Integral inequalities of Hermite-Hadamard type for (α, s)-convex and (α, s, m)-convex functions

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Abstract. In the paper, the authors introduce the concepts "(α, s)-convex function" and "(α, s, m)-convex function" and establish some new integral inequalities of the Hermite-Hadamard type for (α, s)-convex and (α, s, m)-convex functions in terms of the classical Euler beta, gamma, and polygamma functions.

Keywords: integral inequality, (α, s)-convex function, (α, s, m)-convex function, Hermite-Hadamard type, beta function, gamma function, polygamma function.

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1. Introduction

We first recite some definitions of various convex functions.

Definition 1.1. *A function $f : I \subseteq \mathbb{R} = (-\infty, \infty) \to \mathbb{R}$ is said to be convex if*

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y),$$

for all $x, y \in I$ and $\lambda \in [0,1]$.

Definition 1.2 ([24])**.** *For $f : [0,b] \to \mathbb{R}_0 = [0,\infty)$ with $b > 0$ and some $m \in (0,1]$, if*

$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y),$$

for $x, y \in [0,b]$ and $t \in [0,1]$, then we say that f is an m-convex function on $[0,b]$.

Definition 1.3 ([12])**.** *For $f : [0,b] \to \mathbb{R}_0$ with $b > 0$ and $(\alpha, m) \in (0,1]^2$, if*

$$f(tx + m(1-t)y) \le t^\alpha f(x) + m(1-t^\alpha)f(y),$$

for all $x, y \in [0,b]$ and $t \in [0,1]$, then we say that $f(x)$ is an (α,m)-convex function on $[0,b]$.

Definition 1.4 ([4, 9])**.** *Let $s \in (0,1]$ be a real number. A function $f : \mathbb{R} \to \mathbb{R}_0$ is said to be s-convex (in the second sense) if*

$$f(tx + (1-t)y) \le t^s f(x) + (1-t)^s f(y),$$

for all $x, y \in I$ and $t \in [0,1]$.

Definition 1.5 ([28])**.** *For some $s \in [-1,1]$, a function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be extended s-convex if*

$$f(tx + (1-t)y) \le t^s f(x) + (1-t)^s f(y),$$

for all $x, y \in I$ and $t \in (0,1)$.

Definition 1.6 ([33])**.** *For some $(s,m) \in [-1,1] \times (0,1]$, a function $f : [0,b] \to \mathbb{R}_0$ is said to be extended (s,m)-convex if*

$$f(tx + m(1-t)y) \le t^s f(x) + m(1-t)^s f(y),$$

for all $x, y \in I$ and $t \in (0,1)$.

The famous Hermite-Hadamard integral inequality for convex functions and some of its diverse generalizations can be reformulated as follows.

Theorem 1.1. *If $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function on the interval I and $a, b \in I$ with $a < b$, then*

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x \le \frac{f(a)+f(b)}{2}.$$

Theorem 1.2 ([5]). *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$. If $|f'(x)|$ is convex on $[a, b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)\big(|f'(a)|+|f'(b)|\big)}{8}.$$

Theorem 1.3 ([10, Theorems 2.3 and 2.4]). *Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be differentiable on I° and $a, b \in I$ with $a < b$. If $|f'(x)|^p$ is s-convex on $[a, b]$ for some $s \in (0, 1]$ and $p > 1$, then*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{16}\left(\frac{4}{p+1}\right)^{1/p}\big(|f'(a)|+|f'(b)|\big)$$

and

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \le \frac{b-a}{4}\left(\frac{4}{p+1}\right)^{1/p}\Big\{\big[|f'(a)|^{p/(p-1)} + 3|f'(b)|^{p/(p-1)}\big]^{1-1/p} + \big[3|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}\big]^{1-1/p}\Big\}.$$

For more information on integral inequalities of the Hermite-Hadamard type for various kinds of convex functions, please refer to the monograph [6, 8, 13, 14], to recently published papers [2, 3, 5, 7, 11, 15, 16, 17, 18, 19, 20, 21, 22, 23, 25, 26, 28, 29, 30, 31, 32], and to the closely related references therein.

In this paper, we introduce two new concepts "(α, s)-convex function" and "(α, s, m)-convex function" and present inequalities of Hermite-Hadamard type for functions whose twice differentiation are of (α, s, m)-convexity.

2. Two definitions and two lemmas

We introduce the notions of "(α, s)-convex function" and "(α, s, m)-convex function".

Definition 2.1. *For some $s \in [-1, 1]$ and $\alpha \in (0, 1]$, a function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be (α, s)-convex if*

$$f(tx + (1-t)y) \le t^{\alpha s} f(x) + (1 - t^\alpha)^s f(y),$$

for all $x, y \in I$ and $t \in (0, 1)$.

Definition 2.2. *For some $s \in [-1, 1]$ and $(\alpha, m) \in (0, 1]^2$, a function $f : [0, b] \to \mathbb{R}$ is said to be (α, s, m)-convex if*

$$f(tx + m(1-t)y) \le t^{\alpha s} f(x) + m(1 - t^\alpha)^s f(y),$$

for all $x, y \in [0, b]$ and $t \in (0, 1)$.

Remark 2.1. By Definition 2.2, we see that

1. if $s = 1$, then $f(x)$ is an (α, m)-convex function on $(0, b]$;
2. if $\alpha = 1$, then $f(x)$ is an extended (s, m)-convex function on $(0, b]$;
3. if $\alpha = m = 1$, then $f(x)$ is an extended s-convex function on $(0, b]$;
4. if $\alpha = s = m = 1$, then $f(x)$ is a convex function on $(0, b]$.

This means that Definitions (2.1) and (2.2) are significant.

For establishing new integral inequalities of the Hermite-Hadamard type for (α, s)-convex and (α, s, m)-convex functions, we need the following lemmas.

Lemma 2.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° and $a, b \in I$ with $a < b$. If $f'' \in L_1([a, b])$, then*

$$\frac{1}{4}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x = \frac{(b-a)^2}{16} \times \int_0^1 t(1-t)\left[f''\left(at + (1-t)\frac{a+b}{2}\right) + f''\left(t\frac{a+b}{2} + (1-t)b\right)\right]\mathrm{d}\,t.$$

Proof. Integrating by parts and changing variable of integral give

$$\begin{aligned}
&\int_0^1 t(1-t)f''\left(at + (1-t)\frac{a+b}{2}\right)\mathrm{d}\,t \\
&= \frac{2}{b-a}\int_0^1 (1-2t)f'\left(at + (1-t)\frac{a+b}{2}\right)\mathrm{d}\,t \\
&= \frac{4}{(b-a)^2}\left[f(a) + f\left(\frac{a+b}{2}\right)\right] - \frac{8}{(b-a)^2}\int_0^1 f\left(at + (1-t)\frac{a+b}{2}\right)\mathrm{d}\,t \\
&= \frac{4}{(b-a)^2}\left[f(a) + f\left(\frac{a+b}{2}\right)\right] - \frac{16}{(b-a)^3}\int_a^{(a+b)/2} f(x)\,\mathrm{d}\,x
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 t(1-t)f''\left(t\frac{a+b}{2} + (1-t)b\right)\mathrm{d}\,t \\
&= \frac{4}{(b-a)^2}\left[f\left(\frac{a+b}{2} + f(b)\right)\right] - \frac{16}{(b-a)^3}\int_{(a+b)/2}^b f(x)\,\mathrm{d}\,x.
\end{aligned}$$

The proof of Lemma 2.1 is complete. □

Lemma 2.2. *Let $\alpha \in (0, 1]$ and $s \in [-1, 1]$. Then*

1. when $s \in (-1, 1]$, we have

$$\begin{aligned}
M(\alpha, s) &\triangleq \int_0^1 t(1-t)(1-t^\alpha)^s\,\mathrm{d}\,t \\
&= \frac{1}{\alpha}\left[B\left(s+1, \frac{2}{\alpha}\right) - B\left(s+1, \frac{3}{\alpha}\right)\right];
\end{aligned} \tag{2.1}$$

2. when $s = -1$, *we have*

$$\int_0^1 \frac{t(1-t)}{1-t^\alpha} \,\mathrm{d}\, t = \frac{1}{\alpha}\left[\psi\left(\frac{3}{\alpha}\right) - \psi\left(\frac{2}{\alpha}\right)\right],$$

where $\Gamma(x)$, $B(x,y)$, *and* $\psi(x)$ *are the classical Euler gamma, beta, and psi functions defined respectively by*

$$\Gamma(x) = \int_0^1 t^{x-1}e^{-t} \,\mathrm{d}\, t, \quad B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} \,\mathrm{d}\, t, \tag{2.2}$$

and

$$\psi(x) = \frac{\mathrm{d} \ln \Gamma(x)}{\mathrm{d}\, x} = \int_0^\infty \left[\frac{e^{-t}}{t} - \frac{e^{-xt}}{1-e^{-t}}\right] \mathrm{d}\, t \tag{2.3}$$

for $\Re(x), \Re(y) > 0$.

Proof. Let $u = t^\alpha$ for $t \in [0,1]$. If $s \in (-1,1]$, we have

$$\begin{aligned} M(\alpha,s) &= \int_0^1 t(1-t)(1-t^\alpha)^s \,\mathrm{d}\, t \\ &= \frac{1}{\alpha}\int_0^1 \left(u^{2/\alpha-1} - u^{3/\alpha-1}\right)(1-u)^s \,\mathrm{d}\, u = \frac{1}{\alpha}\left[B\left(s+1, \frac{2}{\alpha}\right) - B\left(s+1, \frac{3}{\alpha}\right)\right]. \end{aligned}$$

When $s = -1$, from the formulas

$$\psi(z) + \gamma = \int_0^1 \frac{1-t^{z-1}}{1-t} \,\mathrm{d}\, t, \quad \text{and} \quad \gamma = \int_0^\infty \left(\frac{1}{1+t} - e^{-t}\right)\frac{\mathrm{d}\, t}{t}$$

in [1, p. 259, 6.3.22], it is easily to deduce

$$\int_0^1 \frac{t(1-t)}{1-t^\alpha} \,\mathrm{d}\, t = \frac{1}{\alpha}\int_0^1 \frac{u^{2/\alpha-1} - u^{3/\alpha-1}}{1-u} \,\mathrm{d}\, u = \frac{1}{\alpha}\left[\psi\left(\frac{3}{\alpha}\right) - \psi\left(\frac{2}{\alpha}\right)\right].$$

The proof of Lemma 2.2 is complete. □

3. Integral inequalities of Hermite-Hadamard type

Now we start out to establish some new integral inequalities of the Hermite-Hadamard type for (α, s, m)-convex functions.

Theorem 3.1. *For* $(\alpha, m) \in (0,1]^2$ *and* $s \in (-1,1]$, *let* $f : \left(0, \frac{b^*}{m}\right] \to \mathbb{R}$ *be a twice differentiable function and* $f'' \in L_1([a,b])$ *for* $a, b \in (0, b^*]$ *with* $a < b$. *If* $|f''|^q$ *is an* (α, s, m)*-convex function on* $\left(0, \frac{b^*}{m}\right]$ *for* $q \geq 1$, *then*

$$\begin{aligned} &\left|\frac{1}{4}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\, x\right| \leq \frac{(b-a)^2}{16 \times 6^{1-1/q}} \\ &\times \left\{\left[\frac{1}{(\alpha s+2)(\alpha s+3)}|f''(a)|^q + mM(\alpha,s)\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right]^{1/q}\right. \\ &+ \left.\left[\frac{1}{(\alpha s+2)(\alpha s+3)}\left|f''\left(\frac{a+b}{2}\right)\right|^q + mM(\alpha,s)\left|f''\left(\frac{b}{m}\right)\right|^q\right]^{1/q}\right\}, \end{aligned} \tag{3.1}$$

where $M(\alpha,s)$ is defined by (2.1).

Proof. By Lemma 2.1 and Hölder's integral inequality, we obtain

$$\begin{aligned}&\left|\frac{1}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\\&\le\frac{(b-a)^2}{16}\left[\int_0^1 t(1-t)\left|f''\left(at+(1-t)\frac{a+b}{2}\right)\right|\mathrm{d}\,t\right.\\&\left.+\int_0^1 t(1-t)\left|f''\left(t\frac{a+b}{2}+(1-t)b\right)\right|\mathrm{d}\,t\right]\le\frac{(b-a)^2}{16}\\&\times\left\{\left[\int_0^1 t(1-t)\,\mathrm{d}\,t\right]^{1-1/q}\left[\int_0^1 t(1-t)\left|f''\left(at+(1-t)\frac{a+b}{2}\right)\right|^q\mathrm{d}\,t\right]^{1/q}\right.\\&\left.+\left(\int_0^1 t(1-t)\,\mathrm{d}\,t\right)^{1-1/q}\left[\int_0^1 t(1-t)\left|f''\left(t\frac{a+b}{2}+(1-t)b\right)\right|^q\mathrm{d}\,t\right]^{1/q}\right\}.\end{aligned}\tag{3.2}$$

From the (α,s,m)-convexity of $|f''|^q$ and Lemma 2.2, we arrive at

$$\begin{aligned}&\int_0^1 t(1-t)\left|f''\left(at+(1-t)\frac{a+b}{2}\right)\right|^q\mathrm{d}\,t\\&\le\int_0^1 t(1-t)\left(t^{\alpha s}|f''(a)|^q+m(1-t^\alpha)^s\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right)\mathrm{d}\,t\\&=\frac{1}{(\alpha s+2)(\alpha s+3)}|f''(a)|^q+mM(\alpha,s)\left|f''\left(\frac{a+b}{2m}\right)\right|^q\end{aligned}$$

and

$$\begin{aligned}&\int_0^1 t(1-t)\left|f''\left(t\frac{a+b}{2}+(1-t)b\right)\right|^q\mathrm{d}\,t\\&\qquad\le\frac{1}{(\alpha s+2)(\alpha s+3)}\left|f''\left(\frac{a+b}{2}\right)\right|^q+mM(\alpha,s)\left|f''\left(\frac{b}{m}\right)\right|^q.\end{aligned}\tag{3.3}$$

By the inequalities between (3.2) and (3.3), we conclude the inequality (3.1). The proof of Theorem 3.1 is complete. □

Corollary 3.1. *For $\alpha\in(0,1]$ and $s\in(-1,1]$, let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a twice differentiable function and $f''\in L_1([a,b])$ for $a,b\in I$ with $a<b$. If $|f''|^q$ is an (α,s)-convex function on I for $q\ge1$, then*

$$\begin{aligned}&\left|\frac{1}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\\&\le\frac{(b-a)^2}{16\times6^{1-1/q}}\left\{\left[\frac{1}{(\alpha s+2)(\alpha s+3)}|f''(a)|^q+M(\alpha,s)\left|f''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}\right.\\&\left.+\left[\frac{1}{(\alpha s+2)(\alpha s+3)}\left|f''\left(\frac{a+b}{2}\right)\right|^q+M(\alpha,s)|f''(b)|^q\right]^{1/q}\right\},\end{aligned}$$

where $M(\alpha,s)$ is defined by (2.1).

Proof. This is a special case of Theorem 3.1 for $m = 1$. $\square$

Theorem 3.2. *For $(\alpha, m) \in (0,1]^2$, let $f : \left(0, \frac{b^*}{m}\right] \to \mathbb{R}$ be a twice differentiable function and $f'' \in L_1([a,b])$ for $a, b \in (0, b^*]$ with $a < b$. If $|f''|^q$ is $(\alpha, -1, m)$-convex function on $\left(0, \frac{b^*}{m}\right]$ for $q \ge 1$, then*

$$\left|\frac{1}{4}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|$$
$$\le \frac{(b-a)^2}{16 \times 6^{1-1/q}}\left\{\left[\frac{|f''(a)|^q}{(2-\alpha)(3-\alpha)} + \frac{m\left[\psi(\frac{3}{\alpha}) - \psi(\frac{2}{\alpha})\right]}{\alpha}\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right]^{1/q}\right.$$
$$\left. + \left[\frac{1}{(2-\alpha)(3-\alpha)}\left|f''\left(\frac{a+b}{2}\right)\right|^q + \frac{m\left[\psi(\frac{3}{\alpha}) - \psi(\frac{2}{\alpha})\right]}{\alpha}\left|f''\left(\frac{b}{m}\right)\right|^q\right]^{1/q}\right\},$$

where $\psi(x)$ is defined by (2.3).

Proof. This follows from similar argument to Theorem 3.1. $\square$

Corollary 3.2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function and $f'' \in L_1([a,b])$ for $a, b \in I$ with $a < b$. If $|f''|^q$ is an $(\alpha, -1)$-convex function on I for $\alpha \in (0,1]$ and $q \ge 1$, then*

$$\left|\frac{1}{4}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|$$
$$\le \frac{(b-a)^2}{16 \times 6^{1-1/q}}\left\{\left[\frac{1}{(2-\alpha)(3-\alpha)}|f''(a)|^q + \frac{\psi\left(\frac{3}{\alpha}\right) - \psi\left(\frac{2}{\alpha}\right)}{\alpha}\left|f''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}\right.$$
$$\left. + \left[\frac{1}{(2-\alpha)(3-\alpha)}\left|f''\left(\frac{a+b}{2}\right)\right|^q + \frac{\psi\left(\frac{3}{\alpha}\right) - \psi\left(\frac{2}{\alpha}\right)}{\alpha}|f''(b)|^q\right]^{1/q}\right\},$$

where $\psi(x)$ is defined by (2.3).

Proof. This is a special case of Theorem 3.2 for $m = 1$. $\square$

Theorem 3.3. *For $(\alpha, m) \in (0,1]^2$, $s \in (-1, 1]$, and $\ell \ge 0$, let $f : \left(0, \frac{b^*}{m}\right] \to \mathbb{R}$ be a twice differentiable function and $f'' \in L_1([a,b])$ for $a, b \in (0, b^*]$ with $a < b$. If $|f''|^q$ is an (α, s, m)-convex function on $\left(0, \frac{b^*}{m}\right]$ for $q > 1$ and $q \ge \ell$, then*

$$\left|\frac{1}{4}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|$$
$$\le \frac{(b-a)^2}{16}\left[B\left(\frac{2q-\ell-1}{q-1}, \frac{2q-1}{q-1}\right)\right]^{1-1/q}$$
$$\times \left\{\left[\frac{1}{\alpha s + \ell + 1}|f''(a)|^q + \frac{m}{\alpha}B\left(s+1, \frac{\ell+1}{\alpha}\right)\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right]^{1/q}\right.$$
$$\left. + \left[\frac{1}{\alpha s + \ell + 1}\left|f''\left(\frac{a+b}{2}\right)\right|^q + \frac{m}{\alpha}B\left(s+1, \frac{\ell+1}{\alpha}\right)\left|f''\left(\frac{b}{m}\right)\right|^q\right]^{1/q}\right\},$$

where $B(x, y)$ is the classical beta function defined in (2.2).

Proof. By Lemma 2.1, Hölder's integral inequality, and the (α, s, m)-convexity of $|f''|^q$, one has

$$\begin{aligned}
&\left|\frac{1}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \\
&\le \frac{(b-a)^2}{16}\left(\int_0^1 t^{(q-\ell)/(q-1)}(1-t)^{q/(q-1)}\,\mathrm{d}\,t\right)^{1-1/q}\left\{\left[\int_0^1 t^\ell\left|f''\left(at\right.\right.\right.\right. \\
&\left.\left.\left.\left.+(1-t)\frac{a+b}{2}\right)\right|^q\mathrm{d}\,t\right]^{1/q}+\left[\int_0^1 t^\ell\left|f''\left(t\frac{a+b}{2}+(1-t)b\right)\right|^q\mathrm{d}\,t\right]^{1/q}\right\} \\
&\le \frac{(b-a)^2}{16}\left(\int_0^1 t^{(q-\ell)/(q-1)}(1-t)^{q/(q-1)}\,\mathrm{d}\,t\right)^{1-1/q} \\
&\times\left\{\left[\int_0^1 t^\ell\left(t^{\alpha s}|f''(a)|^q+m(1-t^\alpha)^s\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right)\mathrm{d}\,t\right]^{1/q}\right. \\
&\left.+\left[\int_0^1 t^\ell\left(t^{\alpha s}\left|f''\left(\frac{a+b}{2}\right)\right|^q+m(1-t^\alpha)^s\left|f''\left(\frac{b}{m}\right)\right|^q\right)\mathrm{d}\,t\right]^{1/q}\right\} \\
&=\frac{(b-a)^2}{16}\left[B\left(\frac{2q-\ell-1}{q-1},\frac{2q-1}{q-1}\right)\right]^{1-1/q} \\
&\times\left\{\left[\frac{1}{\alpha s+\ell+1}|f''(a)|^q+\frac{m}{\alpha}B\left(s+1,\frac{\ell+1}{\alpha}\right)\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right]^{1/q}\right. \\
&\left.+\left[\frac{1}{\alpha s+\ell+1}\left|f''\left(\frac{a+b}{2}\right)\right|^q+\frac{m}{\alpha}B\left(s+1,\frac{\ell+1}{\alpha}\right)\left|f''\left(\frac{b}{m}\right)\right|^q\right]^{1/q}\right\}.
\end{aligned}$$

The proof of Theorem 3.3 is thus complete. □

Corollary 3.3. *Under assumptions of Theorem 3.3, if* $\ell = 0$*, then*

$$\begin{aligned}
&\left|\frac{1}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right|\le\frac{(b-a)^2}{16} \\
&\times\left[B\left(\frac{2q-1}{q-1},\frac{2q-1}{q-1}\right)\right]^{1-1/q}\left\{\left[\frac{1}{\alpha s+1}|f''(a)|^q+\frac{m}{\alpha}B\left(s+1,\frac{1}{\alpha}\right)\right.\right. \\
&\left.\left.\times\left|f''\left(\frac{a+b}{2m}\right)\right|^q\right]^{1/q}+\left[\frac{\left|f''\left(\frac{a+b}{2}\right)\right|^q}{\alpha s+1}+\frac{m}{\alpha}B\left(s+1,\frac{1}{\alpha}\right)\left|f''\left(\frac{b}{m}\right)\right|^q\right]^{1/q}\right\},
\end{aligned}$$

where $B(x, y)$ *is the classical beta function defined in* (2.2).

Corollary 3.4. *For* $\alpha \in (0, 1]$*,* $s \in (-1, 1]$*, and* $\ell \ge 0$*, let* $f : I \subseteq \mathbb{R} \to \mathbb{R}$ *be a twice differentiable function and* $f'' \in L_1([a, b])$ *for* $a, b \in I$ *with* $a < b$*. If* $|f''|^q$

is an (α, s)*-convex function on* I *for* $q \geq 1$ *and* $q \geq \ell$*, then*

$$\left|\frac{1}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x\right| \leq \frac{(b-a)^2}{16}\left[B\left(\frac{2q-\ell-1}{q-1},\right.\right.$$
$$\left.\left.\frac{2q-1}{q-1}\right)\right]^{1-1/q}\left\{\left[\frac{1}{\alpha s+\ell+1}|f''(a)|^q+\frac{1}{\alpha}B\left(s+1,\frac{\ell+1}{\alpha}\right)\left|f''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}\right.$$
$$\left.+\left[\frac{1}{\alpha s+\ell+1}\left|f''\left(\frac{a+b}{2}\right)\right|^q+\frac{1}{\alpha}B\left(s+1,\frac{\ell+1}{\alpha}\right)|f''(b)|^q\right]^{1/q}\right\},$$

where $B(x, y)$ *is the classical beta function defined in* (2.2)*.*

Proof. This is a special case of Theorem 3.3 for $m = 1$. □

Theorem 3.4. *For* $(\alpha, m) \in (0, 1]^2$ *and* $s \in (-1, 1]$*, let* $f : \left(0, \frac{b^*}{m}\right] \to \mathbb{R}$ *be an* (α, s, m)*-convex function on* $\left(0, \frac{b^*}{m}\right]$ *for* $b^* > 0$*. If* $f \in L_1([a, b])$ *for* $a, b \in (0, b^*]$ *and* $a < b$*, then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \frac{f(x)+m(2^\alpha-1)^s f\left(\frac{x}{m}\right)}{2^{\alpha s}}\,\mathrm{d}\,x$$

and

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x \leq \max\left\{\frac{1}{\alpha s+1}f(a)+\frac{m}{\alpha}B\left(s+1,\frac{1}{\alpha}\right)f\left(\frac{b}{m}\right),\right.$$
$$\left.\frac{m}{\alpha}B\left(s+1,\frac{1}{\alpha}\right)f\left(\frac{a}{m}\right)+\frac{1}{\alpha s+1}f(b)\right\},$$

where $B(x, y)$ *is the classical beta function defined in* (2.2)*.*

Proof. By the (α, s, m)-convexity of f, we obtain

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2^{\alpha s}}\int_0^1\left[f\big(ta+(1-t)b\big)+m(2^\alpha-1)^s f\left(\frac{(1-t)a+tb}{m}\right)\right]\mathrm{d}\,t.$$

Letting $x = ta + (1 - t)b$ or $x = (1 - t)a + tb$ for all $t \in [0, 1]$ leads to

$$\int_0^1 f(c[ta+(1-t)b])\,\mathrm{d}\,t = \int_0^1 f(c[(1-t)a+tb])\,\mathrm{d}\,t = \frac{1}{b-a}\int_a^b f(cx)\,\mathrm{d}\,x$$

for $c \in \mathbb{R}$.

On the other hand, when putting $x = ta + (1 - t)b$ for all $t \in [0, 1]$, by the (α, s, m)-convexity of f, we can gain

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}\,x \leq \int_0^1\left[t^{\alpha s}f(a)+m(1-t^\alpha)^s f\left(\frac{b}{m}\right)\right]\mathrm{d}\,t$$
$$= \frac{1}{\alpha s+1}f(a)+\frac{m}{\alpha}B\left(s+1,\frac{1}{\alpha}\right)f\left(\frac{b}{m}\right).$$

The proof of Theorem 3.4 is complete. □

Corollary 3.5. *Under conditions of Theorem 3.4, if $m = 1$, then*

$$\frac{2^{\alpha s}}{(2^\alpha - 1)^s + 1} f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a} \int_a^b f(x) \, \mathrm{d}\, x$$
$$\le \max\left\{ \frac{f(a)}{\alpha s + 1} + \frac{1}{\alpha} B\left(s+1, \frac{1}{\alpha}\right) f(b), \frac{1}{\alpha} B\left(s+1, \frac{1}{\alpha}\right) f(a) + \frac{f(b)}{\alpha s + 1} \right\}.$$

Remark 3.1. This paper is a corrected and revised version of the preprint [27].

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