cos \lambda + iB \sin \lambda) e^{-i\lambda}$ and $u_n(t) = \sum_{j=1}^{n} t^{j-1}$.

\textbf{Proof.} The result follows from Theorem 2.8 in the same manner that Theorem 2.4 followed from Theorem 2.1.

For $t = 0$, we have following result proved by Ganesan and Padmanabhan in [7].

Corollary 2.11. For $|z| < R \leq 1$, $\lambda$ real with $|\lambda| < \frac{\pi}{2}$ and $|x| = 1$, we have
\[
e^{i\lambda} \frac{z f'(z)}{f(z)} \in P(A, B)
\]
if and only if
\[
\frac{1}{z} \left[ f^* \left( \left[ \frac{B - e^{i\lambda}(A \cos \lambda + iB \sin \lambda)}{(1 - z)^2} \right] \left( \frac{1 + x e^{i\lambda}(A \cos \lambda + iB \sin \lambda)}{1 + B x} \right) \right) \right] \neq 0.
\]

As a corollary we can derive coefficient inequalities for the class $S^\lambda_q(A, B, t)$.

Corollary 2.12. A function $f \in A$ is in the class $S^\lambda_q(A, B, t)$ if and only if
\[
f(z) = 1 + \sum_{n=2}^{\infty} d_n z^{n-1} \neq 0,
\]
where $d_n = \frac{([n] q - u_n(t)) + ([n] q B - \gamma u_n(t)) \rho a_n}{\rho (B - A)}$ and $\gamma = (A \cos \lambda + iB \sin \lambda) e^{-i\lambda}$.

\textbf{Proof.} A function $f \in S^\lambda_q(A, B, t)$ if and only if
\[
e^{i\lambda} \frac{(1-t) z D_q f(z)}{f(z) - f(tz)} - \frac{i \sin \lambda}{\cos \lambda} \neq 1 + A \rho.
\]
That is
\[
(1 + B \rho)(1 - t)(z D_q f(z)) - (1 + \gamma \rho) [f(z) - f(tz)] \neq 0.
\]
The rest of the proof follows as in Corollary 2.6.

\textbf{Remark 2.13.} As $t = 0$ and $A = 1$, $B = -1$, we get convolution condition characterizing spirallikeness of functions as in Silverman et al. in [13].
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References


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