On some (p, q)-φ relative Gol'dberg type and (p, q)-φ relative Gol'dberg weak type based growth properties of entire functions of several complex variables

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Abstract. The primary concern of this paper is to introduce the notion of (p, q)-φ relative Gol'dberg type, (p, q)-φ relative Gol'dberg weak type of entire functions of several complex variables and study some growth properties based upon them.
Keywords: Entire functions of several complex variables, (p, q)-φ relative Gol'dberg order, (p, q)-φ relative Gol'dberg lower order, (p, q)-φ relative Gol'dberg type, (p, q)-φ relative Gol'dberg weak type.

1. Introduction, definitions and notations

The symbols $\mathbb{C}^n$ and R^n will stand for complex and real n-spaces respectively . In addition, let us assume that the points $(z_1, z_2, \cdots, z_n)$, $(m_1, m_2, \cdots, m_n)$ of $\mathbb{C}^n$ or I^n be represented by their corresponding unsuffixed symbols z, m respectively where I denotes the set of non-negative integers. Then the modulus of z, denoted by $|z|$, is defined as $|z| = (|z_1|^2 + \cdots + |z_n|^2)^{\frac{1}{2}}$. If the coordinates of the vector m are non-negative integers, then the expression $z_1^{m_1} \cdots z_n^{m_n}$ will be denoted by z^m where $\|m\| = m_1 + \cdots + m_n$.

Consider $D \subseteq \mathbb{C}^n$ to be an arbitrary bounded complex n-circular domain with center at the origin of coordinates. Then for any entire function $f(z)$ of n complex variables and $R > 0$, $M_{f,D}(R)$ may be defined as $M_{f,D}(R) = \sup_{z \in D_R} |f(z)|$ where a point $z \in D_R$ if and only if $\frac{z}{R} \in D$. If $f(z)$ is non-constant, then $M_{f,D}(R)$ is strictly increasing and its inverse $M_{f,D}^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exists such that $\lim_{R\to\infty} M_{f,D}^{-1}(R) = \infty$.

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For $k \in \mathbb{N}$, we define $\exp^{[k]} R = \exp(\exp^{[k-1]} R)$ and $\log^{[k]} R = \log(\log^{[k-1]} R)$ where $\mathbb{N}$ is the set of all positive integers. We also denote $\log^{[0]} R = R$, $\log^{[-1]} R = \exp R$, $\exp^{[0]} R = R$ and $\exp^{[-1]} R = \log R$. Further we assume that throughout the present paper p, q and m always denote positive integers. Also throughout the paper an entire function $f(z)$ of n-complex variables will stand for an entire function $f(z)$ for any bounded complete n-circular domain D with center at origin in $\mathbb{C}^n$. Taking this into account, we recall that Datta et al. [6] defined the concept of (p, q)-th Gol'dberg order and (p, q)-th Gol'dberg lower order of an entire function $f(z)$ of n-complex variables where $p \geq q$ in the following way:

$$\rho_D^{(p,q)}(f) = \limsup_{R \to +\infty} \frac{\log^{[p]} M_{f,D}(R)}{\log^{[q]} R}$$
$$\lambda_D^{(p,q)}(f) = \liminf_{R \to +\infty} \frac{\log^{[p]} M_{f,D}(R)}{\log^{[q]} R}.$$

For $p = 2$ and $q = 1$, the symbols $\rho_D^{(2,1)}(f)$ and $\lambda_D^{(2,1)}(f)$ are respectively denoted by $\rho_D(f)$ and $\lambda_D(f)$ which are actually classical growth indicators (see e.g. [9, 10]). However in the line of Gol'dberg (see e.g. [9, 10]), it may be easily established that $\rho_D^{(p,q)}(f)$ and $\lambda_D^{(p,q)}(f)$ are independent of the choice of the domain D, and therefore one can write $\rho^{(p,q)}(f)$ and $\lambda^{(p,q)}(f)$ instead of $\rho_D^{(p,q)}(f)$ and $\lambda_D^{(p,q)}(f)$ respectively.

In [13], Shen et al. introduced the definition of (p, q)-φ order of an entire function. For details about (p, q)-φ order, one may see [13]. Consequently the definition of (p, q)-φ Gol'dberg order and (p, q)-φ Gol'dberg lower order of an entire function $f(z)$ of n-complex variables are established in [4] which are as follows:

Definition 1 ([4]). *Let $\varphi(R) : [0, +\infty) \to (0, +\infty)$ be a non-decreasing unbounded function. Then the (p, q)-φ Gol'dberg order $\rho_D^{(p,q)}(f, \varphi)$ and (p, q)-φ Gol'dberg lower order $\lambda_D^{(p,q)}(f, \varphi)$ of an entire function $f(z)$ of n-complex variables are defined as*

$$\rho_D^{(p,q)}(f, \varphi) = \limsup_{R \to +\infty} \frac{\log^{[p]} M_{f,D}(R)}{\log^{[q]} \varphi(R)}$$
$$\lambda_D^{(p,q)}(f, \varphi) = \liminf_{R \to +\infty} \frac{\log^{[p]} M_{f,D}(R)}{\log^{[q]} \varphi(R)}.$$

The above definition avoids the restriction $p \geq q$. However, an entire function $f(z)$ for which $\rho_D^{(p,q)}(f, \varphi)$ and $\lambda_D^{(p,q)}(f, \varphi)$ are the same is called a function of regular (p, q)-φ Gol'dberg growth. Otherwise, $f(z)$ is said to be irregular (p, q)-φ Gol'dberg growth. For any non-decreasing unbounded function $\varphi(R) : [0, +\infty) \to (0, +\infty)$, if it is assumed that $\lim_{R \to +\infty} \frac{\log^{[q]} \varphi(\alpha R)}{\log^{[q]} \varphi(R)} = 1$ for all $\alpha > 0$, then one can easily verify that $\rho_D^{(p,q)}(f, \varphi)$ and $\lambda_D^{(p,q)}(f, \varphi)$ are independent of the choice of the domain D, and therefore one can use the symbols $\rho^{(p,q)}(f,\varphi)$ and $\lambda^{(p,q)}(f,\varphi)$ instead of $\rho_D^{(p,q)}(f,\varphi)$ and $\lambda_D^{(p,q)}(f,\varphi)$ respectively.

If $\varphi(R) = R$ and $p \geq q$, then Definition 1 coincides with the definition of (p,q)-th Gol'dberg order and (p,q)-th Gol'dberg lower order introduced by Datta et al. [6].

Concerning this we just state the following definition:

Definition 2. *An entire function $f(z)$ of n-complex variables is said to have index-pair (p,q)-φ if $b < \rho^{(p,q)}(f,\varphi) < \infty$ and $\rho^{(p-1,q-1)}(f,\varphi)$ is not a nonzero finite number, where $b = 1$ if $p = q$ and $b = 0$ for otherwise. Moreover if $0 < \rho^{(p,q)}(f,\varphi) < \infty$, then*

$$\begin{cases} \rho^{(p-n,q)}(f,\varphi) = \infty, & \text{for} \quad n < p, \\ \rho^{(p,q-n)}(f,\varphi) = 0, & \text{for} \quad n < q, \\ \rho^{(p+n,q+n)}(f,\varphi) = 1, & \text{for}\ n = 1,2,\cdots. \end{cases}$$

Similarly for $0 < \lambda^{(p,q)}(f,\varphi) < \infty$,

$$\begin{cases} \lambda^{(p-n,q)}(f,\varphi) = \infty, & \text{for} \quad n < p, \\ \lambda^{(p,q-n)}(f,\varphi) = 0, & \text{for} \quad n < q, \\ \lambda^{(p+n,q+n)}(f,\varphi) = 1, & \text{for} \quad n = 1,2,\cdots \quad . \end{cases}$$

Consequently for $\varphi(R) = R$, Definition 2 reduces to the the definition of index-pair (p,q) of an entire function $f(z)$ of n-complex variables. For detail about index-pair (p,q) of an entire function $f(z)$ of n-complex variables, one may see [3].

However for any two entire functions $f(z)$ and $g(z)$ of n-complex variables, Mondal et al. [11] introduced the concept of relative Gol'dberg order of $f(z)$ with respect to $g(z)$. In the case of relative Gol'dberg order, it therefore seems reasonable to define suitably the (p,q)-th relative Gol'dberg order. With this in view one can introduce the following definition in the light of index-pair.

Definition 3 ([3])**.** *Let $f(z)$ and $g(z)$ be any two entire functions of n-complex variables with index-pairs (m,q) and (m,p) respectively. Then the (p,q)-th relative Gol'dberg order $\rho_{g,D}^{(p,q)}(f)$ and (p,q)-th relative Gol'dberg lower order $\lambda_{g,D}^{(p,q)}(f)$ of $f(z)$ with respect to $g(z)$ are defined as*

$$\begin{matrix} \rho_{g,D}^{(p,q)}(f) \\ \lambda_{g,D}^{(p,q)}(f) \end{matrix} = \lim_{R\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p]} M_{g,D}^{-1}(M_{f,D}(R))}{\log^{[q]} R}.$$

Definition 3 avoids the restriction $p \geq q$ of Definition 1.3 of [1]. In view of Theorem 2.1 of [1] one can easily prove that $\rho_{g,D}^{(p,q)}(f)$ and $\lambda_{g,D}^{(p,q)}(f)$ are independent of the choice of the domain D, and therefore one can write $\rho_g^{(p,q)}(f)$ and $\lambda_g^{(p,q)}(f)$ instead of $\rho_{g,D}^{(p,q)}(f)$ and $\lambda_{g,D}^{(p,q)}(f)$.

Further an entire function $f(z)$ of n-complex variables for which $\rho_g^{(p,q)}(f)$ and $\lambda_g^{(p,q)}(f)$ are the same is called a function of regular relative (p,q) Gol'dberg growth with respect to an entire function $g(z)$ of n-complex variables. Otherwise, $f(z)$ is said to be irregular relative (p,q) Gol'dberg growth with respect to $g(z)$.

In [4], Biswas has further introduced the notion of relative (p,q)-th Gol'dberg type $\sigma_{g,D}^{(p,q)}(f)$, relative (p,q)-th Gol'dberg lower type $\overline{\sigma}_{g,D}^{(p,q)}(f)$, relative (p,q)-th Gol'dberg weak type $\tau_{g,D}^{(p,q)}(f)$ and another growth indicator $\overline{\tau}_{g,D}^{(p,q)}(f)$ in the light of index-pair which are as follows:

Definition 4 ([3])**.** *Let $f(z)$ and $g(z)$ be any two entire functions of n-complex variables with index-pairs (m,q) and (m,p) respectively, such that $0<\rho_g^{(p,q)}(f)<+\infty$. Then the relative (p,q)-th Gol'dberg type $\sigma_{g,D}^{(p,q)}(f)$, relative (p,q)-th Gol'dberg lower type $\overline{\sigma}_{g,D}^{(p,q)}(f)$ of $f(z)$ with respect to $g(z)$ are defined as:*

$$\begin{matrix} \sigma_{g,D}^{(p,q)}(f) \\ \overline{\sigma}_{g,D}^{(p,q)}(f) \end{matrix} = \lim_{R\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} M_{g,D}^{-1}(M_{f,D}(R))}{(\log^{[q-1]} R)^{\rho_g^{(p,q)}(f)}}$$

Definition 5 ([3])**.** *Let $f(z)$ and $g(z)$ be any two entire functions of n-complex variables with index-pairs (m,q) and (m,p) respectively, such that $0<\lambda_g^{(p,q)}(f)<+\infty$. Then the relative (p,q)-th Gol'dberg weak type $\tau_{g,D}^{(p,q)}(f)$, and the growth indicator $\overline{\tau}_{g,D}^{(p,q)}(f)$ of $f(z)$ with respect to $g(z)$ are defined as:*

$$\begin{matrix} \overline{\tau}_{g,D}^{(p,q)}(f) \\ \tau_{g,D}^{(p,q)}(f) \end{matrix} = \lim_{R\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} M_{g,D}^{-1}(M_{f,D}(R))}{(\log^{[q-1]} R)^{\lambda_g^{(p,q)}(f)}}$$

Since Gol'dberg has shown that (see [9, 10]) Gol'dberg type depends on the domain D, all the growth indicators defined in Definition 4 and Definition 5 also depend on D (cf. [3]).

Now in order to make some progress in the study of relative Gol'dberg order, in [4], the definition of (p,q)-φ relative Gol'dberg order and the (p,q)-φ relative Gol'dberg lower order in view of index-pair are given which are as follows:

Definition 6 ([4])**.** *Let $\varphi(R):[0,+\infty)\to(0,+\infty)$ be a non-decreasing unbounded function. Also let $f(z)$ and $g(z)$ be any two entire functions of n-complex variables with index-pairs (m,q) and (m,p) respectively. The (p,q)-φ relative Gol'dberg order and the (p,q)-φ relative Gol'dberg lower order of $f(z)$ with respect to $g(z)$ are defined as*

$$\rho_{g,D}^{(p,q)}(f,\varphi) = \limsup_{R\to+\infty} \frac{\log^{[p]} M_{g,D}^{-1}(M_{f,D}(R))}{\log^{[q]} \varphi(R)}$$

and

$$\lambda_{g,D}^{(p,q)}(f,\varphi) = \liminf_{R\to+\infty} \frac{\log^{[p]} M_{g,D}^{-1}(M_{f,D}(R))}{\log^{[q]} \varphi(R)}.$$

Further an entire function $f(z)$ of n-complex variables for which $\rho_{g,D}^{(p,q)}(f,\varphi)$ and $\lambda_{g,D}^{(p,q)}(f,\varphi)$ are the same is called a function of regular (p,q)-φ relative Gol'dberg growth with respect to an entire function $g(z)$ of n-complex variables. Otherwise, $f(z)$ is said to be irregular (p,q)-φ relative Gol'dberg growth with respect to $g(z)$.

In this paper, we assume that the nondecreasing unbounded function $\varphi(R)$: $[0,+\infty) \to (0,+\infty)$ always satisfies $\lim_{R\to+\infty} \frac{\log^{[q]}\varphi(\alpha R)}{\log^{[q]}\varphi(R)} = 1$ for all $\alpha > 0$. Since, Biswas et al. [4] have already shown that $\rho_{g,D}^{(p,q)}(f,\varphi)$ and $\lambda_{g,D}^{(p,q)}(f,\varphi)$ are independent of the choice of the domain D when $\varphi(R) : [0,+\infty) \to (0,+\infty)$ is a nondecreasing unbounded function and satisfies $\lim_{R\to+\infty} \frac{\log^{[q]}\varphi(\alpha R)}{\log^{[q]}\varphi(R)} = 1$ for all $\alpha > 0$, so here we shall always use the notations $\rho_{g}^{(p,q)}(f,\varphi)$ and $\lambda_{g}^{(p,q)}(f,\varphi)$ instead of $\rho_{g,D}^{(p,q)}(f,\varphi)$ and $\lambda_{g,D}^{(p,q)}(f,\varphi)$ respectively.

Now, for the development of such growth indicators, one may introduce (p,q)-φ relative Gol'dberg type $\sigma_{g,D}^{(p,q)}(f,\varphi)$, (p,q)-φ relative Gol'dberg lower type $\overline{\sigma}_{g,D}^{(p,q)}(f,\varphi)$, (p,q)-φ relative Gol'dberg weak type $\tau_{g,D}^{(p,q)}(f,\varphi)$ and another growth indicator $\overline{\tau}_{g,D}^{(p,q)}(f,\varphi)$ with the help of index-pair in the following way:

Definition 7. *Let $\varphi(R) : [0,+\infty) \to (0,+\infty)$ be a non-decreasing unbounded function. Let $f(z)$ and $g(z)$ be any two entire functions of n-complex variables with index-pairs (m,q) and (m,p) respectively, such that $0 < \rho_{g}^{(p,q)}(f,\varphi) < +\infty$. Then the (p,q)-φ relative Gol'dberg type $\sigma_{g,D}^{(p,q)}(f,\varphi)$ and the (p,q)-φ relative Gol'dberg lower type $\overline{\sigma}_{g,D}^{(p,q)}(f,\varphi)$ of $f(z)$ with respect to $g(z)$ are defined as:*

$$\begin{matrix}\sigma_{g,D}^{(p,q)}(f,\varphi) \\ \overline{\sigma}_{g,D}^{(p,q)}(f,\varphi)\end{matrix} = \lim_{R\to+\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log^{[p-1]} M_{g,D}^{-1}(M_{f,D}(R))}{(\log^{[q-1]}\varphi(R))^{\rho_{g}^{(p,q)}(f,\varphi)}}.$$

Definition 8. *Let $\varphi(R) : [0,+\infty) \to (0,+\infty)$ be a non-decreasing unbounded function. Let $f(z)$ and $g(z)$ be any two entire functions of n-complex variables with index-pairs (m,q) and (m,p) respectively, such that $0 < \lambda_{g}^{(p,q)}(f,\varphi) < +\infty$. Then the (p,q)-φ relative Gol'dberg weak type $\tau_{g,D}^{(p,q)}(f,\varphi)$ and the growth indicator $\overline{\tau}_{g,D}^{(p,q)}(f,\varphi)$ of $f(z)$ with respect to $g(z)$ are defined as:*

$$\begin{matrix}\overline{\tau}_{g,D}^{(p,q)}(f,\varphi) \\ \tau_{g,D}^{(p,q)}(f,\varphi)\end{matrix} = \lim_{R\to+\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log^{[p-1]} M_{g,D}^{-1}(M_{f,D}(R))}{(\log^{[q-1]}\varphi(R))^{\lambda_{g}^{(p,q)}(f,\varphi)}}.$$

As earlier it was discussed that the relative (p,q)-th Gol'dberg type, relative (p,q)-th Gol'dberg lower type and all the other growth indicators mentioned in Definition 4 and Definition 5 depend on the particular choice of the domain D, naturally the growth indicators introduced in Definition 7 and Definition 8, in general, also depend upon the domain D.

In this paper we wish to study some growth properties based upon the notion of (p,q)-φ relative Gol'dberg type, (p,q)-φ relative Gol'dberg weak type and the other growth indicators of entire functions of several complex variables.

2. Lemmas

In this section we state some lemmas which will be frequently used to prove the main results of the paper.

Lemma 1 ([5])**.** *Let $g(z)$ be any entire function of n complex variables with regular (m,p)-φ growth and with non zero finite (m,p)-φ Gol'dberg order. Also let $f(z)$ be another entire function of n complex variables with index-pair (m,q). Then*

$$\rho_g^{(p,q)}(f,\varphi)=\frac{\rho^{(m,q)}(f,\varphi)}{\rho^{(m,p)}(g)}\ .$$

Further when $0<\lambda^{(m,q)}(f,\varphi)<\infty$, then

$$\lambda_g^{(p,q)}(f,\varphi)=\frac{\lambda^{(m,q)}(f,\varphi)}{\lambda^{(m,p)}(g)}\ .$$

The above lemma can easily be derived from the conclusion of Theorem 1 already mentioned in [5]. In the similar fashion, from Theorem 1 in [5] the following lemma may be proved:

Lemma 2 ([5])**.** *let $f(z)$ be any entire function of regular (m,q)-φ Gol'dberg growth with non zero finite (m,q)-φ Gol'dberg order. Also let $g(z)$ be another entire function with $0<\lambda^{(m,p)}(g)<\infty$. Then*

$$\rho_g^{(p,q)}(f,\varphi)=\frac{\lambda^{(m,q)}(f,\varphi)}{\lambda^{(m,p)}(g)}\ .$$

Further if $0<\rho^{(m,p)}(g)<\infty$, then

$$\lambda_g^{(p,q)}(f,\varphi)=\frac{\rho^{(m,q)}(f,\varphi)}{\rho^{(m,p)}(g)}\ .$$

3. Main results

In this section we present the main results of the paper.

Theorem 1. *Suppose that $g(z)$ and $f(z)$ are any two entire functions of n complex variables. Also let $g(z)$ be of regular (m,p)-Gol'dberg growth. Then*

$$\left[\frac{\overline{\sigma}_D^{(m,q)}(f,\varphi)}{\sigma_D^{(m,p)}(g)}\right]^{\frac{1}{\rho^{(m,p)}(g)}}\leq\overline{\sigma}_{g,D}^{(p,q)}(f,\varphi)$$
$$\leq\min\left\{\left[\frac{\overline{\sigma}_D^{(m,q)}(f,\varphi)}{\overline{\sigma}_D^{(m,p)}(g)}\right]^{\frac{1}{\rho^{(m,p)}(g)}},\left[\frac{\sigma_D^{(m,q)}(f,\varphi)}{\sigma_D^{(m,p)}(g)}\right]^{\frac{1}{\rho^{(m,p)}(g)}}\right\}$$

$$\leq \max\left\{\left[\frac{\overline{\sigma}_D^{(m,q)}(f,\varphi)}{\overline{\sigma}_D^{(m,p)}(g)}\right]^{\frac{1}{\rho^{(m,p)}(g)}}, \left[\frac{\sigma_D^{(m,q)}(f,\varphi)}{\sigma_D^{(m,p)}(g)}\right]^{\frac{1}{\rho^{(m,p)}(g)}}\right\}$$

$$\leq \sigma_{g,D}^{(p,q)}(f,\varphi) \leq \left[\frac{\sigma_D^{(m,q)}(f,\varphi)}{\overline{\sigma}_D^{(m,p)}(g)}\right]^{\frac{1}{\rho^{(m,p)}(g)}}.$$

Proof. In view of the first part of Lemma 1, we obtain from the definitions of $\sigma_{g,D}^{(p,q)}(f,\varphi)$ and $\overline{\sigma}_{g,D}^{(p,q)}(f,\varphi)$ that

$$\log \sigma_{g,D}^{(p,q)}(f,\varphi) = \limsup_{R\to+\infty}[\log^{[p]} M_{g,D}^{-1}(R) - \rho_g^{(p,q)}(f,\varphi)\cdot \log^{[q]}\varphi(M_{f,D}^{-1}(R))]$$

$$i.e.,\ \log \sigma_{g,D}^{(p,q)}(f,\varphi) = \limsup_{R\to+\infty}\frac{1}{\rho^{(m,p)}(g)}[\rho^{(m,p)}(g)\cdot\log^{[p]} M_{g,D}^{-1}(R) - \rho^{(m,q)}(f,\varphi)\cdot\log^{[q]}\varphi(M_{f,D}^{-1}(R))] \tag{3.1}$$

and

$$\log \overline{\sigma}_{g,D}^{(p,q)}(f,\varphi) = \liminf_{R\to+\infty}\frac{1}{\rho^{(m,p)}(g)}[\rho^{(m,p)}(g)\cdot\log^{[p]} M_{g,D}^{-1}(R) - \rho^{(m,q)}(f,\varphi)\cdot\log^{[q]}\varphi(M_{f,D}^{-1}(R))]. \tag{3.2}$$

Now from the definitions of $\sigma_D^{(m,q)}(f,\varphi)$ and $\overline{\sigma}_D^{(m,q)}(f,\varphi)$, it follows that

$$\log \sigma_D^{(m,q)}(f,\varphi) = \limsup_{R\to+\infty}[\log^{[m]} R - \rho^{(m,q)}(f,\varphi)\cdot\log^{[q]}\varphi(M_{f,D}^{-1}(R))] \tag{3.3}$$

and

$$\log \overline{\sigma}_D^{(m,q)}(f,\varphi) = \liminf_{R\to+\infty}[\log^{[m]} R - \rho^{(m,q)}(f,\varphi)\cdot\log^{[q]}\varphi(M_{f,D}^{-1}(R))]. \tag{3.4}$$

Similarly, from the definitions of $\sigma_D^{(m,p)}(g)$ and $\overline{\sigma}_D^{(m,p)}(g)$, we get that

$$\log \sigma_D^{(m,p)}(g) = \limsup_{R\to+\infty}[\log^{[m]} R - \rho^{(m,p)}(g)\cdot\log^{[p]} M_{g,D}^{-1}(R)] \text{ and} \tag{3.5}$$

$$\log \overline{\sigma}_D^{(m,p)}(g) = \liminf_{R\to+\infty}[\log^{[m]} R - \rho^{(m,p)}(g)\cdot\log^{[p]} M_{g,D}^{-1}(R)]. \tag{3.6}$$

Therefore, from (3.2), (3.4) and (3.5), it follows that

$$\log \overline{\sigma}_{g,D}^{(p,q)}(f,\varphi) = \liminf_{R\to+\infty}\frac{1}{\rho^{(m,p)}(g)}[\log^{[m]} R - \rho^{(m,q)}(f,\varphi)\cdot\log^{[q]}\varphi(M_{f,D}^{-1}(R)) - (\log^{[m]} R - \rho^{(m,p)}(g)\cdot\log^{[p]} M_{g,D}^{-1}(R))]$$

$$i.e., \log \overline{\sigma}_{g,D}^{(p,q)}(f,\varphi) \geq \frac{1}{\rho^{(m,p)}(g)} [\liminf_{R\to+\infty} (\log^{[m]} R - \rho^{(m,q)}(f,\varphi)$$
$$\cdot \log^{[q]} \varphi(M_{f,D}^{-1}(R))) - \limsup_{R\to+\infty} (\log^{[m]} R - \rho^{(m,p)}(g) \cdot \log^{[p]} M_{g,D}^{-1}(R))]$$
$$i.e., \log \overline{\sigma}_{g,D}^{(p,q)}(f,\varphi) \geq \frac{1}{\rho^{(m,p)}(g)} (\log \overline{\sigma}_{D}^{(m,q)}(f,\varphi) - \log \sigma_{D}^{(m,p)}(g)) \ . \tag{3.7}$$

Similarly, from (3.1), (3.3) and (3.6), we have

$$\log \sigma_{g,D}^{(p,q)}(f,\varphi) = \limsup_{R\to+\infty} \frac{1}{\rho^{(m,p)}(g)} [\log^{[m]} R - \rho^{(m,q)}(f,\varphi)$$
$$\cdot \log^{[q]} \varphi(M_{f,D}^{-1}(R)) - (\log^{[m]} R - \rho^{(m,p)}(g) \cdot \log^{[p]} M_{g,D}^{-1}(R))]$$
$$i.e., \log \sigma_{g,D}^{(p,q)}(f,\varphi) \leq \frac{1}{\rho^{(m,p)}(g)} [\limsup_{R\to+\infty} (\log^{[m]} R - \rho^{(m,q)}(f,\varphi)$$
$$\cdot \log^{[q]} \varphi(M_{f,D}^{-1}(R))) - \liminf_{R\to+\infty} (\log^{[m]} R - \rho^{(m,p)}(g) \cdot \log^{[p]} M_{g,D}^{-1}(R))]$$
$$i.e., \log \sigma_{g,D}^{(p,q)}(f,\varphi) \leq \frac{1}{\rho^{(m,p)}(g)} (\log \sigma_{D}^{(m,q)}(f,\varphi) - \log \overline{\sigma}_{D}^{(m,p)}(g)) \ . \tag{3.8}$$

Again, in view of (3.2), (3.3), (3.4), (3.5) and (3.6), we obtain that

$$\log \overline{\sigma}_{g,D}^{(p,q)}(f,\varphi) = \liminf_{R\to+\infty} \frac{1}{\rho^{(m,p)}(g)} [\log^{[m]} R - \rho^{(m,q)}(f,\varphi) \cdot \log^{[q]} \varphi(M_{f,D}^{-1}(R))$$
$$- (\log^{[m]} R - \rho^{(m,p)}(g) \cdot \log^{[p]} M_{g,D}^{-1}(R))]$$
$$i.e., \log \overline{\sigma}_{g,D}^{(p,q)}(f,\varphi) \leq \frac{1}{\rho^{(m,p)}(g)} \times$$
$$\min[\liminf_{R\to+\infty} (\log^{[m]} R - \rho^{(m,q)}(f,\varphi) \cdot \log^{[q]} \varphi(M_{f,D}^{-1}(R)))$$
$$+ \limsup_{R\to+\infty} (\log^{[m]} R - \rho^{(m,p)}(g) \cdot \log^{[p]} M_{g,D}^{-1}(R)),$$
$$\limsup_{R\to+\infty} (\log^{[m]} R - \rho^{(m,q)}(f,\varphi) \cdot \log^{[q]} \varphi(M_{f,D}^{-1}(R)))$$
$$+ \liminf_{R\to+\infty} (\log^{[m]} R - \rho^{(m,p)}(g) \cdot \log^{[p]} M_{g,D}^{-1}(R))]$$
$$i.e., \log \overline{\sigma}_{g,D}^{(p,q)}(f,\varphi) \leq \frac{1}{\rho^{(m,p)}(g)} \times$$
$$\min[\liminf_{R\to+\infty} (\log^{[m]} R - \rho^{(m,q)}(f,\varphi) \cdot \log^{[q]} \varphi(M_{f,D}^{-1}(R)))$$

$$- \liminf_{R\to+\infty}(\log^{[m]} R - \rho^{(m,p)}(g)\cdot \log^{[p]} M_{g,D}^{-1}(R)),$$
$$\limsup_{R\to+\infty}(\log^{[m]} R - \rho^{(m,q)}(f,\varphi)\cdot \log^{[q]} \varphi(M_{f,D}^{-1}(R)))$$
$$- \limsup_{R\to+\infty}(\log^{[m]} R - \rho^{(m,p)}(g)\cdot \log^{[p]} M_{g,D}^{-1}(R))]$$
$$i.e., \log \overline{\sigma}_{g,D}^{(p,q)}(f,\varphi) \le \frac{1}{\rho^{(m,p)}(g)} \min[\log \overline{\sigma}_{D}^{(m,q)}(f,\varphi) \tag{3.9}$$
$$- \log \overline{\sigma}_{D}^{(m,p)}(g), \log \sigma_{D}^{(m,q)}(f,\varphi) - \log \sigma_{D}^{(m,p)}(g)] \ .$$

Further from (3.1), (3.3), (3.4), (3.5) and (3.6), we obtain that

$$\log \sigma_{g,D}^{(p,q)}(f,\varphi) = \limsup_{R\to+\infty} \frac{1}{\rho^{(m,p)}(g)}[\log^{[m]} R - \rho^{(m,q)}(f,\varphi)\cdot \log^{[q]} \varphi(M_{f,D}^{-1}(R))$$
$$- (\log^{[m]} R - \rho^{(m,p)}(g)\cdot \log^{[p]} M_{g,D}^{-1}(R))]$$
$$i.e., \log \sigma_{g,D}^{(p,q)}(f,\varphi) \ge \frac{1}{\rho^{(m,p)}(g)} \times$$
$$\max[\liminf_{R\to+\infty}(\log^{[m]} R - \rho^{(m,q)}(f,\varphi)\cdot \log^{[q]} \varphi(M_{f,D}^{-1}(R)))$$
$$+ \limsup_{R\to+\infty}(\log^{[m]} R - \rho^{(m,p)}(g)\cdot \log^{[p]} M_{g,D}^{-1}(R)),$$
$$\limsup_{R\to+\infty}(\log^{[m]} R - \rho^{(m,q)}(f,\varphi)\cdot \log^{[q]} \varphi(M_{f,D}^{-1}(R)))$$
$$+ \liminf_{R\to+\infty}(\log^{[m]} R - \rho^{(m,p)}(g)\cdot \log^{[p]} M_{g,D}^{-1}(R))]$$
$$i.e., \log \sigma_{g,D}^{(p,q)}(f,\varphi) \ge \frac{1}{\rho^{(m,p)}(g)} \times$$
$$\max[\liminf_{R\to+\infty}(\log^{[m]} R - \rho^{(m,q)}(f,\varphi)\cdot \log^{[q]} \varphi(M_{f,D}^{-1}(R)))$$
$$- \liminf_{R\to+\infty}(\log^{[m]} R - \rho^{(m,p)}(g)\cdot \log^{[p]} M_{g,D}^{-1}(R)),$$
$$\limsup_{R\to+\infty}(\log^{[m]} R - \rho^{(m,q)}(f,\varphi)\cdot \log^{[q]} \varphi(M_{f,D}^{-1}(R)))$$
$$- \limsup_{R\to+\infty}(\log^{[m]} R - \rho^{(m,p)}(g)\cdot \log^{[p]} M_{g,D}^{-1}(R))]$$
$$i.e., \log \sigma_{g,D}^{(p,q)}(f,\varphi) \ge \frac{1}{\rho^{(m,p)}(g)} \max[\log \overline{\sigma}_{D}^{(m,q)}(f,\varphi) \tag{3.10}$$
$$- \log \overline{\sigma}_{D}^{(m,p)}(g), \log \sigma_{D}^{(m,q)}(f,\varphi) - \log \sigma_{D}^{(m,p)}(g)] \ .$$

Thus the theorem follows from (3.7), (3.8), (3.9) and (3.10) . □

Theorem 2. *Let $g(z)$ and $f(z)$ be any two entire functions of n complex variables such that $0 < \lambda^{(m,p)}(g) < \infty$ and $f(z)$ be of regular (m,q)-φ Gol'dberg* *growth with non zero finite (m,q)-φ Gol'dberg order. Then*

$$\begin{aligned}&\left[\frac{\tau_D^{(m,q)}(f,\varphi)}{\overline{\tau}_D^{(m,p)}(g)}\right]^{\frac{1}{\lambda^{(m,p)}(g)}} \leq \overline{\sigma}_{g,D}^{(p,q)}(f,\varphi)\\ &\leq \min\left\{\left[\frac{\tau_D^{(m,q)}(f,\varphi)}{\tau_D^{(m,p)}(g)}\right]^{\frac{1}{\lambda^{(m,p)}(g)}},\left[\frac{\overline{\tau}_D^{(m,q)}(f,\varphi)}{\overline{\tau}_D^{(m,p)}(g)}\right]^{\frac{1}{\lambda^{(m,p)}(g)}}\right\}\\ &\leq \max\left\{\left[\frac{\tau_D^{(m,q)}(f,\varphi)}{\tau_D^{(m,p)}(g)}\right]^{\frac{1}{\lambda^{(m,p)}(g)}},\left[\frac{\overline{\tau}_D^{(m,q)}(f,\varphi)}{\overline{\tau}_D^{(m,p)}(g)}\right]^{\frac{1}{\lambda^{(m,p)}(g)}}\right\}\\ &\leq \sigma_{g,D}^{(p,q)}(f,\varphi) \leq \left[\frac{\overline{\tau}_D^{(m,q)}(f,\varphi)}{\tau_D^{(m,p)}(g)}\right]^{\frac{1}{\lambda^{(m,p)}(g)}}.\end{aligned}$$

Proof. From the definitions of $\overline{\tau}_D^{(m,q)}(f,\varphi)$ and $\tau_D^{(m,q)}(f,\varphi)$, we obtain that

$$\begin{aligned}\log\overline{\tau}_D^{(m,q)}(f,\varphi) &= \limsup_{R\to+\infty}[\log^{[m]}R-\lambda^{(m,q)}(f,\varphi)\cdot\log^{[q]}\varphi(M_f^{-1}(R))] \text{ and}\\ \log\tau_D^{(m,q)}(f,\varphi) &= \liminf_{R\to+\infty}[\log^{[m]}R-\lambda^{(m,q)}(f,\varphi)\cdot\log^{[q]}\varphi(M_f^{-1}(R))]\ .\end{aligned}$$

Similarly, from the definitions of $\overline{\tau}_D^{(m,p)}(g)$ and $\tau_D^{(m,p)}(g)$, it follows that

$$\begin{aligned}\log\overline{\tau}_D^{(m,p)}(g) &= \limsup_{R\to+\infty}[\log^{[m]}R-\lambda^{(m,p)}(g)\cdot\log^{[p]}M_g^{-1}(R)] \text{ and}\\ \log\tau_D^{(m,p)}(g) &= \liminf_{R\to+\infty}[\log^{[m]}R-\lambda^{(m,p)}(g)\cdot\log^{[p]}M_g^{-1}(R)]\ .\end{aligned}$$

Now, using the same method of Theorem 1, one may easily prove the conclusion of the present theorem with the help of Lemma 2, (3.1), (3.2) and the above inequalities. □

Similarly in the line of Theorem 1 and Theorem 2 and using the second part of Lemma 1 and Lemma 2, the following two theorems may easily be proved and so their proofs are omitted:

Theorem 3. *Suppose that $g(z)$ and $f(z)$ are any two entire functions of n complex variables such that $g(z)$ be of regular (m,p)-Gol'dberg growth with non zero finite (m,p)-th Gol'dberg order and $0<\lambda^{(m,q)}(f,\varphi)<\infty$. Then*

$$\left[\frac{\tau_D^{(m,q)}(f,\varphi)}{\overline{\tau}_D^{(m,p)}(g)}\right]^{\frac{1}{\lambda^{(m,p)}(g)}} \leq \tau_{g,D}^{(p,q)}(f,\varphi)$$

$$\leq \min\left\{\left[\frac{\tau_D^{(m,q)}(f,\varphi)}{\tau_D^{(m,p)}(g)}\right]^{\frac{1}{\lambda^{(m,p)}(g)}}, \left[\frac{\overline{\tau}_D^{(m,q)}(f,\varphi)}{\overline{\tau}_D^{(m,p)}(g)}\right]^{\frac{1}{\lambda^{(m,p)}(g)}}\right\}$$

$$\leq \max\left\{\left[\frac{\tau_D^{(m,q)}(f,\varphi)}{\tau_D^{(m,p)}(g)}\right]^{\frac{1}{\lambda^{(m,p)}(g)}}, \left[\frac{\overline{\tau}_D^{(m,q)}(f,\varphi)}{\overline{\tau}_D^{(m,p)}(g)}\right]^{\frac{1}{\lambda^{(m,p)}(g)}}\right\}$$

$$\leq \overline{\tau}_{g,D}^{(p,q)}(f,\varphi) \leq \left[\frac{\overline{\tau}_D^{(m,q)}(f,\varphi)}{\tau_D^{(m,p)}(g)}\right]^{\frac{1}{\lambda^{(m,p)}(g)}}.$$

Theorem 4. *Let $g(z)$ and $f(z)$ be any two entire functions of n complex variables such that $0 < \rho^{(m,p)}(g) < \infty$ and $f(z)$ be of regular (m,q)-φ Gol'dberg growth with non zero finite (m,q)-φ Gol'dberg order. Then*

$$\left[\frac{\overline{\sigma}_D^{(m,q)}(f,\varphi)}{\sigma_D^{(m,p)}(g)}\right]^{\frac{1}{\rho^{(m,p)}(g)}} \leq \tau_{g,D}^{(p,q)}(f,\varphi)$$

$$\leq \min\left\{\left[\frac{\overline{\sigma}_D^{(m,q)}(f,\varphi)}{\overline{\sigma}_D^{(m,p)}(g)}\right]^{\frac{1}{\rho^{(m,p)}(g)}}, \left[\frac{\sigma_D^{(m,q)}(f,\varphi)}{\sigma_D^{(m,p)}(g)}\right]^{\frac{1}{\rho^{(m,p)}(g)}}\right\}$$

$$\leq \max\left\{\left[\frac{\overline{\sigma}_D^{(m,q)}(f,\varphi)}{\overline{\sigma}_D^{(m,p)}(g)}\right]^{\frac{1}{\rho^{(m,p)}(g)}}, \left[\frac{\sigma_D^{(m,q)}(f,\varphi)}{\sigma_D^{(m,p)}(g)}\right]^{\frac{1}{\rho^{(m,p)}(g)}}\right\}$$

$$\leq \overline{\tau}_{g,D}^{(p,q)}(f,\varphi) \leq \left[\frac{\sigma_D^{(m,q)}(f,\varphi)}{\overline{\sigma}_D^{(m,p)}(g)}\right]^{\frac{1}{\rho^{(m,p)}(g)}}.$$

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