Minimal left ideals in some endomorphism semirings of semilattices

Barbora Batíková
Department of Mathematics
CULS
Kamýcká 129, 165 21 Praha 6-Suchdol
Czech Republic
batikova@tf.czu.cz

Tomáš Kepka
Department of Algebra
MFF UK
Sokolovská 83, 186 75 Praha 8
Czech Republic
kepka@karlin.mff.cuni.cz

Petr Němec
Department of Mathematics
CULS
Kamýcká 129, 165 21 Praha 6-Suchdol
Czech Republic
nemec@tf.czu.cz

Abstract. Minimal left ideals in some endomorphism semirings of semilattices are investigated.

Keywords: semiring, semimodule, semilattice, ideal, endomorphism.

1. Preliminaries

1.1 Semirings. A semiring is an algebraic structure possessing two associative binary operations (addition and multiplication), where the addition is commutative and the multiplication distributes over the addition.

Let $S$ be a semiring. A non-empty subset $I$ of $S$ is called

- a left (right) ideal if $(I + I) \cup SI \subseteq I$ and $(I + I) \cup IS \subseteq I$;
- an ideal if $(I + I) \cup SI \cup IS \subseteq I$;
- a bi-ideal if $(S + I) \cup SI \cup IS \subseteq I$.

A (left, right, bi-) ideal $I$ is called minimal if $|I| \geq 2$ and $K = I$ whenever $K$ is a (left, right, bi-) ideal such that $K \subseteq I$ and $|K| \geq 2$. The semiring $S$ is called
(left-, right-, bi-) ideal-simple if it is non-trivial and has no proper non-trivial (left, right, bi-) ideal.

1.2 Semilattices. A semilattice $M (= M(\langle \rangle))$ is a commutative idempotent semigroup. A non-empty subset $A$ of $M$ is an ideal if $M + A = A$. Such an ideal is called prime if the set $M \setminus A$ is a subsemilattice of $M$. We have the basic order on $M$ defined by $a \leq b$ iff $a + b = b$.

An element $w \in M$ is neutral (absorbing) if $x + w = x$ for every $x \in M$. Such an element is unique (provided that it exists - this fact is denoted by $0_M \in M$) and is denoted by $0 = 0_M$ in the sequel. The fact that no such element exists is denoted by $0_M \notin M$.

1.3 Semimodules. Let $S$ be a semiring. A (left $S$-) semimodule is a commutative semigroup $M (= M(\langle \rangle))$ together with a scalar multiplication $S \times M \to M$. The semimodule $M$ is called

- minimal if $|M| \geq 2$ and $N = M$ whenever $N$ is a non-trivial subsemimodule of $M$;
- simple if it has just two congruence relations;
- faithful if for all $a, b \in S$, $a \neq b$, there is at least one $x \in M$ with $ax \neq bx$.

1.4 Endomorphism semirings. Let $M (= M(\langle \rangle))$ be a commutative semigroup. The set $E = E(M) = \text{End}(M)$ of endomorphisms of $M$ is a semiring via $(f + g)(x) = f(x) + g(x)$ and $(fg)(x) = f(g(x))$ for all $f, g \in E$ and $x \in M$. The semiring $E$ is unitary and the multiplicatively neutral element is the identity automorphism $\text{id}_M$ of $M$.

2. 0-preserving endomorphisms

Let $M$ be a non-trivial semilattice such that $0_M \in M$. The set $E_0 = \{ f \in E \mid f(0_M) = 0_M \}$ is a unitary subsemiring of $E$ and the constant endomorphism $\xi$, $\xi(M) = \{0_M\}$, is the zero element of the semiring $E_0$. For all $u, v \in M$, we have $q_{u,v} \in E_0$, where $q_{u,v}(x) = 0_M$ for $x \leq u$ and $q_{u,v}(y) = v$ for $y \notin u$. Thus $q_{u,v}(M) = \{0_M, v\}$ for $u \neq o_M$, $q_{o_M,v} = \xi$ (if $o_M \in M$) and $q_{u,0_M} = \xi$.

Now, let $E$ be a subsemiring of $E_0$ such that $q_{u,v} \in E$ for all $u, v \in M$. For every $u \in N = M \setminus \{o_M\}$, put $T_u = \{ q_{u,v} \mid v \in M \}$. The following observations are quite easy.

Lemma 2.1. (i) $q_{u,v_1} + q_{u,v_2} = q_{u,v_1 + v_2}$.
(ii) $f q_{u,v} = q_{u,f(v)}$ for every $f \in E_0$.
(iii) $q_{u,v_1} q_{pu, v_2} = \xi$ for $v_2 \leq u$.
$q_{u,v_1} q_{u,v_2} = q_{u,v_1}$ for $v_2 \notin u$.
(v) $q_{u,v} q_{u,v} = \xi$ for $v \leq u$.
(vi) $q_{u,v} q_{u,v} = q_{u,v}$ for $v \notin u$. □
Proposition 2.2. The set $T_u$ is a minimal left ideal of the semiring $E$ for every $u \in N$. The mapping $v \mapsto q_{u,v}$ is an isomorphism of the semimodule $EM$ onto the semimodule $ET_u$.

Proof. It follows immediately from 2.1 that $T_u$ is a left ideal of $E$ and that the map $v \mapsto q_{u,v}$ is an isomorphism of the semimodules. Furthermore, if $v \neq 0_M$ then $q_{0_M,v_1}q_{u,v} = q_{u,v_1}$ for every $v_1 \in M$. Now, it is clear that $T_u$ is a minimal left ideal of $E$.

Proposition 2.3. (i) $T_uq_{u,v} = \{\xi\}$ for $v \leq u$.
(ii) $T_uq_{u,v} = T_u$ for $v \not\leq u$.

Proof. Use 2.1(iii),(iv).

Proposition 2.4. (i) The set $\{q_{u,v} \mid v \leq u\}$ is an ideal of the semiring $T_u$ and it is the greatest proper ideal of that semiring.
(ii) The $\{q_{u,v} \mid v \not\leq u\} \cup \{\xi\}$ is just the set of the multiplicative idempotents of $T_u$ ($\xi = q_{u,0_M}$).
(iii) The semiring $T_u$ is multiplicatively idempotent iff $u = 0_M$.
(iv) The semiring $T_u$ is (left-) ideal-simple iff $u = 0_M$.

Proof. It is easy.

Proposition 2.5. The semimodule $ET_u$ is simple.

Proof. Let $\alpha \neq \text{id}$ be a congruence of $T_u$. Then $(q_{u,v_1}, q_{u,v_2}) \in \alpha$ for some $v_1 < v_2$, and hence $(\xi, q_{u,v_2}) = (q_{v_1,v_2}q_{u,v_1}^t, q_{v_1,v_2}^tq_{u,v_2}) \in \alpha$. Consequently, $(\xi, q_{u,v_2}) = (q_{0_M,v_3}^t, q_{0_M,v_3}^tq_{u,v_2}) \in \alpha$ for every $v_3 \in M$. Thus $\alpha = T_u \times T_u$. 

Proposition 2.6. The semimodule $T_u$ is faithful.

Proof. Use 2.1(ii).

Lemma 2.7. If $I$ is a non-trivial ideal of $E$ then $\bigcup_{u \in N} T_u \subseteq I$.

Proof. We have $IT_u \subseteq I \cap T_u$ and the latter set is a left ideal of $E$. If $T_u \not\subseteq I$ then $IT_u = I \cap T_u = \{\xi\}$ and it means that $q_{u,f(v)} = fq_{u,v} = \xi$ for every $f \in I$ and $v \in M$. Since $u \neq 0_M$, we get $f(v) = 0_M$ and $f = \xi$. Thus $I = \{\xi\}$, a contradiction.

Corollary 2.8. The set $\{q_{u_1,v_1}f_1 + \cdots + q_{u_n,v_n}f_n \mid n \geq 1, u_i \in N, v_i \in M, f_i \in E\}$ is just the smallest non-trivial ideal of the semiring $E$.

2.9 Let $K$ be a minimal left ideal of the semiring $E$ such that $K \neq T_u$ for every $u \in N$.

Lemma 2.9.1. $K \cap T_u = \{\xi\}$ for every $u \in N$.

Proof. It is obvious.
Lemma 2.9.2. \( Ef = K \) for every \( f \in K \setminus \{ \xi \} \).

**Proof.** The set \( K_1 = \{ f \in K \mid Ef = \{ \xi \} \} \) is a left ideal of \( E \). If \( K_1 = K \) then \( EK = \{ \xi \} \) and \( K = \{ \xi, f_0 \} \) is two-element. Now, \( q_{0_M, f_0} = \xi \), and hence \( f_0 = \xi \), a contradiction. It follows that \( K_1 = \{ \xi \} \) and \( Ef = K \), since \( Ef \) is a left ideal of \( E \).

Lemma 2.9.3. There are subsemilattices \( A \) and \( B \) of \( M \) such that:

(i) \( A \cup B = M \) and \( A \cap B = \emptyset \).

(ii) \( A = \{x \in M \mid f(x) \neq 0_M\} \) and \( B = \{x \in M \mid f(x) = 0_M\} \) for every \( f \in K \setminus \{\xi\} \).

(iii) \( A + M = A \) (i.e., \( A \) is a prime ideal of the semilattice \( M \)).

(iv) \( o_B \notin B \) (i.e., the prime ideal \( A \) is not principal).

**Proof.** If \( f, g \in K \setminus \{\xi\} \) then \( g = hf \) for some \( h \in E \) by 2.9.2 and we see that \( g(x) \neq 0_M \) implies \( f(x) \neq 0_M \). Clearly, \( A + M = A \) and \( B + B = B \). We have \( 0_M \in B \) and \( A \neq \emptyset \), since \( f \neq \xi \). If \( u = o_B \in B \) then \( u \neq o_M \) and \( q_{0_M, f} = q_{u, v} \in K \) for every \( v \in M \), a contradiction.

Lemma 2.9.4. If \( f \in K \setminus \{\xi\} \) and \( v \in M \) then \( q_{0_M, f} = q_{A, v} \in K \), where \( q_{A, v}(B) = \{0_M\} \) and \( q_{A, v}(A) = \{v\} \).

**Proof.** It is obvious.

Proposition 2.9.5. (i) \( K = \{q_{A, v} \mid v \in M\} \).

(ii) The mapping \( v \mapsto q_{A, v} \) is an isomorphism of the semimodule \( EM \) onto the semimodule \( EK \).

(iii) \( Kq_{A, v} = \{\xi\} \) for every \( v \in B \).

(iv) \( Kq_{A, v} = K \) for every \( v \in A \).

(v) The set \( \{q_{A, v} \mid v \in B\} \) is an ideal of the semiring \( K \) and it is the greatest proper left ideal of that semiring.

(vi) The set \( \{q_{A, v} \mid v \in A\} \cup \{\xi\} \) is just the set of the multiplicative idempotents of \( K \) (\( \xi = q_{A, 0_M}\)).

(vii) The semiring \( K \) is neither multiplicatively idempotent nor ideal-simple.

**Proof.** According to 2.9.4, \( T_A = \{q_{A, v} \mid v \in M\} \subseteq K \). The rest is clear (see 2.2, 2.4 and 2.4).

Theorem 2.10. (i) The set \( T_u = \{q_{u, v} \mid v \in M\} \) is a minimal left ideal of the semiring \( E \) for every \( u \in N \).

(ii) If \( K \) is a minimal left ideal of \( E \) such that \( K \neq T_u \) for every \( u \in N \) then there is a non-principal prime ideal \( A \) of the semilattice \( M \) such that \( K = T_A = \{q_{A, v} \mid v \in M\} \).

(iii) If \( A \) is a non-principal prime ideal of \( M \) such that \( Q_{A, v} \in E \) for at least one \( v \in M \setminus \{0_M\} \) then \( T_A \subseteq E \) and \( T_A \) is a minimal left ideal of \( E \) such that \( T_A \neq T_u \) for every \( u \in N \).
(iv) If \( K \) is a minimal left ideal of \( E \) then \( _EK \) is a faithful, simple and minimal \((\text{left } E)-\) semimodule and the semimodules \( _EK \) and \( _EM \) are isomorphic.

(v) If \( K \) is a minimal left ideal of \( E \) then the semiring \( K \) is multiplicatively idempotent if and only if \( K = T_{0M} \).

**Proof.** Combine the foregoing results. \( \square \)

**Corollary 2.11.** Denote by \( A \) the set of prime ideals \( A \) of \( M \) such that \( T_A \subseteq E \) (see 2.10(ii),(iii)). Then:

(i) \( \{ T_A \mid a \in A \} \) is just the set of \((\text{pairwise distinct})\) minimal left ideals of the semiring \( E \).

(ii) \( A_u = \{ x \in M \mid x \not\preceq u \} \in A \) for every \( u \in N \).

(iii) The set \( \sum T_A, A \in A, \) is just the smallest non-trivial ideal of the semiring \( E \). \( \square \)

3. 1-preserving endomorphisms (a)

Let \( M \) be a non-trivial semilattice such that \( o_M \in M \). The set \( E_1 = E_1(M) = \{ f \in E \mid f(o_M) = o_M \} \) is a unitary subsemiring of \( E \) and the constant endomorphism \( \zeta = \zeta_M, \zeta(M) = \{ o_M \}, \) is the bi-absorbing element of the semiring \( E_1 \). For all \( u, v \in M, u \neq o_M, \) we have \( p_{u,v} \in E_1, \) where \( p_{u,v}(x) = v \) for \( x \leq u \) and \( p_{u,v}(y) = o_M \) for \( y \not\preceq u \). Thus \( p_{u,v}(M) = \{ v, o_M \} \) and \( p_{u,o_M} = \zeta \).

Now, let \( F \) be a subsemiring of \( E_1 \) such that \( p_{u,v} \in F \) for all \( u, v \in M, u \neq o_M \). For every \( u \in N = M \setminus \{ o_M \}, \) put \( V_u = \{ p_{u,v} \mid v \in M \} \).

The following observations are quite easy.

**Lemma 3.1.** (i) \( p_{u,v_1} + p_{u,v_2} = p_{u,v_1+v_2} \).

(ii) \( fp_{u,v} = p_{u,f(v)} \) for every \( f \in E_1 \).

(iii) \( p_{u,v_1}p_{u,v_2} = \zeta \) for \( v_2 \not\preceq u \).

(iv) \( p_{u,v_1}p_{u,v_2} = p_{u,v_1} \) for \( v_2 \leq u \).

(v) \( p_{u,v}p_{u,v} = \zeta \) for \( v \not\preceq u \).

(vi) \( p_{u,v}p_{u,v} = p_{u,v} \) for \( v \leq u \). \( \square \)

**Proposition 3.2.** The set \( V_u \) is a minimal left ideal of the semiring \( F \) for every \( u \in N \). The mapping \( v \mapsto p_{u,v} \) is an isomorphism of the semimodule \( _FM \) onto the semimodule \( _FV_u \).

**Proof.** It follows immediately from 3.1 that \( V_u \) is a left ideal of \( F \) and the map \( v \mapsto p_{u,v} \) is an isomorphism of the semimodules. Furthermore, if \( v \neq o_M \) then \( p_{v,v_1}p_{u,v} = p_{u,v_1} \) for every \( v_1 \in M \). Now, it is clear that \( V_u \) is a minimal left ideal of \( F \). \( \square \)

**Proposition 3.3.** (i) \( V_u p_{u,v} = \{ \zeta \} \) for \( v \not\preceq u \).

(ii) \( V_u p_{u,v} = V_u \) for \( v \leq u \).

**Proof.** Use 3.1 (iii), (iv). \( \square \)
Proposition 3.4. (i) The set \( \{ p_{u,v} \mid v \not\leq u \} \) is a bi-ideal of the semiring \( V_u \) and it is the greatest proper left ideal of that semiring.

(ii) The set \( \{ p_{u,v} \mid v \leq u \} \cup \{ \zeta \} \) is just the set of multiplive idempotents of \( V_u (\zeta = p_{u,o_M}) \).

(iii) The semiring \( V_u \) is multiplicatively idempotent iff \( u = o_N \in N \).

(iv) The semiring \( V_u \) is (left-) ideal-simple iff \( u = o_N \in N \).

Proof. It is easy. \( \square \)

Proposition 3.5. The semimodule \( fV_u \) is simple.

Proof. Let \( \alpha \neq \text{id} \) be a congruence of \( fV_u \). Then \( (p_{u,v_1}, p_{u,v_2}) \in \alpha \) for some \( v_1 < v_2 \) and hence \( (p_{u,v_2}, \zeta) = (p_{v_1,v_2}p_{u,v_1}, p_{v_1,v_2}p_{u,v_2}) \in \alpha \). Consequently, \( (p_{u,v_1}, \zeta) \in \alpha \) and, finally, \( (p_{u,v_3}, \zeta) = (p_{v_1,v_3}p_{u,v_1}, p_{v_1,v_3}\zeta) \in \alpha \) for every \( v_3 \in M \). Thus \( \alpha = V_u \times V_u \).

Proof. Use 3.1(ii). \( \square \)

Lemma 3.7. If \( I \) is a non-trivial ideal of \( F \) then \( \bigcup_{u \in N} V_u \subseteq I \).

Proof. We can proceed similarly as in the proof of 2.7. \( \square \)

Corollary 3.8. The set \( \{ p_{u_1,v_1}f_1 + \ldots + p_{u_n,v_n}f_n \mid n \geq 1, u_i \in N, v_i \in M, f_i \in F \} \) is just the smallest non-trivial ideal of the semiring \( F \). \( \square \)

3.9 Let \( K \) be a minimal left ideal of the semiring \( F \) such that \( K \neq V_u \) for every \( u \in N \).

Lemma 3.9.1. \( K \cap V_u = \{ \zeta \} \) for every \( u \in N \).

Proof. It is obvious. \( \square \)

Lemma 3.9.2. \( Ff = K \) for every \( f \in K \setminus \{ \zeta \} \).

Proof. If \( f \in F \setminus \{ \zeta \} \) then \( f(v) \neq o_M \) for at least one \( v \in N \) and \( p_{f(v),v} \neq \zeta \). Now, we can proceed similarly as in the proof of 2.9.2. \( \square \)

Lemma 3.9.3. There is a proper ideal \( A \) of the semilattice \( M \) such that \( A = \{ x \in M \mid f(x) = o_M \} \) for every \( f \in K \setminus \{ \zeta \} \) and we put \( B = M \setminus A \).

Proof. Use 3.9.2. \( \square \)

Lemma 3.9.4. Let \( f \in K \setminus \zeta \), \( v \in M \), and let \( o_m \neq w \in f(M) \). Then \( p_{w,v}f = p_{A,v}K \), where \( p_{A,v}(B) = \{ v \} \) and \( p_{A,v}(A) = \{ o_M \} \).

Proof. We have \( g = p_{w,v}f \in K \) and, by 3.9.3, \( A = \{ x \in M \mid g(x) = o_M \} = \{ x \in M \mid f(x) \not\leq w \} \). Thus \( B = \{ y \in M \mid g(y) \neq o_M \} = \{ y \in M \mid f(y) \leq w \} \) and \( g(B) = \{ v \} \). \( \square \)
Proposition 3.9.5. (i) $A$ is a prime ideal of $M$, $o_B \notin B$ and $K = \{ p_{A,v} \mid v \in M \} = V_A$.

(ii) The mapping $v \mapsto p_{A,v}$ is an isomorphism of the semimodule $FM$ onto the semimodule $FK$.

(iii) $Kp_{A,v} = \{ \zeta \}$ for $v \in A$.

(iv) $Kp_{A,v} = K$ for $v \in B$.

(v) The set $\{ p_{A,v} \mid v \in A \}$ is a bi-ideal of the semiring $K$ and it is the greatest proper left ideal of that semiring.

(vi) The set $\{ p_{A,v} \mid v \in B \} \cup \{ \zeta \}$ is just the set of multiplicative idempotents of $K$ ($\zeta = p_{A,o_M}$).

(vii) The semiring $K$ is neither multiplicatively idempotent nor bi-ideal-simple.

Proof. According to 3.9.4, $V_A = \{ p_{A,v} \mid v \in M \} \subseteq K$. The rest is clear (see 3.2, 3.3 and 3.4).

Theorem 3.10. (i) The set $V_u = \{ p_{u,v} \mid v \in M \}$ is a minimal left ideal of the semiring $F$ for every $u \in N$.

(ii) If $K$ is a minimal left ideal of $F$ such that $K \neq V_u$ for every $u \in N$ then there is a non-principal prime ideal $A$ of the semilattice $M$ such that $K = V_A = \{ p_{A,v} \mid v \in M \}$.

(iii) If $A$ is a non-principal prime ideal of $M$ such that $p_{A,v} \in F$ for at least one $v \in M \setminus \{ o_M \}$ then $V_A \subseteq F$ and $V_A$ is a minimal left ideal of $F$ such that $V_A \neq V_u$ for every $u \in N$.

(iv) If $K$ is a minimal left ideal of $F$ then $FK$ is a faithful, simple and minimal $F$-semimodule and the semimodules $FK$ and $FM$ are isomorphic.

(v) If $K$ is a minimal left ideal of $F$ then the semiring $K$ is multiplicatively idempotent if and only if $o_N \in N$ and $K = V_{o_N}$.

Proof. See the foregoing results.

Corollary 3.11. Denote by $B$ the set of prime ideals $A$ of $M$ such that $V_A \subseteq F$ (see 3.10(ii),(iii)). Then:

(i) $\{ V_A \mid A \in B \}$ is just the set of (pair-wise distinct) minimal left ideals of the semiring $F$.

(ii) $A_u = \{ x \in M \mid x \notin u \} \in B$ for every $u \in N$.

(iii) The set $\bigcup V_A$, $A \in B$, is just the smallest non-trivial ideal of the semiring $F$.

Proof. See the foregoing results.

4. 1-preserving endomorphisms (b)

Let $M$ be an antichain, i.e., a semilattice containing at least three elements such that $o_M \in M$ and $x + y = o_M$ for all $x, y \in M$, $x \neq y$, and put $N = M \setminus \{ o_M \}$. Now, a transformation $f$ of $M$ belongs to $E_1$ if and only if $f(o_M) = o_M$ and $f(x) \neq f(y)$ for all $x, y \in M$ such that $x \neq y$ and $f(x) \neq o_M$. 
Proposition 4.1. Let $E$ be a subsemiring of of $E_1$ such that $\zeta \in E$. The following conditions are equivalent:

(i) The semimodule $EM$ is minimal.

(ii) $N \subseteq E(x) = \{ f(x) \mid f \in E \}$ for every $x \in N$.

(iii) $E(x) = M$ for every $x \in N$.

(iv) The semimodule $EM$ is simple.

Proof. Clearly, the first three conditions are equivalent. Now, let $\alpha$ be a congruence of $EM$ and $P = \{ x \in M \mid (x, o_M) \in \alpha \}$. Then $P$ is a subsemimodule of $M$ and if $M$ is minimal then $P = \{ o_M \}$. On the other hand, if $(x, y) \in \alpha$, $x \neq y$, then $(x, o_M) = (x + x, x + y) \in \alpha$ and $x \in P$. Thus (i) implies (iv). Conversely, for every $x \in M$, the relation $\alpha_x = (E(x) \times E(x)) \cup \text{id}_M$ is a congruence of $EM$. If $\alpha = M \times M$ then $E(x) = M$. If $\alpha_x = \text{id}_M$ then $E(x) = \{ o_M \}$ and $x \in Q = \{ y \in M \mid F(y) = \{ o_M \} \}$. Of course, $Q$ is a subsemimodule of $EM$ and $\beta = (Q \times Q) \cup \text{id}_M$ is a congruence of $EM$. If $\beta = \text{id}_M$ then $x = o_M$. If $\beta = M \times M$ then $Q = M$, $E(M) = \{ o_M \}$ and $EM$ is not simple. Thus (iv) implies (iii). \qed

Lemma 4.2. Let $f, g \in E_1$. Then $f + g = \zeta$ iff $f(x) = o_M$ whenever $x \in M$ is such that $f(x) = g(x)$.

Proof. It is obvious. \qed

In the remaining part of this section, let $E$ be a non-trivial subsemiring of $E_1$ satisfying the equivalent conditions of 4.1 and let $K$ be a left ideal of $E$.

Lemma 4.3. Let $w \in M$. The mapping $\tau_w : f \mapsto f(w)$ is a homomorphism of the semimodule $EK$ into the semimodule $EM$. If $K(w) \neq \{ o_M \}$ then $\tau(K) = M$.

Proof. It is easy. \qed

Proposition 4.4. If the semimodule $EK$ is simple then $K$ is a minimal left ideal of $E$ and the semimodules $EK$ and $EM$ are isomorphic.

Proof. Since $EK$ is simple, the left ideal $K$ is non-trivial and, taking $f \in K \setminus \{ o_M \}$, we find $w \in M$ with $f(w) \neq o_M$. Then $K(w) = M$ and $\tau_w$ is an isomorphism of the semimodules (use 4.3). \qed

Lemma 4.5. For every $w \in M$, the set $L_w = \{ f \in E \mid f(w) = o_M \}$ is a left ideal of the semiring $E$.

Proof. It is obvious. \qed

Lemma 4.6. Let $w \in M$ be such that $K \cap L_w = \{ \zeta \}$ (e.g., if $K$ is minimal and $K \not\subseteq L_w$). If $f, g \in K$ are such that $f(w) \neq g(w)$ then $f + g = \zeta$. 

Proof. It is obvious. \qed
Proof. We have \( f + g \in K \cap L_w \).

Lemma 4.7. If \( K \) is minimal then \( f + g = \zeta \) for all \( f, g \in K \), \( f \neq g \).

Proof. Since \( f \neq g \), there is \( w \in M \) with \( f(w) \neq g(w) \). Clearly, \( K \nsubseteq L_w \) and 4.6 applies.

Theorem 4.8. If \( K \) is a minimal left ideal of the semiring \( E \) then the semimodules \( EK \) and \( EM \) are isomorphic.

Proof. Since \( K \) is non-trivial, there is \( w \in M \) with \( K(w) \neq M \) (equivalently, \( K \nsubseteq L_w \)). If \( f, g \in K \) are such that \( f(w) = g(w) \) then \( f(w) = (f + g)(w) = g(w) \) and either \( f(w) = o_M = g(w) \) and \( f = \zeta = g \), or \( (f + g)(w) \neq o_M \), \( f + g \neq \zeta \) and \( f = g \) by 4.7.

Remark 4.9. Assume that \( K \) is a minimal left ideal and let \( w \in M \) be such that \( K(w) = M \) (equivalently, \( K \nsubseteq L_w \)). If \( f \in K \) then \( Kf = \{\zeta\} \) iff \( K \subseteq L_f(w) \). On the other hand, if \( Kf \neq \{\zeta\} \) then \( Kf = K, K(f(w)) = M \) and \( K \cap L_f(w) = \{\zeta\} \).

Lemma 4.10. If \( f \in E_1 \) is such that \( f^2 = f \) then \( f(x) \in \{x, o_M\} \) for every \( x \in M \).

Proof. It is obvious.

Lemma 4.11. Let \( K \) be a minimal left ideal, \( A = \{w \in N \mid K \subseteq L_w\} \) and let \( f \in K \) be such that \( f^2 = f \neq \zeta \). Then:

(i) \( f(x) = x \) for every \( x \in M \setminus A \).

(ii) \( f \) is right multiplicatively neutral in \( K \).

(iii) If \( A = \emptyset \) then \( f = \text{id}_M \) and \( K = E \).

(iv) If \( A \neq \emptyset \) then \( K = \bigcap L_w, w \in A \).

Proof. (i) This follows from 4.10.

(ii) If \( g \in K \) then \( gf(x) = g(x) \) for every \( x \in M \setminus A \) and \( gf(y) = o_M = g(y) \) for every \( y \in A \). Thus \( gf = g \).

(iii) By (i), \( f = \text{id}_M \).

(iv) If \( g \in \bigcap L_w \) then \( gf = g \).

Corollary 4.12. Let \( K \) be a minimal left ideal of \( E \). Then:

(i) \( K \) contains at most two multiplicatively idempotent elements.

(ii) \( K \) is not multiplicatively idempotent.
5. 1-preserving endomorphisms (c)

Let $K$ and $L$ be semilattices containing at least three elements and such that $o_K \in K$ and $o_L \in L$. The set $Q = \{(o_K) \times L\} \cup (K \times \{o_L\})$ is an ideal of the cartesian product $K \times L$ and we put $M = (K \times L) / Q \cup \{(o_M) \cup (K' \times L')\}$.

It is obvious.

Proof. According to 5.1 and 5.2, the set $K \times L$ is a subsemiring of $K \times L$ and we put $M = (K \times L) / Q \cup \{(o_M) \cup (K' \times L')\}$.

If $f \in E_1(K)$ and $g \in E_1(L)$ then $(f \times g)(Q) \subseteq Q$ and we put $f \ast g = (f \times g) / Q \in E_1(M)$. Thus $(f_1 \ast g_1) + (f_2 \ast g_2) = (f_1 + f_2) \ast (g_1 + g_2)$, $(f_1 \ast g_1)(f_2 \ast g_2) = f_1 f_2 \ast g_1 g_2$, and hence $E_1(K) \ast E_1(L)$ is a subsemiring of $E_1(M)$. In fact, if $E$ is a subsemiring of $E_1(K)$ and $F$ is a subsemiring of $E_1(L)$ then $E \ast F$ is a subsemiring of $E_1(M)$.

In the remaining part of this section, assume that $L$ is an antichain and take a non-trivial subgroup $G$ of $\text{Aut}(L)$ such that $f(x) \neq g(x)$ for all $x \in L'$, $f, g \in G$, $f \neq g$. Let $S$ be a subsemiring of $E_1(K) \ast (G \cup \{\zeta_L\})$ such that $p_{u,v} \ast g \in S$ for all $u \in K'$, $v \in K$ and $g \in G$ (see the preceding two sections).

One checks easily that $o_S = \zeta_M = p_{u,o_K} \ast g \in S$ is the bi-absorbing element of $S$ and $AA \subseteq A$, where $A = \{f \in E_1(K) \mid f \ast g \in S, g \in G\}$. Furthermore, $S \subseteq E_1(K) \ast G$ and $B = \{f \in E_1(K) \mid f \ast \text{id}_M \in S\}$ is a subsemiring of $E_1(K)$ such that $p_{u,v} \in B$ for all $u \in K'$ and $v \in K$.

Lemma 5.1. Let $u \in K'$, $v_1, v_2 \in K$ and $g_1, g_2 \in G$. Then:

(i) If $g_1 = g_2$ then $(p_{u,v_1} \ast g_1) + (p_{u,v_2} \ast g_2) = (p_{u,v_1 + v_2} \ast g_1)$.

(ii) If $g_1 \neq g_2$ then $(p_{u,v_1} \ast g_1) + (p_{u,v_2} \ast g_2) = \zeta_M$.

Proof. We have $(p_{u,v_1} \ast g_1) + (p_{u,v_2} \ast g_2) = (p_{u,v_1 + v_2} \ast (g_1 + g_2))$.

Lemma 5.2. Let $u \in K'$, $v \in K$, $g \in G$ and let $f \in E_1(K)$ and $h \in G$ be such that $f \ast h \in S$. Then $(f \ast h)(p_{u,v} \ast g) = p_{u,f(v)} \ast (hg)$.

Proof. It is obvious.

Proposition 5.3. For every $u \in K'$, the set $W_u = \{p_{u,v} \ast g \mid v \in K, g \in G\}$ is a minimal left ideal of the semiring $S$.

Proof. According to 5.1 and 5.2, the set $W_u$ is a left ideal of $S$. If $v \in K'$, $w \in K$ and $g, h \in S$ then $p_{u,w} \ast hg^{-1}(p_{u,v} \ast g) = p_{u,w} \ast h$. Thus $S(p_{u,v} \ast g) = W_u$.

Lemma 5.4. Let $f_1, f_2 \in E_1(K)$ and $g_1, g_2 \in G$ be such that $f_1 \ast g_1 = f_2 \ast g_2$. Then $f_1 = f_2$ and if $f_1 \neq \zeta_K$ then $g_1 = g_2$.

Proof. Let $x_0 \in K$ be such that $f_1(x_0) \neq f_2(x_0)$. Then $x_0 \in K'$, $(f_1(x_0), g_1(y)) \in Q$ and $(f_2(x_0), g_2(y)) \in Q$ for every $y \in L'$. If $f_1(x_0) \neq o_K$ then $g_1(y) = o_L$, a contradiction. Thus $f_1(x_0) = o_K$ and, similarly, $f_2(x_0) = o_K$. It follows that $f_1 = f_2$. The rest is clear.

Lemma 5.5. Let $u \in K'$, $v_1, v_2 \in K$ and $g_1, g_2 \in G$. Then $p_{u,v_1} \ast g_1 = p_{u,v_2} \ast g_2$ iff $v_1 = v_2$ and either $g_1 = g_2$ or $v_1 = o_K$. 

Proof. This follows from 5.4. \[\square\]

**Proposition 5.6.** Let \( u \in K' \) and \( z \in L' \). Define a mapping \( \nu_{u,z} : W_u \to M \) by \( \nu_{u,z}(p_{u,v} * g) = (v, g(z)) \) for \( v \in K' \) and \( \nu_{u,z}(o_M) = o_M \) (see 5.5). Then \( \nu_{u,z} \) is an injective homomorphism of the semimodule \( sW_u \) into the semimodule \( sM \). The homomorphism \( \nu_{u,z} \) is an isomorphism of the semimodules iff \( G(z) = L' \) (i.e., the group \( G \) operates transitively on \( L' \)).

**Proof.** Use 5.1, . . . , 5.5. \[\square\]

**Proposition 5.7.** Let \( u \in K' \). The semimodule \( sW_u \) is simple, minimal and faithful.

**Proof.** The semimodule is minimal due to 5.3. By 5.2, \((f * h)(p_{u,v} * id_L) = p_{u,f(v)} * h\) and, using 5.5, we conclude easily the our semimodule is faithful as well.

It remains to show that \( sW_u \) is simple. For, let \( \alpha \neq id \) be a congruence of the semimodule. Then \( (p_{u,v1} * g1, p_{u,v2} * g2) \in \alpha \), where \( p_{u,v1} * g1 \neq p_{u,v2} * g2 \) and we can assume that \( v_1 \in K' \). If \( g1 \neq g2 \) then \((p_{u,v1} * g1, \zeta) \in \alpha\) follows from 5.1(ii). But \( S(p_{u,v1} * g1) = W_u \) by 5.3, and hence \( \alpha = W_u \times W_u \). Assume, therefore, that \( g1 = g2 = g \) and \( v_2 \neq v_1 \). Then \( (p_{u,v1} * g, p_{u,v3} * g) \in \alpha \), where \( v_1 < v_3 = v_1 + v_2 \). From this, \((p_{u,v1} * g, \zeta) = ((p_{u,v1} * g)(p_{u,v1} * g), (p_{v1,vb4} * g)(p_{u,v1} * g)) \in \alpha\) for every \( v_4 \in K \). Thus \( \alpha = W_u \times W_u \). \[\square\]

**Remark 5.8.** \( S \) is a subsemiring of \( E_1(M) \), and so the (left \( S \)-) semimodule \( sM \) is faithful. In view of 5.6, the semimodule \( sM \) is minimal iff \( G \) operates transitively on \( L' \), and then the semimodules \( sM \) and \( sW_u \) are isomorphic, so that \( sM \) is simple by 5.7. If \( y \in L' \) then \( R = (K' \times G(y)) \cup \{o_M\} \) is an ideal of the semimodule \( sM \) and \((R \times R) \cup id_M \) is a congruence of \( sM \). If \( \tau = M \times M \) (e.g., if \( sM \) is simple) then \( R = M \) and \( G \) operates transitively on \( L' \).

**Proposition 5.9.** (i) \( W_u(p_{u,v} * g) = \{\zeta\} \) for \( v \not\leq u \).

(ii) \( W_u(p_{u,v} * g) = W_u \) for \( v \leq u \).

**Proof.** It is easy. \[\square\]

**Proposition 5.10.** (i) The set \( \{p_{u,v} \ast g \mid v \not\leq u\} \) is a bi-ideal of the semiring \( W_u \) and it is the greatest proper left ideal of that semiring.

(ii) The set \( \{p_{u,v} \ast id_L \mid v \leq u\} \cup \{\zeta\} \) is just the set of multiplicative idempotents of \( W_u \).

(iii) The semiring \( W_u \) is not multiplicatively idempotent.

(iv) The semiring \( W_u \) is not bi-ideal-simple.

**Proof.** Easy. \[\square\]
5.11 Let $R$ be a minimal left ideal of the semiring $S$ such that $R \neq W_u$ for every $u \in N$.

**Lemma 5.11.1.** $R \cap W_u = \{\zeta\}$.

**Proof.** It is obvious.

**Lemma 5.11.2.** $S(f \ast g) = R$ for every $f \ast g \in R \setminus \{\zeta\}$.

**Proof.** We have $f \in E_1(K) \setminus \{\zeta\}$, and hence $f(v) \neq o_K$ for at least one $v \in N$ and $(p_{f(v),v} \ast \id_M)(f \ast g) = (p_{f(v),v}f \ast g) \neq \zeta$. Now, we can proceed similarly as in the proof of 2.9.2.

**Lemma 5.11.3.** There is a proper ideal $A$ of the semilattice $K$ such that $A = \{ x \in K \mid f(x) \neq o_K \}$ for every $f \ast g \in R \setminus \{\zeta\}$.

**Proof.** Use 5.11.2.

**Lemma 5.11.4.** Let $f \ast g \in R \setminus \{\zeta\}$, $v \in M$ and let $o_K \neq w \in f(K)$. Then $p_{w,v}f \ast g = p_{A,v}g = K$, where $B = K \setminus A$, $p_{A,v}(B) = \{v\}$ and $p_{A,v}(A) = \{o_K\}$.

**Proof.** We have $p_{w,v}f \ast g = (p_{w,v} \ast \id_L)(f \ast g) \in R$ and, by 5.11.3, $A = \{ x \in K \mid p_{w,v}f(x) = o_K \} = \{ x \in K \mid f(x) \neq w \}$. Thus $B = \{ y \in K \mid f(y) \leq w \}$ and $p_{w,v}f(B) = \{v\}$.

**Proposition 5.11.5.** (i) $A$ is a prime ideal of the semilattice $K$, $o_B \notin B$ and $W_A = \{ p_{A,v}g \mid v \in K, g \in G \} = R$.

(ii) The mapping $\nu_{A,z} : R \to M$, where $z \in L'$ and $\nu_{A,z}(p_{A,v}g) = (v,g(z))/Q$ is an injective homomorphism of the semimodule $\mathcal{S} \! R$ into the semimodule $\mathcal{S} \! M$. This homomorphism is an isomorphism of the semimodules iff $G$ operates transitively on $L'$.

(iii) $R(p_{A,v}g) = \{\zeta\}$ for $v \in A$.

(iv) $R(p_{A,v}g) = R$ for $v \in B$.

(v) The set $\{ p_{A,v}g \mid v \in A, g \in G \}$ is a bi-ideal of the semiring $R$ and it is the greatest proper left ideal of that semiring.

(vi) The set $\{ p_{A,v} \ast \id_L \mid v \in B \} \cup \{\zeta\}$ is just the set of multiplicative idempotents of $R$.

(vii) The semiring $R$ is neither multiplicatively idempotent nor bi-ideal-simple.

**Proof.** (i) First, take $f \ast g \in R \setminus \{\zeta\}$. By 5.11, we get $p_{A,v} \ast g \in R$ for every $v \in K$. If $v \in K'$ and $h \in G$ then $p_{A,v} \ast h = (p_{A,v} \ast hg^{-1})(p_{A,v} \ast g) \in R$. Thus $W_A \subseteq R$. On the other hand, $W_A$ is a non-trivial left ideal of $S$ and, $R$ being minimal, we find that $R = W_A$.

(ii) We can proceed similarly as in the proof of 5.6.

The remaining assertions are easy to check.
Theorem 5.12. (i) The set \( W_u = \{ p_{u,v} \ast g \mid v \in K, g \in G \} \) is a minimal left ideal of the semiring \( S \) for every \( u \in K' \).

(ii) if \( R \) is a minimal left ideal of \( S \) such that \( R \neq W_u \) for every \( u \in K' \) then there is a non-principal prime ideal \( A \) of the semilattice \( K \) such that \( R = W_A = \{ p_{A,v} \ast g \mid v \in K, g \in G \} \), where \( p_{A,v}(A) = \{ o_K \} \) and \( p_{A,v}(K \setminus A) = \{ v \} \).

(iii) If \( A \) is a non-principal prime ideal of \( K \) such that \( p_{A,v} \ast g \in S \) for at least one \( v \in K' \) and at least one \( g \in G \) then \( W_A \subseteq S \) and \( W_A \) is an minimal left ideal of \( S \) such that \( W_A \neq W_u \) for every \( u \in K' \).

(iv) If \( P \) is a minimal left ideal of \( S \) then \( SP \) is a faithful, simple and minimal (left \( S \)-) semimodule. Besides, if \( z \in L' \) then \( SP \) is isomorphic to the subsemimodule \( S(K \times G(z))/Q \) of \( SM \).

(v) If \( P_1 \) and \( P_2 \) are minimal left ideals of \( S \) then the semimodules \( SP_1 \) and \( SP_2 \) are isomorphic.

(vi) If \( P \) is a minimal left ideal of \( S \) then the semiring \( \mathcal{P} \) is neither multiplicatively idempotent nor bi-ideal-simple.

Proof. Combine the foregoing results.

Corollary 5.13. Denote by \( C \) the set of prime ideals \( A \) of \( K \) such that \( W_A \subseteq S \). Then:

(i) The set \( \{ W_A \mid A \in C \} \) is just the set of (pair-wise distinct) minimal left ideals of the semiring \( S \).

(ii) \( A_u = \{ x \in K \mid x \not\leq u \} \in C \) for every \( u \in K' \).

(iii) The set \( \bigcup W_A, A \in C \), is just the smallest non-trivial ideal of the semiring \( S \).

Proposition 5.14. The following conditions are equivalent:

(i) There is a minimal left ideal \( R \) of \( S \) such that \( R(+) \) is an antichain.

(ii) For every minimal left ideal \( P \) of the semiring \( S \), the semilattice \( P(+) \) is an antichain.

(iii) The semilattice \( K \) is an antichain.

(iv) The semilattice \( M \) is an antichain.

Proof. It is easy.

Lemma 5.15. The following conditions are equivalent for a minimal left ideal \( P \) of \( S \):

(i) The set \( P \backslash \{ \zeta \} \) is a subsemigroup of the multiplicative semigroup of \( P \).

(ii) \( K' + K' = K' \) and \( P = W_A \), where \( A = \{ o_K \} \).

Proof. It is easy.
6. 0,1-preserving endomorphisms

Let $M$ be a non-trivial semilattice such that $0_M,o_M \in M$. The set $E_{0,1} = \{ f \in E \mid f(0_M) = 0_M, f(o_M) = o_M \}$ is a unitary subsemiring of $E$. For every $u \in N \setminus \{ o_M \}$, we have $r_u \in E_{0,1}$, where $r_u(x) = 0_M$ for $x \leq u$ and $r_u(y) = o_M$ for $y \not\leq u$. Clearly, $r_{0_u}$ is additively absorbing.

Now, let $E$ be a subsemiring of $E_{0,1}$ such that $r_u \in E$ for every $u \in N$. For every pair $(u,v) \in N \times N$, $u < v$, the set $X_{(u,v)} = \{ r_u,r_v \}$ is a minimal left ideal of $E$.

6.1 Let $K$ be a minimal left ideal of $E$ such that $K \neq X_{(u,v)}$ for all $u,v \in N$, $u < v$.

**Lemma 6.1.1.** Let $u \in N$, $f \in K$ and $A = A_f,u = \{ x \in M \mid f(x) \not\leq u \}$. Then:

(i) $A$ is a prime ideal of $M$.

(ii) $r_u,f$ and $r_A \in K$, where $r_A(A) = \{ o_M \}$ and $B = M \setminus A$.

(iii) If $u = o_B \in B$ then $r_A = r_{0_u}$.

**Proof.** It is easy. □

**Lemma 6.1.2.** Let $A_1,A_2$ be prime ideals of $M$ such that $r_{A_1},r_{A_2} \in K$. Then either $A_1 \subseteq A_2$ or $A_2 \subseteq A_1$.

**Proof.** Let, on the contrary, $A_1 \not\subseteq A_2 \not\subseteq A_1$. The set $A_3 = A_1 \cup A_2$ is a prime ideal and $r_{A_3} = r_{A_1} + r_{A_2} \in K$. Now, the sets $K_1 = \{ r_{A_1},r_{A_2} \}$ and $K_2 = \{ r_{A_2},r_{A_3} \}$ are two-element left ideals contained in $K$. Consequently, $K_1 = K_2 = K$, $r_{A_1} = r_{A_2}$ and $A_1 = A_2$, a contradiction. □

**Corollary 6.1.3.** Let $A_1,A_2$ be two different prime ideals such that $r_{A_1},r_{A_2} \in K$. Then $K = \{ r_{A_1},r_{A_2} \}$ and either $A_1 \subset A_2$ or $A_2 \subset A_1$. □

Now, assume that $r_A \in K$ for exactly one prime ideal $A$ and put $B = M \setminus A$ (see 6.1.1).

**Lemma 6.1.4.** $f(A) = \{ o_M \}$ for every $f \in K$.

**Proof.** We have $A = \{ x \in M \mid f(x) \not\leq u \}$ for every $u \in N$ (use 6.1.1). Thus $f(A) = \{ o_M \}$. □

**Lemma 6.1.5.** $f(B) = \{ 0_M \}$ for every $f \in K$.

**Proof.** We have $B = M \setminus A = \{ y \in M \mid f(y) \leq 0_M \}$. □

**Corollary 6.1.6.** $K = \{ r_A \}$.

**Theorem 6.2.** (i) For all $u,v \in N$, $u < v$, the set $\{ r_u,r_v \}$ is a minimal left ideal of $E$.

(ii) If $K$ is a minimal left ideal of $E$ then there are prime ideals $A_1,A_2$ of $M$ such that $A_1 \subset A_2$ and $K = \{ r_{A_1},r_{A_2} \}$.
**Proof.** See 6.1. □

**Theorem 6.3.** Denote by $D$ the set of prime ideals of $M$ such that $r_A \in E$ (see 6.1). Then:

(i) Minimal left ideals of the semiring $E$ are just the two-element sets $\{r_{A_1}, r_{A_2}\}$, $A_1, A_2 \in D$, $A_1 \subset A_2$.

(ii) $A_u = \{ x \in M \mid x \not\preceq u \} \in D$ for every $u \in N$.

(iii) $P = M \setminus \{0_M\} \in D$ and $r_P = r_{0_M}$.

(iv) For every $A \in D$, $A \neq P$, the two-element set $\{r_A, r_{0_M}\}$ is a minimal left ideal of $E$.

(v) $\{r_A \mid A \in D\}$ is an ideal and it is the smallest right ideal of $E$.

**Proof.** It is easy (use 6.2). □

7. Non-preserving endomorphisms

Let $M$ be a non-trivial semilattice and let $E$ be a subsemiring of $E(M)$ containing all constant endomorphisms $\mu_u$, $\mu_u(M) = \{u\}$, $u \in M$. Now, the set $Q = \{ \mu_u \mid u \in M \}$ is an ideal and, in fact, it is the smallest left ideal of the semiring $E$. The two-element sets $\{\mu_u, \mu_v\}$, where $u < v$, are minimal right ideals and there are no more. The semimodules $E_M$ and $EQ$ are isomorphic via $u \mapsto \mu_u$.

**References**


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