# Prime-valent one-regular graphs of order $8p$

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**Abstract.** A graph is *one-regular* and *arc-transitive* if its automorphism group acts on its arcs regularly and transitively, respectively. In this paper, we classify one-regular graphs of prime valency and order $8p$ for each prime $p$. By analyzing the structure of the full automorphism group of such graphs and using the classification of arc-transitive graphs of order $2p$, we prove that there are only two infinite families of such graphs: one is the cycle $C_{8p}$ with valency 2, the other is the $\mathbb{Z}_p-$cover $CQ_p$ of hypercube $Q_3$ with valency 3 and $3|(p-1)$.

**Keywords:** one-regular graph, arc-transitive graph, covering graph.

## 1. Introduction

Throughout this paper graphs are assumed to be finite, simple, connected and undirected. For group-theoretic concepts or graph-theoretic terms not defined here we refer the reader to [14, 16] or [1, 2], respectively. Let $G$ be a permutation group on a set $\Omega$ and $v \in \Omega$. Denote by $G_v$ the stabilizer of $v$ in $G$, that is, the subgroup of $G$ fixing the point $v$. We say that $G$ is *semiregular* on $\Omega$ if $G_v = 1$ for every $v \in \Omega$ and *regular* if $G$ is transitive and semiregular.

For a graph $X$, denote by $V(X)$, $E(X)$ and $\mathrm{Aut}(X)$ its vertex set, its edge set and its full automorphism group, respectively. A graph $X$ is said to be $G$-*vertex-transitive* if $G \leq \mathrm{Aut}(X)$ acts transitively on $V(X)$. $X$ is simply called *vertex-transitive* if it is $\mathrm{Aut}(X)$-vertex-transitive. An $s$-*arc* in a graph is an ordered $(s+1)$-tuple $(v_0, v_1, \cdots, v_{s-1}, v_s)$ of vertices of the graph $X$ such that $v_{i-1}$ is adjacent to $v_i$ for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. In particular, a 1-arc is just an arc and a 0-arc is a vertex. For a subgroup $G \leq \mathrm{Aut}(X)$, a graph $X$ is said to be $(G, s)$-*arc-transitive* or $(G, s)$-*regular* if $G$ is transitive or regular on the set of $s$-arcs in $X$, respectively. A $(G, s)$-arc-transitive graph is said to be $(G, s)$-*transitive* if it is not $(G, s+1)$-arc-transitive. In particular, a $(G, 1)$-arc-transitive graph is called $G$-*symmetric*. A graph $X$ is simply called $s$-*arc-transitive*, $s$-*regular* or $s$-*transitive* if it is $(\mathrm{Aut}(X), s)$-arc-transitive, $(\mathrm{Aut}(X), s)$-regular or $(\mathrm{Aut}(X), s)$-transitive, respectively.

We denote by $C_n$ and $K_n$ the cycle and the complete graph of order $n$, respectively. Denote by $D_{2n}$ the dihedral group of order $2n$. As we all known that there is only one connected 2-valent graph of order $n$, that is, the cycle

$C_n$, which is 1-regular with full automorphism group $D_{2n}$. Let $p$ be a prime. Classifying $s$-transitive and $s$-regular graphs has received considerable attention. The classification of $s$-transitive graphs of order $p$ and $2p$ was given in [3] and [4], respectively. Wang [15] characterized the prime-valent $s$-transitive graphs of order $4p$. The classification of cubic, pentavalent and heptavalent $s$-transitive graphs of order $8p$ was given in [8], [13] and [12], respectively.

For 2-valent case, $s$-transitivity always means 1-regularity, and for cubic case, $s$-transitivity always means $s$-regularity by Miller [7]. However, for the other prime-valent case, this is not true, see for example [9] for pentavalent case and [10] for heptavalent case. Thus, characterization and classification of prime-valent $s$-regular graphs is very interesting and also reveals the $s$-regular global and local actions of the permutation groups on $s$-arcs of such graphs. In particular, 1-regular action is the most simple and typical situation. In this paper, we classify prime-valent one-regular graphs of order $8p$ for each prime $p$.

## 2. Preliminary results

Let $X$ be a connected $G$-symmetric-transitive graph with $G \leq \mathrm{Aut}(X)$, and let $N$ be a normal subgroup of $G$. The *quotient graph* $X_N$ of $X$ relative to $N$ is defined as the graph with vertices the orbits of $N$ on $V(X)$ and with two orbits adjacent if there is an edge in $X$ between those two orbits. In view of [11, Theorem 9], we have the following:

**Proposition 2.1.** *Let $X$ be a connected $G$-symmetric graph with $G \leq \mathrm{Aut}(X)$ and prime valency $q \geq 3$, and let $N$ be a normal subgroup of $G$. Then one of the following holds:*

(1) *$N$ is transitive on $V(X)$;*
(2) *$X$ is bipartite and $N$ is transitive on each part of the bipartition;*
(3) *$N$ has $r \geq 3$ orbits on $V(X)$, $N$ acts semiregularly on $V(X)$, the quotient graph $X_N$ is a connected $q$-valent $G/N$-symmetric graph.*

To extract a classification of connected prime-valent symmetric graphs of order $2p$ for a prime $p$ from Cheng and Oxley [4], we introduce the graphs $G(2p, q)$. Let $V$ and $V'$ be two disjoint copies of $\mathbb{Z}_p$, say $V = \{0, 1, \cdots, p-1\}$ and $V' = \{0', 1', \cdots, (p-1)'\}$. Let $q$ be a positive integer dividing $p-1$ and $H(p, q)$ the unique subgroup of $Z_p^*$ of order $q$. Define the graph $G(2p, q)$ to have vertex set $V \cup V'$ and edge set $\{xy' \mid x - y \in H(p, q)\}$.

**Proposition 2.2.** *Let $X$ be a connected $q$-valent symmetric graph of order $2p$ with $p, q$ primes. Then $X$ is isomorphic to $K_{2p}$ with $q = 2p-1$, $K_{p,p}$ or $G(2p, q)$ with $q|(p-1)$. Furthermore, if $(p, q) \neq (11, 5)$ then* $\mathrm{Aut}(G(2p, q)) = (\mathbb{Z}_p \rtimes \mathbb{Z}_q) \rtimes \mathbb{Z}_2$*; if $(p, q) = (11, 5)$ then* $\mathrm{Aut}(G(2p, q)) = \mathrm{PGL}(2, 11)$.

From [6, pp.12-14], we can deduce the non-abelian simple groups whose orders have at most three different prime divisors.

**Proposition 2.3.** *Let $G$ be a non-abelian simple group. Suppose that the order $|G|$ has at most three different prime divisors. Then $G$ is called $K_3$ simple group and isomorphic to one of the following groups.*

Table 1: **Non-abelian simple $\{2,3,p\}$-groups**

| Group | Order | Group | Order |
|---|---|---|---|
| $A_5$ | $2^2\cdot 3\cdot 5$ | $\mathrm{PSL}(2,17)$ | $2^4\cdot 3^2\cdot 17$ |
| $A_6$ | $2^3\cdot 3^2\cdot 5$ | $\mathrm{PSL}(3,3)$ | $2^4\cdot 3^3\cdot 13$ |
| $\mathrm{PSL}(2,7)$ | $2^3\cdot 3\cdot 7$ | $\mathrm{PSU}(3,3)$ | $2^5\cdot 3^3\cdot 7$ |
| $\mathrm{PSL}(2,8)$ | $2^3\cdot 3^2\cdot 7$ | $\mathrm{PSU}(4,2)$ | $2^6\cdot 3^4\cdot 5$ |

## 3. Classification

This section is devoted to classifying prime-valent one-regular graphs of order $8p$ for each prime $p$. Let $q$ be a prime. In what follows, we always denote by $X$ a connected $q$-valent one-regular graph of order $8p$. Set $A=\mathrm{Aut}(X)$, $v\in V(X)$. Then the vertex stabilizer $A_v\cong\mathbb{Z}_q$ and hence $|A|=8pq$. Clearly, if $q=2$, then $X\cong C_{8p}$ with $A\cong D_{16p}$.

Let $q=3$. Then by [8, Theorem 5.1], we can have the classification of cubic one-regular graphs of order $8p$. For convenience, we use the same notation $CQ_p$ as in [8] to denote the cyclic $\mathbb{Z}_p$-covering graph of the three-dimensional hypercube $Q_3$.

**Lemma 3.1.** *If $q=3$, then $X\cong CQ_p$ and $A\cong\mathbb{Z}_p\rtimes(\mathrm{A}_4\times\mathbb{Z}_2)$ with $3|(p-1)$.*

For $q=5$ or $7$, by [13, Theorem 3.1] and [12, Theorem 1.1], it is easy to see that there is no new graph. For $p=2$ or $3$, by [5], there is no prime-valent one-regular graph of order 16 or 24. Thus, we treat with the case $p\geq 5$ and $q>7$ by proving the following lemma.

**Lemma 3.2.** *Let $p\geq 5$ and $q>7$. Then there is no new graph.*

**Proof.** Recall that $|A|=8pq$ and $A_v\cong\mathbb{Z}_q$. By Proposition 2.3, the order of each $K_3$ simple group has a divisor 3. It forces that $A$ is solvable. We divide the proof into the following two cases: $p=q$ and $p\neq q$.

**Case 1:** Suppose that $p=q$. Then $|A|=8p^2$ and $A_v\cong\mathbb{Z}_p$.

Let $P$ be a Sylow $p$-subgroup of $A$. Then $|P|=p^2$. Note that $p=q>7$. Thus, by Sylow Theorem, we have that $P$ is normal in $A$. This means that $P$ is the only Sylow $p$-subgroup of $A$. Since $A_v\cong\mathbb{Z}_p$, we have that $A_v\leq P$, that is, $A_v=P_v\neq 1$. By Proposition 2.1, $P$ is transitive or has two orbits on $V(X)$. Clearly, both are impossible because $|P|=p^2$ and $|V(X)|=8p$.

**Case 2:** Suppose that $p \neq q$. Then $|A| = 8pq$ and $A_v \cong \mathbb{Z}_q$.

Since $|A| = 8pq$ and $A_v \cong \mathbb{Z}_q$, we have that $A_v$ is a Sylow $q$-subgroup of $A$. It forces that the Sylow $q$-subgroups of $A$ cannot be normal in $A$. Recall that $A$ is solvable. Thus, all normal subgroups of $A$ are solvable. It follows that $A$ has a maximal normal $r$-subgroup with $r = 2$ or $p$.

Assume that $A$ has a maximal normal $p$-subgroup $M$. Then $M \cong \mathbb{Z}_p$ because $|A| = 8pq$. Clearly, $M$ acting on $V(X)$ has 8 orbits. By Proposition 2.1, $X_M$ is a $q$-valent symmetric graph of order 8. By [5], $X_M \cong Q_3$ with $q = 3$ or $K_8$ with $q = 7$. This contradicts our hypothesis. Thus, the Sylow $p$-subgroup of $A$ cannot be normal in $A$.

Assume that $A$ has a maximal normal 2-subgroup $N$. Since $N$ is a 2-subgroup, we have that $N$ acting on $V(X)$ has at least $p$ orbits. By Proposition 2.1, $X_N$ is a $q$-valent symmetric graph of order $8p/|N|$. Recall that $q > 7$ is an odd prime and there is no graph of odd order and odd valency. Thus, $|N| \neq 8$ and $|N| = 2$ or 4.

Let $|N| = 2$. Then $X_N$ is a $q$-valent symmetric graph of order $4p$ and $|A/N| = 4pq$. Note that $A$ is solvable. Thus, $A/N$ is also solvable. Since $A_v \cong \mathbb{Z}_q$, we have that $A/N$ has no normal $q$-subgroup. If $A/N$ has a normal $p$-subgroup $K/N \cong \mathbb{Z}_p$. Then $|K| = 2p$. Since $p \geq 5$, we have that $K$ has a normal Sylow $p$-subgroup $M$ by Sylow Theorem. It forces that $M$ is characteristic in $K$ and hence normal in $A$. By the above argument, this is impossible. This implies that $A/N$ has a non-trivial normal 2-subgroup, this contradicts the maximality of $N$.

Let $|N| = 4$. Then $X_N$ is a $q$-valent symmetric graph of order $2p$ and $|A/N| = 2pq$. Recall that $q \neq p$ and $q \neq 5$. By Proposition 2.2, $X_N \cong K_{2p}$ with $q = 2p - 1$ or $G(2p, q)$ with $A \cong (\mathbb{Z}_p \rtimes \mathbb{Z}_q) \rtimes \mathbb{Z}_2$. For the former, $q = 2p - 1$ is a prime. Thus, $A/N \lesssim S_{2p}$ and $A/N$ is 2-transitive on $V(X)$. By Burnside's Theorem, any 2-transitive permutation group is almost simple or affine. Since $A/N$ is solvable, we have that $A/N$ is affine. It forces that $A/N$ must have a normal subgroup isomorphic to $\mathbb{Z}_p$. By the above argument, this is impossible. For the later, $A/N \cong (\mathbb{Z}_p \rtimes \mathbb{Z}_q) \rtimes \mathbb{Z}_2$. It is easy to see that $A/N$ has a normal Sylow $p$-subgroup. Similarly, since $p \geq 5$, we can easily deduce that $A$ has a normal Sylow $p$-subgroup, a contradiction. $\square$

Combining the above arguments with the cases $q = 2, 5, 7$ and $p = 2, 3$, and by Lemmas 3.1-3.2, we have the following result.

**Theorem 3.3.** *Let $p, q$ be two primes and $X$ a connected $q$-valent one-regular graph of order $8p$. Then $X \cong C_{8p}$ with $\mathrm{Aut}(X) \cong D_{16p}$ or $X \cong CQ_p$ with $\mathrm{Aut}(X) \cong \mathbb{Z}_p \rtimes (\mathrm{A}_4 \times \mathbb{Z}_2)$ and $3|(p-1)$.*

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