On a maximal subgroup of the Conway group $Co_3$

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Abstract. This paper is dealing with a split extension group of the form $3^5:(2 \times M_{11})$, which is a maximal subgroup of the Conway simple group $Co_3$. We refer to this extension by $G$. We firstly determine the conjugacy classes of $G$ using the coset analysis technique. The structures of inertia factor groups were determined through deep investigation on the maximal subgroups of the maximal subgroups of $2 \times M_{11}$. We found the inertia factors to be the groups $2 \times M_{11}$, $A_6^2$ (non-split) and $(S_3 \times S_3):2$. We then determine the Fischer matrices of $G$ and apply the Clifford-Fischer theory to compute the ordinary character table of this group. The Fischer matrices of $G$ are all integer valued, with sizes ranging from 1 to 4. The full character table of $G$ is $37 \times 37$ complex valued matrix and is given at the end of this paper.

Keywords: group extensions, Mathieu group, inertia groups, Fischer matrices, character table.

1. Introduction

the Conway group $Co_3$ is of order $495\,766\,656\,000$. From the Atlas [15] we can see that $Co_3$ has 14 conjugacy classes of maximal subgroups. The fifth largest maximal subgroup is a group of the form $3^5:(2 \times M_{11})$. We refer to this group by $G$ and clearly it has order $243 \times 2 \times |M_{11}| = 3\,849\,120$ and index $128\,800$ in

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In fact the normalizer of an elementary abelian group $3^5$ in $C_3$ will have the form of $\mathcal{G}$; that is $N_{C_3}(3^5) = 3^3(2 \times M_{11})$. Using GAP [16] we were able to construct this split extension group in terms of permutations of 33 points. We used the coset analysis technique to construct the conjugacy classes of $\mathcal{G}$, where correspond to the 20 classes of $2 \times M_{11}$, we obtained 37 classes of $\mathcal{G}$. Then by looking on the maximal subgroups of the maximal subgroups of $2 \times M_{11}$, we were able to determine the structures of the inertia factor groups. Then we computed the Fischer matrices of the extension and we found to be integer valued matrices with sizes ranging from 1 to 4. Finally we were able to compute the ordinary character table of $\mathcal{G}$ using Clifford-Fischer theory and we supplied it at the end of this paper.

The character table of any finite group extension $\mathcal{G} = N \cdot G$ (here $N$ is the kernel of the extension and $G$ is isomorphic to $\mathcal{G}/N$) produced by Clifford-Fischer Theory is in a special format that could not be achieved by direct computations using GAP or Magma [14]. Also there is an interesting interplay between the coset analysis and Clifford-Fischer Theory. Indeed the size of each Fischer matrix is $c(g_i)$, the number of $\mathcal{G}$-classes corresponding to $[g_i]_\mathcal{G}$ obtained via the coset analysis technique. That is computations of the conjugacy classes of $\mathcal{G}$ using the coset analysis technique will determine the sizes of all Fischer matrices.

For the notation used in this paper and the description of Clifford-Fischer theory technique, we follow [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13].

By the Atlas of Wilson [25], we see that $C_3$ has an absolutely irreducible module of dimension 22 over $\mathbb{F}_2$. Using the two $22 \times 22$ matrices over $\mathbb{F}_2$ that generate $C_3$ together with the program for obtaining maximal subgroups of $C_3$ given in [25] we were able to locate our group $\mathcal{G}$ inside $C_3$ in terms of $22 \times 22$ matrices over $\mathbb{F}_2$. The following two elements $a$ and $b$ generate $\mathcal{G}$.

$$a = \begin{pmatrix}
0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
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0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\
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0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}$.
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\[
b = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
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1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
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0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \\
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0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1
\end{pmatrix}
\]

where $o(a) = 4$, $o(b) = 3$ and $o(ab) = 22$.

For the sake of computations, we used few GAP commands to convert the representation of our group $\mathcal{G}$ from matrix into permutation representation, where we were able to represent $\mathcal{G}$ in terms of the set $\{1, 2, \cdots, 33\}$. Thus in terms of permutations and using $(c_1, c_2, \cdots, c_k)$ to denote the $k$-cycle $(c_1 c_2 \cdots c_k)$, the group $\mathcal{G}$ is generated by $\mathcal{G}_1$ and $\mathcal{G}_2$, where

\[
\mathcal{G}_1 = (1, 4, 23, 29)(2, 15, 19, 10)(3, 16, 24, 33)(6, 25, 26, 31)(7, 12, 17, 21)
(8, 9, 30, 32)(13, 20),
\]
\[
\mathcal{G}_2 = (1, 3, 12)(2, 22, 18)(4, 25, 32)(5, 11, 21)(6, 23, 13)(7, 9, 28)(8, 19, 33)
(14, 27, 31)(15, 16, 20).
\]

Since $\mathcal{G}$ can be constructed in GAP, it is easy to obtain all its maximal subgroups. In fact $\mathcal{G}$ has 7 conjugacy classes of maximal subgroups. The second largest maximal subgroup of $\mathcal{G}$ is a group of the form $3^4;(A_6^2):S_3 := T$, with order 349920 and index 11 in $\mathcal{G}$. The group $T$ itself has also 7 conjugacy classes of maximal subgroups and the second largest maximal subgroup of $T$ is a group of the form $3^4;2^3 \times (A_6^2)$ $\cong 3^4;(2 \times M_{10}) := M$. The order of $M$ is 116640 and its index in $T$ is 3 (and hence has index 33 in $\mathcal{G}$). In fact $M$ gives rise to the above permutation representation of $\mathcal{G}$ on 33 points.

**Remark 1.** From the Atlas of Wilson [25] we see that $Co_3$ can be generated in terms of permutations of 276 points and thus any maximal subgroup (including our group $\mathcal{G}$) will be generated in terms of permutations acting on 276 points. However starting with the 22-dimensional matrix representation over $\mathbb{F}_2$ for $Co_3$ (available in [25]), then generating $\mathcal{G}$ in terms of $22 \times 22$ matrices over $\mathbb{F}_2$ (for example the above generators $a$ and $b$) and then using few GAP commands, we were able to generate $\mathcal{G}$ in terms of permutations on only 33 points. Indeed this will work faster for the required computations.

Having $\mathcal{G}$ being constructed in GAP, it is easy to obtain all its normal subgroups. In fact $\mathcal{G}$ possesses three proper normal subgroups of orders 243, 486 and 1924560. The normal subgroup of order 243 is an elementary abelian group isomorphic to $N$. 

In GAP one can check for the complements of \( N \) in \( \overline{G} \), where in our case we obtained only one complement isomorphic to \( 2 \times M_{11} \) and together with \( N \) gives the split extension in consideration.

2. The conjugacy classes of \( \overline{G} \)

In this section we compute the conjugacy classes of the group \( \overline{G} \) using the coset analysis technique (see Basheer [2], Basheer and Moori [3, 4, 6] or Moori [18] and [19] for more details) as we are interested to organize the classes of \( 2 \times M_{11} \) corresponding to the classes of \( \overline{G} \). Firstly note that \( M_{11} \) has 10 conjugacy classes (see the Atlas) and thus \( 2 \times M_{11} \) will have 20 conjugacy classes. Corresponding to these 20 classes of \( 2 \times M_{11} \), we obtained 37 classes in \( \overline{G} \).

In Table 1, we list the conjugacy classes of \( \overline{G} \), where in this table:

- \( k_i \) is the number of orbits \( Q_{i1}, Q_{i2}, \cdots, Q_{ik_i} \) for the action of \( N \) on the coset \( Ng_i = Ng_i \), where \( g_i \) is a representative of a class of the complement \(( \cong 2 \times M_{22}) \) of \( N \) in \( \overline{G} \). In particular, the action of \( N \) on the identity coset \( Ng \) produces 243 orbits each consists of singleton. Thus for \( \overline{G} \), we have \( k_1 = 243 \).

- \( f_{ij} \) is the number of orbits fused together under the action of \( C_G(g_{ij}) \) on \( Q_{1}, Q_{2}, \cdots, Q_k \). In particular, the action of \( C_G(1_G) = G \) on the orbits \( Q_{1}, Q_{2}, \cdots, Q_k \) affords three orbits of lengths 1, 110 and 132 (with corresponding point stabilizers \( 2 \times M_{11} \), \( 3^2 : QD_{16} \) and \( S_5 \), where \( QD_{16} \) is the quasi-dihedral group of order 16. Thus \( f_{11} = 1, f_{12} = 110 \) and \( f_{13} = 132 \).

- \( m_{ij} \)'s are weights (attached to each class of \( \overline{G} \)) that will be used later in computing the Fischer matrices of \( \overline{G} \). These weights are computed through the formula

\[
m_{ij} = [N \overline{G}(N \overline{g}_i) : C_{\overline{G}}(g_{ij})] = [N] [C_G(g_i)] [C_{\overline{G}}(g_{ij})],
\]

where \( N \) is the kernel of an extension \( \overline{G} \) that is in consideration.

| \( g_i | \overline{G} \) | \( k_i \) | \( f_{ij} \) | \( m_{ij} \) | \( [g_i]|\overline{G} \) | \( [g_i]|\overline{G} \) | \( C_{\overline{G}}(g_{ij}) \) |
|---|---|---|---|---|---|---|
| \( g_1 = 1A \) | \( k_1 = 243 \) | \( f_{11} = 1 \) | \( m_{11} = 1 \) | \( g_{11} \) | 1 | 1 | 3849120 |
| | \( f_{12} = 110 \) | \( m_{12} = 110 \) | \( g_{12} \) | 3 | 110 | 34992 |
| | \( f_{13} = 132 \) | \( m_{13} = 132 \) | \( g_{13} \) | 3 | 132 | 29160 |
| \( g_2 = 2A \) | \( k_2 = 1 \) | \( f_{22} = 1 \) | \( m_{22} = 243 \) | \( g_{22} \) | 2 | 243 | 15840 |
| \( g_3 = 2B \) | \( k_3 = 27 \) | \( f_{31} = 1 \) | \( m_{31} = 9 \) | \( g_{31} \) | 2 | 1485 | 2592 |
| | \( f_{32} = 6 \) | \( m_{32} = 54 \) | \( g_{32} \) | 6 | 8910 | 432 |
| | \( f_{33} = 8 \) | \( m_{33} = 72 \) | \( g_{33} \) | 6 | 11880 | 324 |
| | \( f_{34} = 12 \) | \( m_{34} = 108 \) | \( g_{34} \) | 6 | 17820 | 216 |
| \( g_4 = 2C \) | \( k_4 = 9 \) | \( f_{41} = 1 \) | \( m_{41} = 27 \) | \( g_{41} \) | 2 | 4455 | 864 |
| | \( f_{42} = 8 \) | \( m_{42} = 216 \) | \( g_{42} \) | 6 | 35640 | 108 | continued on next page
Table 1 (continued from previous page)

| $[g]_G$ | $k_i$ | $f_{ij}$ | $m_{ij}$ | $[g]_S$, $|\sigma(g)_{ij}|$, $|C_{ij}(g)_{ij}|$ |
|---------|------|---------|---------|---------------------------------|
| $g_5 = 3A$ | $k_5 = 9$ | $f_{51} = 1$ | $m_{51} = 27$ | $g_{51}$ | $3$ | $11880$ | $324$ |
| $g_6 = 4A$ | $k_6 = 3$ | $f_{61} = 1$ | $m_{61} = 81$ | $g_{61}$ | $4$ | $80190$ | $48$ |
| $g_7 = 4B$ | $k_7 = 9$ | $f_{71} = 1$ | $m_{71} = 27$ | $g_{71}$ | $4$ | $26730$ | $144$ |
| $g_8 = 5A$ | $k_8 = 3$ | $f_{81} = 1$ | $m_{81} = 81$ | $g_{81}$ | $5$ | $128304$ | $30$ |
| $g_9 = 6A$ | $k_9 = 1$ | $f_{91} = 1$ | $m_{91} = 243$ | $g_{91}$ | $6$ | $106920$ | $36$ |
| $g_{10} = 6B$ | $k_{10} = 3$ | $f_{10.1} = 1$ | $m_{10.1} = 81$ | $g_{10.1}$ | $6$ | $106920$ | $36$ |
| $g_{11} = 6C$ | $k_{11} = 3$ | $f_{11.1} = 1$ | $m_{11.1} = 81$ | $g_{11.1}$ | $6$ | $106920$ | $36$ |
| $g_{12} = 8A$ | $k_{12} = 3$ | $f_{12.1} = 1$ | $m_{12.1} = 81$ | $g_{12.1}$ | $8$ | $80190$ | $48$ |
| $g_{13} = 8B$ | $k_{13} = 3$ | $f_{13.1} = 1$ | $m_{13.1} = 81$ | $g_{13.1}$ | $8$ | $80190$ | $48$ |
| $g_{14} = 8C$ | $k_{14} = 1$ | $f_{14.1} = 1$ | $m_{14.1} = 81$ | $8$ | $240570$ | $16$ |
| $g_{15} = 8D$ | $k_{15} = 1$ | $f_{15.1} = 1$ | $m_{15.1} = 81$ | $8$ | $240570$ | $16$ |
| $g_{16} = 10A$ | $k_{16} = 1$ | $f_{16.1} = 1$ | $m_{16.1} = 243$ | $g_{16.1}$ | $10$ | $384912$ | $10$ |
| $g_{17} = 11A$ | $k_{17} = 1$ | $f_{17.1} = 1$ | $m_{17.1} = 243$ | $g_{17.1}$ | $11$ | $349920$ | $22$ |
| $g_{18} = 11B$ | $k_{18} = 1$ | $f_{18.1} = 1$ | $m_{18.1} = 243$ | $g_{18.1}$ | $11$ | $349920$ | $22$ |
| $g_{19} = 22A$ | $k_{19} = 1$ | $f_{19.1} = 1$ | $m_{19.1} = 243$ | $g_{19.1}$ | $22$ | $349920$ | $22$ |
| $g_{20} = 22B$ | $k_{20} = 1$ | $f_{20.1} = 1$ | $m_{20.1} = 243$ | $g_{20.1}$ | $22$ | $349920$ | $22$ |

3. The inertia factor groups of $\overline{G}$

We recall that knowledge of the appropriate character tables of inertia factor groups is crucial in calculating the full character table of any group extension. Since in our extension $\overline{G}$, the normal subgroup $3^5$ is abelian and the extension splits, it follows by applications of Mackey’s Theorem (see for example Theorem 3.3.4 of Whitley [24]), that every character of $3^5$ is extendible to an ordinary character of its respective inertia group $H_k$. Thus all the character tables of the inertia factor groups that we will use to construct the character tables of $\overline{G}$ are the ordinary ones. Next we determine the structures of the inertia factor groups.

We have seen from Section 2 that the action of $\overline{G} = 3^5:(2 \times M_{11})$ (which can be reduced to the action of $2 \times M_{11}$) on the classes of $N = 3^5$ yielded three orbits of lengths 1, 110 and 132 (and the corresponding point stabilizers were $2 \times M_{11}$, $3^2:QD_{16}$ and $S_5$). By a theorem of Brauer (see for example Theorem 5.1.5 of
Mpono [22]), it follows that the action of $\mathcal{G}$ on $\text{Irr}(N)$ will also produce three orbits. We used Programme C of [23], which can also be found in [20, 21], to determine the lengths of the orbits of $\mathcal{G}$ or just $2 \times M_{11}$ on $\text{Irr}(N)$. We found that the action of $2 \times M_{11}$ on $\text{Irr}(N)$ produces three orbit of lengths 1, 22 and 220. Let $H_1$, $H_2$ and $H_3$ be the respective inertia factor groups of the representatives of characters from the orbits with previous lengths. We notice that these inertia factors have indices 1, 22 and 220 respectively in $2 \times M_{11}$:

\begin{align*}
T_1 &= \text{M}_{11} \\
T_2 &= 2 \times M_{10} \cong 2 \times (A_6^2) \\
T_3 &= 2 \times PSL(2, 11) \\
T_4 &= 2 \times (3^2 : QD_{16}) \\
T_5 &= 2 \times S_5 \\
T_6 &= 2 \times GL(2, 3)
\end{align*}

Now the first inertia factor group $H_1$ of $2 \times M_{11}$ has an index 1 and thus $H_1 = 2 \times M_{11}$ itself. Since we have the character table of $M_{11}$ (see the Atlas) we can easily construct the character table of $2 \times M_{11}$, which we supply below as Table 3.

The second inertia factor group $H_2$ has index 22 in $2 \times M_{11}$. From Table 2 we can see that the only index of a maximal subgroup that divides 22 is 11. It follows that $H_2$ is an index 2 subgroup of $2 \times M_{10} \cong 2 \times (A_6^2)$. Now the group $2 \times (A_6^2)$ has 6 conjugacy classes of maximal subgroups, each class represented by $A_6^2$ (twice), $2 \times A_6$, $2 \times (3^2 : Q_8)$, $2 \times (5 : 4)$ and $2 \times QD_{16}$ with respective orders 720 (3 times), 144, 40 and 32, where $Q_8$ and $QD_{16}$ are the quaternion and quasi-dihedral groups of orders 8 and 16 respectively. Therefore we can see that

\begin{equation}
H_2 \in \{A_6^2, 2 \times A_6\}.
\end{equation}

Using GAP we were able to construct the character tables of $A_6^2$ and $2 \times A_6$, where below in Table 4 we only supply the character table of $A_6^2$ as the character table of $2 \times A_6$ can be constructed easily from the character tables of $Z_2$ and of $A_6$ (available in the Atlas).
where in Table 3, \(A = -i\sqrt{2}\) and \(B = \frac{1}{2} - i\frac{\sqrt{11}}{2}\).

**Table 4: The character table of \(A_6^2\)**

<table>
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<td>1</td>
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<td>-1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
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<td>-1</td>
<td>-1</td>
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</tr>
<tr>
<td>x4</td>
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<td>-1</td>
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<td>-1</td>
<td>1</td>
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<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>x5</td>
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<td>2</td>
<td>1</td>
<td>-2</td>
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<td>0</td>
<td>0</td>
<td>0</td>
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<td>0</td>
</tr>
<tr>
<td>x6</td>
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<td>-2</td>
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<td>0</td>
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<td>0</td>
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<tr>
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<td>-2</td>
<td>-1</td>
<td>0</td>
<td>0</td>
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<td>0</td>
<td>0</td>
<td>0</td>
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<td>0</td>
</tr>
<tr>
<td>x8</td>
<td>10</td>
<td>-2</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

We keep the information that \(|\text{Irr}(A_6^2)| = 8\) and \(|\text{Irr}(2 \times A_6)| = 14\) in mind and we continue to look at the third inertia factor group \(H_3\), where later we will determine both \(H_2\) and \(H_3\) simultaneously.

Now the third inertia factor group \(H_3\) has index 220 in \(2 \times M_{11}\). Again from Table 2 we can see that the only indices of maximal subgroups that divides 220 are 2, 11 and 55. Therefore \(H_3\) is either

- index 110 subgroup of \(M_{11}\),
- index 20 subgroup of \(2 \times (A_6^2)\) or
- index 4 subgroup of \(2 \times (3^2:QD_{16})\).

Now this requires more analysis on the structures of the subgroups of \(M_{11}\), \(2 \times (A_6^2)\) and \(2 \times (3^2:QD_{16})\). We consider each case of the above:

- From the Atlas we can see that an index 110 subgroup of \(M_{11}\) must be either an index 10 subgroup of \(M_{10} \cong A_6^2\), or an index 2 subgroup of \(3^2:QD_{16}\). Now \(A_6^2\) has four maximal subgroups, namely \(A_6\), \(3^2:Q_8\), 5:4 and \(QD_{16}\) of respective orders 360, 72, 20 and 16. Thus \(H_3\) is either a subgroup of \(A_6\) with index 5 or is
isomorphic to $3^2:Q_8$. However the group $A_6$ does not contain a subgroup of order 72 (the largest maximal subgroup of $A_6$ is $A_5$, which is of order 60). Therefore a subgroup of $A_6:2$ of index 10 must necessarily be isomorphic to $3^2:Q_8$.

From another side the group $3^2:QD_{16}$ has four maximal subgroups, namely $3^2:Q_8$, $3^2:8$, $(S_3 \times S_3):2$ and $QD_{16}$ with respective orders 72, 72, 72 and 16. Therefore an index 2 subgroup of $3^2:QD_{16}$ must necessarily be isomorphic to one of the first three maximal subgroups; that is either $3^2:Q_8$, $3^2:8$ or $(S_3 \times S_3):2$.

Now we conclude this case by saying that if $H_3$ is an index 110 subgroup of $M_{11}$, then $H_3 \in \{3^2:Q_8, 3^2:8, (S_3 \times S_3):2\}$.

- We now consider the case that $H_3$ is an index 20 subgroup of $2 \times (A_6:2)$. The $2 \times (A_6:2)$ has 6 conjugacy classes of maximal subgroups, each class represented by $A_6:2$ (2 isomorphic non-conjugate copies), $2 \times A_6$, $2 \times (3^2:Q_8)$, $2 \times (5:4)$ and $2 \times QD_{16}$. Because of the order, only the first four maximal subgroups $(A_6:2, A_6:2, 2 \times A_6$ and $2 \times (3^2:Q_8))$ can contain $H_3$ with respective indices 10, 10, 10 and 2. As of the above case, if $H_3$ is an index 10 in $A_6:2$ (any of the non-conjugate copies), then $H_3$ will be isomorphic to a subgroup of the form $3^2:Q_8$.

The group $2 \times A_6$ has 6 conjugacy classes of maximal subgroups, each class represented by $A_6, 2 \times A_5$ (2 isomorphic non-conjugate copies), $2 \times (3^2:4)$ and $2 \times S_4$ (2 isomorphic non-conjugate copies) with respective orders 360, 120 (twice), 72, 48 (twice). We know that $A_6$ has no index 5 subgroup (see the Atlas). Therefore if $H_3$ is a subgroup of $2 \times A_6$, then it must be isomorphic to $2 \times (3^2:4)$.

The group $2 \times (3^2:Q_8)$ has 8 conjugacy classes of maximal subgroups, each class represented by $2 \times (3^2:4)$ (3 isomorphic non-conjugate copies), $3^2:Q_8$ (4 isomorphic non-conjugate copies) and $2 \times Q_8$ with respective orders 72 (7 times) and 16. Therefore $H_3 \in \{2 \times (3^2:4), 3^2:Q_8\}$.

Now gathering all the possibilities of $H_3$ in this case, we conclude this case by saying that if $H_3$ is an index 20 subgroup of $2 \times (A_6:2)$, then $H_3 \in \{3^2:Q_8, 2 \times (3^2:4)\}$.

- Here we consider the case that $H_3$ is an index 4 of $2 \times (3^2:QD_{16})$. Firstly $2 \times (3^2:QD_{16})$ has 8 conjugacy classes of maximal subgroups, each class represented by $(3^2:Q_8):2$ (4 isomorphic non-conjugate copies), $2 \times (3^2:Q_8)$, $2 \times ((S_3 \times S_3):2)$, $2 \times (3^2:8)$ and $2 \times QD_{16}$ with respective orders 144 (7 times) and 32. We look at the cases $(3^2:Q_8):2, 2 \times (3^2:Q_8), 2 \times ((S_3 \times S_3):2)$ and $2 \times (3^2:8)$.

The group $(3^2:Q_8):2$ has 4 conjugacy classes of maximal subgroups, each class represented by $(S_3 \times S_3):2, 3^2:Q_8, 3^2:8$ and $QD_6$ with respective orders 72 (3 times) and 16. Thus possibilities for $H_3$ are $(S_3 \times S_3):2, 3^2:Q_8$ or $3^2:8$.

The group $2 \times (3^2:Q_8)$ has 8 conjugacy classes of maximal subgroups, each class represented by $2 \times (3^2:4), 3^2:Q_8$ (4 times), $2 \times (3^2:4)$ (twice) and $2 \times Q_8$ with respective orders 72 (7 times) and 16. Thus possibilities for $H_3$ are $2 \times (3^2:4)$ or $3^2:Q_8$.

The group $2 \times ((S_3 \times S_3):2)$ has 8 conjugacy classes of maximal subgroups, each class represented by $2 \times (3^2:4), (S_3 \times S_3):2$ (4 times), $2 \times S_3 \times S_3$ (twice) and $2 \times D_6$ with respective orders 72 (7 times) and 16. Thus possibilities for $H_3$ are $2 \times (3^2:4), (S_3 \times S_3):2$ or $2 \times S_3 \times S_3$. 

A.B.M. BASHEER, F. ALI and M.L. ALOTAIBI
The group $2 \times (3^2:8)$ has 4 conjugacy classes of maximal subgroups, each class represented by $2 \times (3^2:4)$, $3^2:8$ (twice) and $8 \times 2$ with respective orders 72 (3 times) and 16. Thus the only possibility for $H_3$ is to be $3^2:8$.

Now we conclude this case by saying that if $H_3$ is an index 4 subgroup of $2 \times (3^2:QD_{16})$, then $H_3 \in \{3^2:Q_8, 2 \times (3^2:4), (S_3 \times S_3):2, 3^2:8, 2 \times S_3 \times S_3, 3^2:8\}$.

Now combining the possibilities of $H_3$ in the above three cases, we deduce that

$$H_3 \in \{3^2:Q_8, 3^2:8, (S_3 \times S_3):2, 2 \times (3^2:4), 2 \times S_3 \times S_3\}.$$ 

**Remark 2.** Using GAP we were able to obtain that $|\text{Irr}(3^2:Q_8)| = 6$, $|\text{Irr}(3^2:8)| = |\text{Irr}((S_3 \times S_3):2)| = 9$, $|\text{Irr}(2 \times (3^2:4))| = 12$ and $|\text{Irr}(2 \times S_3 \times S_3)| = 18$. Also we recall that $|\text{Irr}(A_6^2)| = 8$ and $|\text{Irr}(2 \times A_6)| = 14$. These information will help us in determining $\tilde{H}_2$ and eliminating many possibilities for $H_3$ as we will see in Proposition 3.1 below.

**Proposition 3.1.** The group $H_2$ is $A_6^2$ while $H_3$ is either $3^2:8$ or $(S_3 \times S_3):2$.

**Proof.** By Table 1 we can see that the number of conjugacy classes of $\overline{G}$ is 37 and thus $|\text{Irr}(\overline{G})| = 37$. We have also indicated before that since the extension $\overline{G}$ splits and the kernel $3^5$ is abelian, then all the character tables of the inertia factor groups that we will use to construct the character table of $\overline{G}$ are the ordinary ones. Now by Equations (2) and (3) we have

$$(H_2,H_3) \in \{(A_6^2,3^2:Q_8), (A_6^2,3^2:8), (A_6^2,(S_3 \times S_3):2), (A_6^2,2 \times (3^2:4)), (A_6^2,2 \times S_3 \times S_3), (2 \times A_6,3^2:Q_8), (2 \times A_6,3^2:8), (2 \times A_6,3 \times S_3:2),(2 \times A_6,2 \times (3^2:4)), (2 \times A_6,2 \times S_3 \times S_3)\}$$

and respectively we have

$$(4) \ (|\text{Irr}(H_2)|, |\text{Irr}(H_3)|) \in \{(8,6), (8,9), (8,9), (8,12), (8,18), (14,6), (14,9), (14,9), (14,12), (14,18)\}.$$ 

We know that $|\text{Irr}(\overline{G})| = \sum_{i=1}^{3} |\text{Irr}(H_i)|$. From Table 3 we have that $|\text{Irr}(H_1)| = 20$. It follows that $|\text{Irr}(H_2)| + |\text{Irr}(H_3)| = |\text{Irr}(\overline{G})| - |\text{Irr}(H_1)| = 37 - 20 = 17$. Now the only possible pairs that satisfy the relation $|\text{Irr}(H_2)| + |\text{Irr}(H_3)| = 17$ are $(8,9)$ and $(8,9)$; that is $(H_2,H_3) \in \{(A_6^2,3^2:8), (A_6^2,(S_3 \times S_3):2)\}$. In either case we can see that $H_2 = A_6^2$ and $H_3$ is either $3^2:8$ or $(S_3 \times S_3):2$ as claimed.

The character table supplied in Table 4 is the character table of the second inertia factor group $H_2$. We now proceed to determine $H_3$, which by Proposition 3.1 is either $3^2:8$ or $(S_3 \times S_3):2$. Using GAP we constructed the character tables of both $3^2:8$ and $(S_3 \times S_3):2$, which we show in Tables 5 and 6 respectively.

**Proposition 3.2.** The third inertia factor group $H_3$ is $(S_3 \times S_3):2$. 
Table 5: The character table of $3^2:8$

<table>
<thead>
<tr>
<th>$(C_2 	imes 2)(b)$</th>
<th>2 4 6 8 10 12 14 16 18 20</th>
<th>2 4 6 8 10 12 14 16 18 20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>1 1 1 1 1 1 1 1 1 1</td>
<td>1 1 1 1 1 1 1 1 1 1</td>
</tr>
<tr>
<td>$x_2$</td>
<td>1 1 1 1 1 1 1 1 1 1</td>
<td>1 1 1 1 1 1 1 1 1 1</td>
</tr>
<tr>
<td>$x_3$</td>
<td>1 1 1 1 1 1 1 1 1 1</td>
<td>1 1 1 1 1 1 1 1 1 1</td>
</tr>
<tr>
<td>$x_4$</td>
<td>1 1 1 1 1 1 1 1 1 1</td>
<td>1 1 1 1 1 1 1 1 1 1</td>
</tr>
<tr>
<td>$x_5$</td>
<td>1 1 1 1 1 1 1 1 1 1</td>
<td>1 1 1 1 1 1 1 1 1 1</td>
</tr>
<tr>
<td>$x_6$</td>
<td>1 1 1 1 1 1 1 1 1 1</td>
<td>1 1 1 1 1 1 1 1 1 1</td>
</tr>
<tr>
<td>$x_7$</td>
<td>1 1 1 1 1 1 1 1 1 1</td>
<td>1 1 1 1 1 1 1 1 1 1</td>
</tr>
<tr>
<td>$x_8$</td>
<td>1 1 1 1 1 1 1 1 1 1</td>
<td>1 1 1 1 1 1 1 1 1 1</td>
</tr>
<tr>
<td>$x_9$</td>
<td>1 1 1 1 1 1 1 1 1 1</td>
<td>1 1 1 1 1 1 1 1 1 1</td>
</tr>
</tbody>
</table>

where in Table 5, $A = -i$ and $B = -E(8)$.

Table 6: The character table of $(S_3 \times S_3):2$

<table>
<thead>
<tr>
<th>$(C_{(S_3 \times S_3):2}(g))$</th>
<th>1 2 4 6 8 10 12 14 16 18 20</th>
<th>1 2 4 6 8 10 12 14 16 18 20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>1 1 1 1 1 1 1 1 1 1 1</td>
<td>1 1 1 1 1 1 1 1 1 1 1</td>
</tr>
<tr>
<td>$x_2$</td>
<td>1 1 1 1 1 1 1 1 1 1 1</td>
<td>1 1 1 1 1 1 1 1 1 1 1</td>
</tr>
<tr>
<td>$x_3$</td>
<td>1 1 1 1 1 1 1 1 1 1 1</td>
<td>1 1 1 1 1 1 1 1 1 1 1</td>
</tr>
<tr>
<td>$x_4$</td>
<td>1 1 1 1 1 1 1 1 1 1 1</td>
<td>1 1 1 1 1 1 1 1 1 1 1</td>
</tr>
<tr>
<td>$x_5$</td>
<td>1 1 1 1 1 1 1 1 1 1 1</td>
<td>1 1 1 1 1 1 1 1 1 1 1</td>
</tr>
<tr>
<td>$x_6$</td>
<td>1 1 1 1 1 1 1 1 1 1 1</td>
<td>1 1 1 1 1 1 1 1 1 1 1</td>
</tr>
<tr>
<td>$x_7$</td>
<td>1 1 1 1 1 1 1 1 1 1 1</td>
<td>1 1 1 1 1 1 1 1 1 1 1</td>
</tr>
<tr>
<td>$x_8$</td>
<td>1 1 1 1 1 1 1 1 1 1 1</td>
<td>1 1 1 1 1 1 1 1 1 1 1</td>
</tr>
<tr>
<td>$x_9$</td>
<td>1 1 1 1 1 1 1 1 1 1 1</td>
<td>1 1 1 1 1 1 1 1 1 1 1</td>
</tr>
</tbody>
</table>

Proof. By Table 1 we can see that the class $g_{10} \in [6B]_{2 \times M_{11}}$ affords 2 conjugacy classes in $G$. This implies that $c(g_{10}) = 2$ and hence the corresponding Fischer matrix $F_{10}$ to this class will be of size 2. By description of Fischer matrices (for example see [2]), the first row of $F_{10}$ will correspond to the first inertia factor group $H_1 = 2 \times M_{11}$. Thus the second row of $F_{10}$ must correspond to either $H_2$ or $H_3$ depending on the fusions. From Tables 4 and 5 we can see that neither $H_2$ nor $3^2:8$ has an element of order 6 so that it fuses into $[6B]_{2 \times M_{11}}$. This argument shows that $H_3 \neq (S_3 \times S_3):2$. Since by Proposition 3.1, $H_3 \neq (S_3 \times S_3):2$; it follows that $H_3 = (S_3 \times S_3):2$. Hence the result.

We summarize that the inertia factor groups are $H_1 = G = 2 \times M_{11}$, $H_2 = A_6:2$ (non-split) and $H_3 = (S_3 \times S_3):2$.

Next we turn to determine the fusions of classes of the inertia factor groups of the extension into the classes of $G = 2 \times M_{11}$. We have used the permutation characters of $G$ on the inertia factor groups and the centralizer sizes to determine the fusions of these inertia factors into $G$. We list these fusions in Tables 7 and 8.

Table 7: The fusion of $H_2$ into $2 \times M_{11}$

| $(g)|H_2$ | $[x]|2 \times M_{11}$ | $(g)|H_2$ | $[x]|2 \times M_{11}$ |
|----------|---------------------|----------|---------------------|
| 1a       | 1A                  | 4b       | 4B                  |
| 2a       | 2B                  | 5a       | 5A                  |
| 3a       | 3A                  | 8a       | 8A                  |
| 4a       | 4A                  | 8b       | 8B                  |
Table 8: The fusion of $H_3$ into $2 \times M_{11}$

<table>
<thead>
<tr>
<th>$g$</th>
<th>$H_3$</th>
<th>$x \rightarrow 2 \times M_{11}$</th>
<th>$g$</th>
<th>$H_3$</th>
<th>$x \rightarrow 2 \times M_{11}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1a</td>
<td>1A</td>
<td>3b</td>
<td>3A</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2a</td>
<td>2B</td>
<td>4a</td>
<td>4B</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2b</td>
<td>2B</td>
<td>6a</td>
<td>6B</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2c</td>
<td>2C</td>
<td>6B</td>
<td>6C</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3a</td>
<td>3A</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4. Fischer matrices of $\overline{G}$

In this section, we use the arithmetical properties of Fischer matrices, given in Proposition 3.6 of [3], to calculate some of the entries of the Fischer matrices and also to build an algebraic system of equations. To build these systems of equations, we firstly recall that we label the top and bottom of the columns of the Fischer matrix $F_i$, corresponding to $g_i$, by the sizes of the centralizers of $g_{ij}, 1 \leq j \leq c(g_i)$ in $\overline{G}$ and $m_{ij}$ respectively. In Table 1 we supplied $|C_{\overline{G}}(g_{ij})|$ and $m_{ij}, 1 \leq i \leq 20, 1 \leq j \leq c(g_i)$. Also having obtained the fusions of the inertia factor groups into $2 \times M_{11}$, we are able to label the rows of the Fischer matrices as described in [2, 3].

Since the size of the Fischer matrix $F_i$ is $c(g_i)$, it follows from Table 1 that the sizes of the Fischer matrices of $\overline{G}$ range between 1 and 4 for all $i \in \{1, 2, \cdots, 20\}$. Now with the help of the symbolic mathematical package Maxima [17], we were able to solve the systems of equations and hence we have computed all the Fischer matrices of $\overline{G}$, where we found that all these matrices are integer valued. Below we list these matrices.
5. The character tables of $\overline{G}$

Throughout Sections 2, 3 and 4 we have found:

- the conjugacy classes of $\overline{G}$ (Table 1),
- the inertia factor groups $H_1$, $H_2$ and $H_3$ of $\overline{G}$ and their character tables (Tables 3, 4 and 6). Also we obtained the fusions of classes of the inertia factors $H_2$ and $H_3$ of $\overline{G}$ into $2 \times M_{11}$ (Tables 7 and 8),
- the Fischer matrices of $\overline{G}$ (Section 4).

It follows by [2, 3] that the full character table of $\overline{G}$ can be constructed easily in the format of Clifford-Fischer theory. This table will be partitioned into 60 parts corresponding to the 20 cosets and the three inertia factor groups. The full character table of $\overline{G}$ is $37 \times 37$ $\mathbb{C}$-valued matrix. In Table 9, we supply the character table of $\overline{G}$ in the format of Clifford-Fischer Theory. In this table we have also included the fusions of the conjugacy classes of $\overline{G}$ into the conjugacy classes of the Conway group $Co_3$, where the classes of $Co_3$ as in the Atlas. Finally we would like to remark that the accuracy of this character table has been tested using GAP.

Table 9: The character table of $\overline{G}$

<table>
<thead>
<tr>
<th>$\overline{G}$</th>
<th>$1$</th>
<th>$A$</th>
<th>$2A$</th>
<th>$2B$</th>
<th>$2C$</th>
<th>$3A$</th>
<th>$4A$</th>
<th>$4D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\overline{H}_1$</td>
<td>1 little</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\overline{H}_2$</td>
<td>1 A</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\overline{H}_3$</td>
<td>1 B</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

The character table is partitioned into 60 parts corresponding to the 20 cosets and the three inertia factor groups.
where in Table 9, $A = -i\sqrt{2}$, $B = 2i\sqrt{2}$ and $C = \frac{1}{2} - i\frac{\sqrt{11}}{2} = b11$.

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**References**


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