On drift parameter estimation for mean-reversion type nonlinear nonhomogeneous stochastic differential equations

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Abstract. This paper is concerned with the drift parameter estimation for mean-reversion type nonlinear nonhomogeneous stochastic differential equations. Firstly, the Girsanov transformation is used to simplify the equation because of the expression of the drift coefficient. Secondly, we find a closed interval on which the likelihood function is continuous and does not attain the maximum at two endpoints of this interval. Then, we prove that the maximum likelihood estimator exists in the interval when the sample size is large enough. Finally, the strong consistency of the estimator and the asymptotic normality of the error of estimation are proved.

Keywords: drift parameter estimation, mean-reversion, nonlinear nonhomogeneous, Girsanov transformation, consistency, asymptotic normality.

1. Introduction

Stochastic differential equations are very important for model building in physics, engineering, medical science and life sciences. Recently, stochastic differential equations, especially mean-reversion type stochastic differential equations, have been widely used to describe the price dynamics of a financial asset such as interest rate, discount bonds and futures, ([2, 6, 8, 13, 16]). For example, the Black-Scholes option pricing model described by a geometric Brownian motion ([3]), the Vasicek and Cox-Ingersoll-Ross models developed based on two specific mean-reversion diffusion processes ([17, 4, 5]). Therefore, stochastic differential equations are the basic stochastic modeling instruments in the modern financial theory. However, in practice, part or all of the parameters in stochastic models are always unknown, but the observed values are known. Hence, the unknown parameters are needed to be estimated based on observations. The parameters in the model may be finite-dimensional or infinite-dimensional. They have to be estimated either from one or many continuous realizations of the process if a continuous realization is possible, or from a discrete sampled data set for the process if the process is not continuously observable.

In the past decades, some methods have been used to estimate the parameters in stochastic differential equations based on continuous observations. For example, Kutoyants [10] and Wei [19] discussed the consistency and asymptotic normality of the maximum likelihood estimator based on likelihood ratio function and maximum probability estimation respectively. Barczy [1] analyzed the consistency of the maximum likelihood estimator and given the sufficient condition for the estimation error to satisfy the normal distribute. Kan [9] used filter theory to prove the consistency of the Bayes estimator. Mendy [14] applied least squares method to estimate the parameter in Ornstein-Uhlenbeck process driven by fractional Brownian motion. Wen [20] studied the local asymptotic normality of the maximum likelihood estimator for McKean-Vlasov stochastic differential equation. Other methods such as M-estimation method ([21, 22]) and sequential estimation method ([7]) were used to estimate the parameters as well.

Although the parameter estimation for mean-reversion type stochastic differential equations has been studied by some authors such as Li [12] and Wei [18], the drift and diffusion coefficients do not depend on the time t, so the ergodic theorem could be used. However, the nonlinear nonhomogeneous stochastic differential equation could describe the stochastic phenomenon much better and the ergodic theorem could not be applied. Therefore, it is of great importance to study this topic. In this paper, parameter estimation is investigated for nonlinear nonhomogeneous stochastic differential equation. Since the stochastic differential equation is nonhomogeneous, the ergodic theorem used in Li [12] and Wei [18] can not be applied in this paper. Hence, the Girsanov transformation is used firstly to simplify the equation. Then, we find a closed interval on which the likelihood function is continuous and does not attain the maximum at two endpoints of this interval. The existence and strong consistency of the maximum likelihood estimator are proved and the asymptotic normality of the error of estimation is discussed by using the central limit theorem, Lepingle's law of large numbers and the law of large numbers for square-integrable martingales.

This paper is organized as follows. In Section 2 the stochastic differential equation is simplified and a new family of probability measures has been indexed, the likelihood function is given as well. The main results are given in Section 3. In this section, the existence and strong consistency of the maximum likelihood estimator are proved and the asymptotic normality of the error of estimation is analyzed. The conclusion is given in Section 4.

2. Problem formulation and preliminaries

We study the parameter estimation problem for the following stochastic differential equation:

$$\begin{cases} dX_t = (\gamma + \mu(t, X_t))a(t, X_t, \theta)dt + b(t, X_t)dB_t, \\ X_0 = x_0 \in \mathbb{R}, \end{cases} \tag{1}$$

where $\theta \in \Theta$ is an unknown parameter, $\gamma \neq 0$ is a constant, $\mu(t, x)$ is twice differentiable with respect to x and differentiable with respect to t, both $a(t, x, \theta)$

and $b(t,x)$ are continuous with respect to t and x, $(B_t, t \geq 0)$ is a standard Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, \mathrm{P})$. Suppose Equation (1) satisfies the conditions that ensure the existence and uniqueness of the solution.

Because of the complexity of Equation 1, the Girsanov transformation will be used to simplify the drift coefficient. From now on, we will work under the assumptions as follows:

Assumption 1. $|\mu(t,x)a(t,x,\theta) - \mu(t,y)a(t,y)| \leq K|x-y|$, K is a positive constant.

Assumption 2. $Q_\theta(\int_0^T a^2(t,X_t,\theta)dt < \infty) = 1$ where Q is the probability measure defined on $(\Omega, \mathcal{F}_T)$.

Assumption 3. $\mathbb{E}_\theta[\int_0^T (a'(t,X_t,\theta)dt)^2)] < \infty$ and $\mathbb{E}_\theta[\int_0^T (a''(t,X_t,\theta)dt)^2)] < \infty$.

Assumption 4. for any θ, there exists a neighborhood V_θ of θ such that $Q_\theta(\int_0^\infty \frac{(a(t,X_t,\theta')-a(t,X_t,\theta))^2}{b^2(t,X_t)}dt = \infty) = 1$, for all $\theta' \in V_\theta - \theta$.

Firstly, we introduce the Girsanov theorem below.

Lemma 1 ([11])**.** *Let $Y(t)$ be an Itô process of the form*

$$dY(t) = a(t,\omega)dt + dB(t); t \leq T,$$

where $T \leq \infty$ is a given constant and $B(t)$ is Brownian motion. Put

$$M_t = \exp(-\int_0^t a(s,\omega)dB_s - \frac{1}{2}\int_0^t a^2(s,\omega)ds); t \leq T.$$

Assume that $a(s,\omega)$ satisfies Novikov's condition

$$\mathbb{E}[\exp(\frac{1}{2}\int_0^T a^2(s,\omega)ds)] < \infty,$$

where $\mathbb{E}$ is the expectation with respect to P. Define the measure Q on $(\Omega, \mathcal{F}_T)$ by

$$dQ(\omega) = M_T dP(\omega).$$

Then $Y(t)$ is a Brownian motion with respect to the probability law Q, for $t \leq T$.

According to the Girsanov theorem, Equation 1 could be written as

$$dX_t = ((\gamma + \mu(t,X_t))a(t,X_t,\theta) + b(t,X_t)\alpha(t,X_t))dt + b(t,X_t)d\widetilde{B}_t, \tag{2}$$

where $\alpha(t,X_t)$ satisfies the Novikov condition, $\widetilde{B}_t = B_t - \int_0^t \alpha(s,X_s)ds$ is an $\{\mathcal{F}_t\}_{t\in[0,1]}$-Brownian motion under the probability Q_t.

Let

$$\mu(t,x)a(t,x,\theta)+b(t,x)\alpha(t,x)=0,$$

hence

$$\mu(t,X_t)a(t,X_t,\theta)+b(t,X_t)\alpha(t,X_t)=0.$$

Therefore, Equation 2 becomes

$$dX_t=\gamma a(t,X_t,\theta)dt+b(t,X_t)d\widetilde{B}_t. \tag{3}$$

According to the Girsanov theorem, the continuous-time log-likelihood function has the following expression

$$\ell_T(\theta)=\gamma\int_0^T \frac{a(t,X_t,\theta)}{b^2(t,X_t)}dX_t-\frac{1}{2}\gamma^2\int_0^T \frac{a^2(t,X_t,\theta)}{b^2(t,X_t)}dt. \tag{4}$$

3. Main results and proofs

In the following theorem, the strongly consistency of the maximum likelihood estimator is proved.

Theorem 1. *Under Assumptions 1-4, when $T\to\infty$,*

$$\widehat{\theta_T}\overset{a.s.}{\to}\theta.$$

Proof. It is easy to check that, for any $\delta>0$ such that $\theta\pm\delta\in\Theta$,

$$\begin{aligned}\ell_T(\theta\pm\delta)-\ell_T(\theta)&=\gamma\int_0^T \frac{(a(t,X_t,\theta\pm\delta)-a(t,X_t,\theta))}{b^2(t,X_t)}dX_t\\ &-\frac{1}{2}\gamma^2\int_0^T \frac{(a^2(t,X_t,\theta\pm\delta)-a^2(t,X_t,\theta))}{b^2(t,X_t)}dt\\ &=\gamma\int_0^T \frac{(a(t,X_t,\theta\pm\delta)-a(t,X_t,\theta))}{b(t,X_t)}d\widetilde{B}_t\\ &-\frac{1}{2}\gamma^2\int_0^T \frac{(a(t,X_t,\theta\pm\delta)-a(t,X_t,\theta))^2}{b^2(t,X_t)}dt.\end{aligned}$$

Then we can obtain that

$$\frac{\ell_T(\theta\pm\delta)-\ell_T(\theta)}{\gamma^2\int_0^T \frac{(a(t,X_t,\theta\pm\delta)-a(t,X_t,\theta))^2}{b^2(t,X_t)}dt}=\frac{\gamma\int_0^T \frac{(a(t,X_t,\theta\pm\delta)-a(t,X_t,\theta))}{b(t,X_t)}d\widetilde{B}_t}{\gamma^2\int_0^T \frac{(a(t,X_t,\theta\pm\delta)-a(t,X_t,\theta))^2}{b^2(t,X_t)}dt}-\frac{1}{2}. \tag{5}$$

Under Assumption 4, by using Lepingle's law of large numbers, when $T\to\infty$, it follows that

$$\frac{\gamma\int_0^T \frac{(a(t,X_t,\theta\pm\delta)-a(t,X_t,\theta))}{b(t,X_t)}d\widetilde{B}_t}{\gamma^2\int_0^T \frac{(a(t,X_t,\theta\pm\delta)-a(t,X_t,\theta))^2}{b^2(t,X_t)}dt}\overset{a.s.}{\to}0. \tag{6}$$

Hence, when $T \to \infty$, it can be check that

$$\frac{\ell_T(\theta \pm \delta) - \ell_T(\theta)}{\gamma^2 \int_0^T \frac{(a(t,X_t,\theta\pm\delta)-a(t,X_t,\theta))^2}{b^2(t,X_t)} dt} \overset{a.s.}{\to} -\frac{1}{2}. \tag{7}$$

Note that $\gamma^2 \int_0^T \frac{(a(t,X_t,\theta\pm\delta)-a(t,X_t,\theta))^2}{b^2(t,X_t)} dt > 0 a.s.$ for large T by Assumption 4. Therefore, for almost every δ and θ, there exists T_0 such that for $T \geq T_0$

$$\ell_T(\theta \pm \delta) < \ell_T(\theta). \tag{8}$$

Since $\ell_T(\theta)$ is continuous on the compact set $[\theta - \delta, \theta + \delta]$, it has a local maximum and the maximum is attained at an element $\widehat{\theta}_T \in (\theta - \delta, \theta + \delta)$. As $\ell_T(\theta)$ is differentiable with respect to θ, it follows that $\ell'_T(\widehat{\theta}_T) = 0$ and $|\widehat{\theta}_T - \theta| < \delta$.

Therefore, when $T \to \infty$, $\widehat{\theta_T} \overset{a.s.}{\to} \theta$. The proof is complete. $\square$

Let $N_T(\theta) = \int_0^T \frac{(a'(t,X_t,\theta))^2}{b^2(t,X_t)} dt$. In the following theorem, the asymptotic normality of the error of estimation is proved .

Theorem 2. *Under Assumptions 1-4, when $T \to \infty$,*

$$\sqrt{N_T(\theta)}(\widehat{\theta_T} - \theta) \overset{d}{\to} N(0,1).$$

Proof. Expanding $\ell'_T(\theta)$ around $\widehat{\theta_T}$, it follows that

$$\ell'_T(\theta) = \ell'_T(\widehat{\theta_T}) + \ell''_T(\widetilde{\theta}_T)(\theta - \widehat{\theta_T}), \tag{9}$$

where $\widetilde{\theta}_T$ is between $\widehat{\theta_T}$ and θ.

In view of Theorem 1, it is known that $\ell'_T(\widehat{\theta_T}) = 0$, then

$$\ell'_T(\theta) = \ell''_T(\widetilde{\theta}_T)(\theta - \widehat{\theta_T}). \tag{10}$$

Since $\widehat{\theta_T} \overset{a.s.}{\to} \theta$ when $T \to \infty$ and $\ell''_T(\theta)$ is continuous in θ, it follows that when $T \to \infty$,

$$\ell''_T(\widetilde{\theta}_T)(\theta - \widehat{\theta_T}) \overset{P}{\to} \ell''_T(\theta). \tag{11}$$

By using the central limit theorem and the law of large numbers for square-integrable martingales, it is easy to check that when $T \to \infty$, $N_T(\theta) \to \infty$ and $\frac{\ell'_T(\theta)}{\sqrt{N_T(\theta)}} \overset{d}{\to} N(0,1)$.

Since

$$\frac{\ell'_T(\theta)}{\sqrt{N_T(\theta)}} = \frac{\ell''_T(\widetilde{\theta}_T)(\theta - \widehat{\theta_T})}{\sqrt{N_T(\theta)}}, \tag{12}$$

when $T \to \infty$, we can obtain that

$$\frac{\ell'_T(\theta)}{\sqrt{N_T(\theta)}} \xrightarrow{P} \frac{\ell''_T(\theta)(\theta - \widehat{\theta_T})}{\sqrt{N_T(\theta)}}. \tag{13}$$

Hence, when $T \to \infty$, it follows that

$$\frac{\ell''_T(\theta)(\theta - \widehat{\theta_T})}{\sqrt{N_T(\theta)}} \xrightarrow{d} N(0,1). \tag{14}$$

As

$$\frac{\ell''_T(\theta)}{N_T(\theta)} \xrightarrow{P} -1, \tag{15}$$

it can be check that

$$\sqrt{N_T(\theta)}(\widehat{\theta_T} - \theta) \xrightarrow{d} N(0,1).$$

The proof is complete. □

4. Conclusion

The current work concerns the drift parameter estimation for mean-reversion type nonlinear nonhomogeneous stochastic differential equations. The Girsanov theorem has been used to simplify the equations and a new family of probability measures has been indexed. The existence and strong consistency of the estimator and asymptotic normality of the error of estimation have been proved by applying the central limit theorem, Lepingle's law of large numbers and the law of large numbers for square-integrable martingales. Further topics will include the parameter estimation for stochastic differential equations driven by lévy process.

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