On a maximal subgroup of the Conway group Co_3

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Abstract. This paper is dealing with a split extension group of the form $3^5{:}(2\times M_{11})$, which is a maximal subgroup of the Conway simple group Co_3. We refer to this extension by $\overline{G}$. We firstly determine the conjugacy classes of $\overline{G}$ using the coset analysis technique. The structures of inertia factor groups were determined through deep investigation on the maximal subgroups of the maximal subgroups of $2 \times M_{11}$. We found the inertia factors to be the groups $2 \times M_{11}$, $A_6^{\cdot}2$ (non-split) and $(S_3 \times S_3){:}2$. We then determine the Fischer matrices of $\overline{G}$ and apply the Clifford-Fischer theory to compute the ordinary character table of this group. The Fischer matrices of $\overline{G}$ are all integer valued, with sizes ranging from 1 to 4. The full character table of $\overline{G}$ is 37×37 complex valued matrix and is given at the end of this paper.

Keywords: group extensions, Matheiu group, inertia groups, Fischer matrices, character table.

1. Introduction

the Conway group Co_3 is of order 495 766 656 000. From the Atlas [15] we can see that Co_3 has 14 conjugacy classes of maximal subgroups. The fifth largest maximal subgroup is a group of the form $3^5{:}(2 \times M_{11})$. We refer to this group by $\overline{G}$ and clearly it has order $243 \times 2 \times |M_{11}| = 3\ 849\ 120$ and index 128 800 in

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Co_3. In fact the normalizer of an elementary abelian group 3^5 in Co_3 will have the form of $\overline{G}$; that is $N_{Co_3}(3^5) = 3^5{:}(2 \times M_{11})$. Using GAP [16] we were able to construct this split extension group in terms of permutations of 33 points. We used the coset analysis technique to construct the conjugacy classes of $\overline{G}$, where correspond to the 20 classes of $2 \times M_{11}$, we obtained 37 classes of $\overline{G}$. Then by looking on the maximal subgroups of the maximal subgroups of $2 \times M_{11}$, we were able to determine the structures of the inertia factor groups. Then we computed the Fischer matrices of the extension and we found to be integer valued matrices with sizes ranging from 1 to 4. Finally we were able to compute the ordinary character table of $\overline{G}$ using Clifford-Fischer theory and we supplied it at the end of this paper.

The character table of any finite group extension $\overline{G} = N{\cdot}G$ (here N is the kernel of the extension and G is isomorphic to $\overline{G}/N$) produced by Clifford-Fischer Theory is in a special format that could not be achieved by direct computations using GAP or Magma [14]. Also there is an interesting interplay between the coset analysis and Clifford-Fischer Theory. Indeed the size of each Fischer matrix is $c(g_i)$, the number of $\overline{G}$-classes corresponding to $[g_i]_G$ obtained via the coset analysis technique. That is computations of the conjugacy classes of $\overline{G}$ using the coset analysis technique will determine the sizes of all Fischer matrices.

For the notation used in this paper and the description of Clifford-Fischer theory technique, we follow [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13].

By the Atlas of Wilson [25], we see that Co_3 has an absolutely irreducible module of dimension 22 over $\mathbb{F}_2$. Using the two 22×22 matrices over $\mathbb{F}_2$ that generate Co_3 together with the program for obtaining maximal subgroups of Co_3 given in [25] we were able to locate our group $\overline{G}$ inside Co_3 in terms of 22×22 matrices over $\mathbb{F}_2$. The following two elements a and b generate $\overline{G}$.

$$a = \begin{pmatrix}
0&1&1&0&1&0&0&1&1&0&1&0&1&0&0&0&1&0&0&0&1&0\\
0&0&1&1&1&0&0&1&1&0&1&1&1&1&1&1&0&1&1&0&1&0\\
0&1&0&0&1&0&0&1&1&0&1&0&1&0&0&0&1&0&0&0&1&0\\
0&1&0&1&1&0&0&1&1&0&1&0&1&0&0&0&1&0&0&0&1&0\\
0&1&0&0&0&0&1&0&0&0&0&0&0&0&1&0&0&0&0&0&0&1\\
1&0&1&0&0&1&1&0&0&0&0&1&1&1&0&0&1&0&1&1&1&1\\
0&0&1&1&1&0&0&1&1&1&1&1&1&1&0&1&0&1&1&1&1&1\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&1&0&1&1&0&0&1\\
1&0&0&1&0&1&0&0&1&0&0&0&1&0&0&0&1&1&1&0&0&0\\
0&1&1&0&0&0&1&1&0&0&0&0&0&0&0&0&1&1&0&0&0&1\\
0&0&1&0&0&1&1&0&0&1&1&0&0&1&0&1&1&0&1&1&1&0\\
1&0&1&1&1&1&0&0&0&1&0&1&1&0&0&0&1&1&1&1&0&0\\
0&0&1&0&0&0&0&0&1&1&1&1&1&0&1&1&0&1&0&1&0&0\\
1&0&1&1&1&1&0&1&0&0&0&1&1&0&0&1&1&0&1&1&0&0\\
0&1&0&0&1&0&0&1&1&0&1&0&1&0&0&0&1&0&0&0&1&1\\
0&1&0&1&0&0&1&1&1&1&1&0&0&0&1&0&1&1&0&0&0&1\\
0&1&0&0&1&0&0&1&1&0&1&0&1&0&0&0&0&0&0&0&1&0\\
0&0&0&1&0&0&0&0&1&0&1&0&0&0&0&1&0&1&0&0&0&1\\
0&1&1&0&0&0&1&0&0&0&0&1&0&1&0&1&1&0&1&0&0&1\\
1&1&0&1&1&0&0&1&0&0&0&0&1&0&0&1&1&0&0&1&1&1\\
0&1&0&0&1&0&0&1&1&0&1&0&1&0&0&0&1&0&0&0&0&0\\
1&0&1&1&0&0&0&0&1&1&1&0&0&1&0&0&1&0&1&1&0&1
\end{pmatrix},$$

$$b = \begin{pmatrix}
0&0&1&0&0&1&0&0&0&1&1&0&0&0&1&0&1&0&1&0&0&1\\
0&1&0&1&1&0&0&1&1&0&0&1&0&0&0&0&0&0&0&1&0&0\\
1&1&0&1&0&1&0&0&0&1&1&0&0&0&0&0&0&1&0&0&0&1\\
0&1&0&1&0&1&0&1&1&1&1&0&1&1&0&0&0&0&1&1&0&0\\
0&1&0&1&1&0&1&1&0&0&0&1&0&1&0&0&0&0&1&1&1&0\\
1&0&1&1&1&1&0&1&1&0&1&0&0&0&0&1&0&0&1&0&0&0\\
0&1&1&0&1&0&1&1&0&1&0&1&0&0&1&1&1&1&0&1&1&0\\
0&1&1&0&1&0&0&1&0&1&0&0&0&0&1&1&1&1&0&1&1&0\\
0&0&1&1&1&0&0&1&1&0&0&1&0&0&0&1&0&1&0&1&0&0\\
0&1&0&0&0&0&0&1&0&1&1&1&1&1&1&0&1&0&1&0&0&1\\
0&1&1&0&1&0&0&1&0&1&1&1&0&0&1&1&1&1&0&1&1&0\\
0&1&1&0&0&0&1&0&1&1&1&1&1&0&0&1&0&1&0&1&0&1\\
0&0&1&0&0&0&1&0&0&1&1&1&1&1&1&1&0&1&1&0&1&0\\
0&1&1&0&1&0&0&1&0&1&0&1&0&1&1&1&1&1&0&1&1&0\\
0&0&1&1&0&1&0&0&1&1&0&0&0&1&1&1&1&1&0&0&1&1\\
0&1&1&0&1&0&0&1&1&1&0&1&0&0&1&1&1&1&0&1&1&0\\
0&0&1&0&1&1&1&1&1&0&0&1&1&0&0&0&0&0&1&0&1&0\\
0&1&1&1&1&0&1&1&0&0&1&1&1&0&0&1&0&1&1&0&1&1\\
1&1&1&1&0&1&0&1&0&1&0&0&1&0&1&1&0&0&1&1&1&0\\
1&1&1&0&1&1&1&1&0&1&0&0&0&1&1&0&0&0&0&0&0&1\\
0&0&0&0&0&1&1&0&1&0&0&1&1&1&1&0&1&1&1&0&1&0\\
0&1&0&1&1&1&1&0&0&0&0&0&1&0&1&1&0&0&0&0&1&1
\end{pmatrix},$$

where $o(a) = 4$, $o(b) = 3$ and $o(ab) = 22$.

For the sake of computations, we used few GAP commands to convert the representation of our group $\overline{G}$ from matrix into permutation representation, where we were able to represent $\overline{G}$ in terms of the set $\{1, 2, \cdots, 33\}$. Thus in terms of permutations and using $(c_1, c_2, \cdots, c_k)$ to denote the k-cycle $(c_1\ c_2\ \cdots\ c_k)$, the group $\overline{G}$ is generated by $\overline{g}_1$ and $\overline{g}_2$, where

$$\begin{aligned}
\overline{g}_1 &= (1,4,23,29)(2,15,19,10)(3,16,24,33)(6,25,26,31)(7,12,17,21)\\
&\quad (8,9,30,32)(13,20),\\
\overline{g}_2 &= (1,3,12)(2,22,18)(4,25,32)(5,11,21)(6,23,13)(7,9,28)(8,19,33)\\
&\quad (14,27,31)(15,16,20).
\end{aligned}$$

Since $\overline{G}$ can be constructed in GAP, it is easy to obtain all its maximal subgroups. In fact $\overline{G}$ has 7 conjugacy classes of maximal subgroups. The second largest maximal subgroup of $\overline{G}$ is a group of the form $(3^4{:}(A_6^{\cdot}2)){:}S_3 := T$, with order 349920 and index 11 in $\overline{G}$. The group T itself has also 7 conjugacy classes of maximal subgroups and the second largest maximal subgroup of T is a group of the form $3^4{:}(2 \times (A_6^{\cdot}2)) \cong 3^4{:}(2 \times M_{10}) := M$. The order of M is 116640 and its index in T is 3 (and hence has index 33 in $\overline{G}$). In fact M gives rise to the above permutation representation of $\overline{G}$ on 33 points.

Remark 1. From the Atlas of Wilson [25] we see that Co_3 can be generated in terms of permutations of 276 points and thus any maximal subgroup (including our group $\overline{G}$) will be generated in terms of permutations acting on 276 points. However starting with the 22-dimensional matrix representation over $\mathbb{F}_2$ for Co_3 (available in [25]), then generating $\overline{G}$ in terms of 22×22 matrices over $\mathbb{F}_2$ (for example the above generators a and b) and then using few GAP commands, we were able to generate $\overline{G}$ in terms of permutations on only 33 points. Indeed this will work faster for the required computations.

Having $\overline{G}$ being constructed in GAP, it is easy to obtain all its normal subgroups. In fact $\overline{G}$ possesses three proper normal subgroups of orders 243, 486 and 1924560. The normal subgroup of order 243 is an elementary abelian group isomorphic to N.

In GAP one can check for the complements of N in $\overline{G}$, where in our case we obtained only one complement isomorphic to $2 \times M_{11}$ and together with N gives the split extension in consideration.

2. The conjugacy classes of $\overline{G}$

In this section we compute the conjugacy classes of the group $\overline{G}$ using the coset analysis technique (see Basheer [2], Basheer and Moori [3, 4, 6] or Moori [18] and [19] for more details) as we are interested to organize the classes of $\overline{G}$ corresponding to the classes of $2\times M_{11}$. Firstly note that M_{11} has 10 conjugacy classes (see the Atlas) and thus $2 \times M_{11}$ will have 20 conjugacy classes. Corresponding to these 20 classes of $2 \times M_{11}$, we obtained 37 classes in $\overline{G}$.

In Table 1, we list the conjugacy classes of $\overline{G}$, where in this table:

• k_i is the number of orbits $Q_{i1}, Q_{i2}, \cdots, Q_{ik_i}$ for the action of N on the coset $N\overline{g}_i = Ng_i$, where g_i is a representative of a class of the complement ($\cong 2 \times M_{22}$) of N in $\overline{G}$. In particular, the action of N on the identity coset N produces 243 orbits each consists of singleton. Thus for $\overline{G}$, we have $k_1 = 243$.

• f_{ij} is the number of orbits fused together under the action of $C_G(g_i)$ on $Q_1, Q_2, \cdots, Q_k$. In particular, the action of $C_G(1_G) = G$ on the orbits $Q_1, Q_2, \cdots, Q_k$ affords three orbits of lengths 1, 110 and 132 (with corresponding point stabilizers $2 \times M_{11}$, $3^2{:}QD_{16}$ and S_5, where QD_{16} is the quasi-dihedral group of order 16. Thus $f_{11} = 1$, $f_{12} = 110$ and $f_{13} = 132$.

• m_{ij}'s are weights (attached to each class of $\overline{G}$) that will be used later in computing the Fischer matrices of $\overline{G}$. These weights are computed through the formula

$$m_{ij} = [N_{\overline{G}}(N\overline{g}_i) : C_{\overline{G}}(g_{ij})] = |N|\frac{|C_G(g_i)|}{|C_{\overline{G}}(g_{ij})|}, \qquad (1)$$

where N is the kernel of an extension $\overline{G}$ that is in consideration.

Table 1: The conjugacy classes of $\overline{G}$

$[g_i]_G$	k_i	f_{ij}	m_{ij}	$[g_{ij}]_{\overline{G}}$	$o(g_{ij})$	$\|[g_{ij}]_{\overline{G}}\|$	$\|C_{\overline{G}}(g_{ij})\|$
		$f_{11} = 1$	$m_{11} = 1$	g_{11}	1	1	3849120
$g_1 = 1A$	$k_1 = 243$	$f_{12} = 110$	$m_{12} = 110$	g_{12}	3	110	34992
		$f_{13} = 132$	$m_{13} = 132$	g_{13}	3	132	29160
$g_2 = 2A$	$k_2 = 1$	$f_{22} = 1$	$m_{22} = 243$	g_{22}	2	243	15840
		$f_{31} = 1$	$m_{31} = 9$	g_{31}	2	1485	2592
$g_3 = 2B$	$k_3 = 27$	$f_{32} = 6$	$m_{32} = 54$	g_{32}	6	8910	432
		$f_{33} = 8$	$m_{33} = 72$	g_{33}	6	11880	324
		$f_{34} = 12$	$m_{34} = 108$	g_{34}	6	17820	216
$g_4 = 2C$	$k_4 = 9$	$f_{41} = 1$	$m_{41} = 27$	g_{41}	2	4455	864
		$f_{42} = 8$	$m_{42} = 216$	g_{42}	6	35640	108

continued on next page

Table 1 (continued from previous page)

$[g_i]_G$	k_i	f_{ij}	m_{ij}	$[g_{ij}]_{\overline{G}}$	$o(g_{ij})$	$\lvert[g_{ij}]_{\overline{G}}\rvert$	$\lvert C_{\overline{G}}(g_{ij})\rvert$
		$f_{51} = 1$	$m_{51} = 27$	g_{51}	3	11880	324
$g_5 = 3A$	$k_5 = 9$	$f_{52} = 2$	$m_{52} = 54$	g_{52}	3	23760	162
		$f_{53} = 2$	$m_{53} = 54$	g_{53}	9	23760	162
		$f_{54} = 4$	$m_{54} = 108$	g_{54}	9	47520	81
$g_6 = 4A$	$k_6 = 3$	$f_{61} = 1$	$m_{61} = 81$	g_{61}	4	80190	48
		$f_{62} = 2$	$m_{62} = 162$	g_{62}	12	160380	24
		$f_{71} = 1$	$m_{71} = 27$	g_{71}	4	26730	144
$g_7 = 4B$	$k_7 = 9$	$f_{72} = 4$	$m_{72} = 108$	g_{72}	12	106920	36
		$f_{73} = 4$	$m_{73} = 108$	g_{73}	12	106920	36
$g_8 = 5A$	$k_8 = 3$	$f_{81} = 1$	$m_{81} = 81$	g_{81}	5	128304	30
		$f_{82} = 2$	$m_{82} = 162$	g_{82}	15	256608	15
$g_9 = 6A$	$k_9 = 1$	$f_{91} = 1$	$m_{91} = 243$	g_{91}	6	106920	36
$g_{10} = 6B$	$k_{10} = 3$	$f_{10,1} = 1$	$m_{10,1} = 81$	$g_{10,1}$	6	106920	36
		$f_{10,2} = 2$	$m_{10,2} = 162$	$g_{10,2}$	18	213840	18
$g_{11} = 6C$	$k_{11} = 3$	$f_{11,1} = 1$	$m_{11,1} = 81$	$g_{11,1}$	6	106920	36
		$f_{11,2} = 2$	$m_{11,2} = 162$	$g_{11,2}$	6	213840	18
$g_{12} = 8A$	$k_{12} = 3$	$f_{12,1} = 1$	$m_{12,1} = 81$	$g_{12,1}$	8	80190	48
		$f_{12,2} = 2$	$m_{12,2} = 162$	$g_{12,2}$	24	160380	24
$g_{13} = 8B$	$k_{13} = 3$	$f_{13,1} = 1$	$m_{13,1} = 81$	$g_{13,1}$	8	80190	48
		$f_{13,2} = 2$	$m_{13,2} = 162$	$g_{13,2}$	24	160380	24
$g_{14} = 8C$	$k_{14} = 1$	$f_{14,1} = 1$	$m_{14,1} = 243$	$g_{14,1}$	8	240570	16
$g_{15} = 8D$	$k_{15} = 1$	$f_{15,1} = 1$	$m_{15,1} = 243$	$g_{15,1}$	8	240570	16
$g_{16} = 10A$	$k_{16} = 1$	$f_{16,1} = 1$	$m_{16,1} = 243$	$g_{16,1}$	10	384912	10
$g_{17} = 11A$	$k_{17} = 1$	$f_{17,1} = 1$	$m_{17,1} = 243$	$g_{17,1}$	11	349920	22
$g_{18} = 11B$	$k_{18} = 1$	$f_{18,1} = 1$	$m_{18,1} = 243$	$g_{18,1}$	11	349920	22
$g_{19} = 22A$	$k_{19} = 1$	$f_{19,1} = 1$	$m_{19,1} = 243$	$g_{19,1}$	22	349920	22
$g_{20} = 22B$	$k_{20} = 1$	$f_{20,1} = 1$	$m_{20,1} = 243$	$g_{20,1}$	22	349920	22

3. The inertia factor groups of $\overline{G}$

We recall that knowledge of the appropriate character tables of inertia factor groups is crucial in calculating the full character table of any group extension. Since in our extension $\overline{G}$, the normal subgroup 3^5 is abelian and the extension splits, it follows by applications of Mackey's Theorem (see for example Theorem 3.3.4 of Whitley [24]), that every character of 3^5 is extendible to an ordinary character of its respective inertia group $\overline{H}_k$. Thus all the character tables of the inertia factor groups that we will use to construct the character tables of $\overline{G}$ are the ordinary ones. Next we determine the structures of the inertia factor groups.

We have seen from Section 2 that the action of $\overline{G} = 3^5{:}(2 \times M_{11})$ (which can be reduced to the action of $2 \times M_{11}$) on the classes of $N = 3^5$ yielded three orbits of lengths 1, 110 and 132 (and the corresponding point stabilizers were $2 \times M_{11}$, $3^2{:}QD_{16}$ and S_5). By a theorem of Brauer (see for example Theorem 5.1.5 of

Mpono [22]), it follows that the action of $\overline{G}$ on $\mathrm{Irr}(N)$ will also produce three orbits. We used Programme C of [23], which can also be found in [20, 21], to determine the lengths of the orbits of $\overline{G}$ or just $2\times M_{11}$ on $\mathrm{Irr}(N)$. We found that the action of $2 \times M_{11}$ on $\mathrm{Irr}(N)$ produces three orbit of lengths 1, 22 and 220. Let H_1, H_2 and H_3 be the respective inertia factor groups of the representatives of characters from the orbits with previous lengths. We notice that these inertia factors have indices 1, 22 and 220 respectively in $2 \times M_{11}$. Now since M_{11} has 5 maximal subgroups, the group $2 \times M_{11}$ will have 6 maximal subgroups, namely M_{11} itself together with the direct product of each maximal subgroup of M_{11} by $\mathbb{Z}_2$. Let $T_1, T_2, \cdots, T_6$ be representatives of the conjugacy classes of the maximal subgroups of $2 \times M_{11}$. In Table 2, we give few information on these maximal subgroups.

Table 2: The maximal subgroups of $2 \times M_{11}$

T_i	$\lvert T_i\rvert$	$[2 \times M_{11} : T_i]$
M_{11}	7920	2
$2 \times M_{10} \cong 2 \times (A_6^{\cdot}2)$	1440	11
$2 \times PSL(2,11)$	1320	12
$2 \times (3^2{:}QD_{16})$	288	55
$2 \times S_5$	240	66
$2 \times GL(2,3)$	96	165

Now the first inertia factor group H_1 of $2 \times M_{11}$ has an index 1 and thus $H_1 = 2 \times M_{11}$ itself. Since we have the character table of M_{11} (see the Atlas) we can easily construct the character table of $2 \times M_{11}$, which we supply below as Table 3.

The second inertia factor group H_2 has index 22 in $2 \times M_{11}$. From Table 2 we can see that the only index of a maximal subgroup that divides 22 is 11. It follows that H_2 is an index 2 subgroup of $2 \times M_{10} \cong 2 \times (A_6^{\cdot}2)$. Now the group $2 \times (A_6^{\cdot}2)$ has 6 conjugacy classes of maximal subgroups, each class represented by $A_6^{\cdot}2$ (twice), $2 \times A_6$, $2 \times (3^2{:}Q_8)$, $2 \times (5{:}4)$ and $2 \times QD_{16}$ with respective orders 720 (3 times), 144, 40 and 32, where Q_8 and QD_{16} are the quaternion and quasi-dihedral groups of orders 8 and 16 respectively. Therefore we can see that

$$H_2 \in \{A_6^{\cdot}2, 2 \times A_6\}. \tag{2}$$

Using GAP we were able to construct the character tables of $A_6^{\cdot}2$ and $2\times A_6$, where below in Table 4 we only supply the character table of $A_6^{\cdot}2$ as the character table of $2 \times A_6$ can be constructed easily from the character tables of $\mathbb{Z}_2$ and of A_6 (available in the Atlas).

Table 3: The character table of $H_1 = G = 2 \times M_{11}$

	$1A$	$2A$	$2B$	$2C$	$3A$	$4A$	$4B$	$5A$	$6A$	$6B$	$6C$	$8A$	$8B$	$8C$	$8D$	$10A$	$11A$	$11B$	$22A$	$22B$
$\lvert C_{H_1}(h)\rvert$	15840	15840	96	96	36	16	16	10	36	12	12	16	16	16	16	10	22	22	22	22
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	−1	1	−1	1	1	−1	1	−1	1	−1	−1	1	1	−1	−1	1	1	−1	−1
χ_3	10	10	2	2	1	2	2	0	1	−1	−1	0	0	0	0	0	−1	−1	−1	−1
χ_4	10	−10	2	−2	1	2	−2	0	−1	−1	1	0	0	0	0	0	−1	−1	1	1
χ_5	10	−10	−2	2	1	0	0	0	−1	1	−1	A	$-A$	A	$-A$	0	−1	−1	1	1
χ_6	10	−10	−2	2	1	0	0	0	−1	1	−1	$-A$	A	$-A$	A	0	−1	−1	1	1
χ_7	10	10	−2	−2	1	0	0	0	1	1	1	A	A	$-A$	$-A$	0	−1	−1	−1	−1
χ_8	10	10	−2	−2	1	0	0	0	1	1	1	$-A$	$-A$	A	A	0	−1	−1	−1	−1
χ_9	11	11	3	3	2	−1	−1	1	2	0	0	−1	−1	−1	−1	1	0	0	0	0
χ_{10}	11	−11	3	−3	2	−1	1	1	−2	0	0	1	−1	−1	1	−1	0	0	0	0
χ_{11}	16	−16	0	0	−2	0	0	1	2	0	0	0	0	0	0	−1	$-B$	$-\overline{B}$	B	$\overline{B}$
χ_{12}	16	−16	0	0	−2	0	0	1	2	0	0	0	0	0	0	−1	$-\overline{B}$	$-B$	$\overline{B}$	B
χ_{13}	16	16	0	0	−2	0	0	1	−2	0	0	0	0	0	0	1	$-\overline{B}$	$-B$	$-\overline{B}$	$-B$
χ_{14}	16	16	0	0	−2	0	0	1	−2	0	0	0	0	0	0	1	$-B$	$-\overline{B}$	$-B$	$-\overline{B}$
χ_{15}	44	44	4	4	−1	0	0	−1	−1	1	1	0	0	0	0	−1	0	0	0	0
χ_{16}	44	−44	4	−4	−1	0	0	−1	1	1	−1	0	0	0	0	1	0	0	0	0
χ_{17}	45	45	−3	−3	0	1	1	0	0	0	0	−1	−1	−1	−1	0	1	1	1	1
χ_{18}	45	−45	−3	3	0	1	−1	0	0	0	0	1	−1	−1	1	0	1	1	−1	−1
χ_{19}	55	55	−1	−1	1	−1	−1	0	1	−1	−1	1	1	1	1	0	0	0	0	0
χ_{20}	55	−55	−1	1	1	−1	1	0	−1	−1	1	−1	1	1	−1	0	0	0	0	0

where in Table 3, $A = -i\sqrt{2}$ and $B = \frac{1}{2} - i\frac{\sqrt{11}}{2}$.

Table 4: The character table of $A_6^{\cdot}2$

	$1a$	$2a$	$3a$	$4a$	$4b$	$5a$	$8a$	$8b$
$\lvert C_{H_2}(h)\rvert$	720	16	9	8	4	5	8	8
χ_1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	−1	1	−1	−1
χ_3	9	1	0	1	−1	−1	1	1
χ_4	9	1	0	1	1	−1	−1	−1
χ_5	10	2	1	−2	0	0	0	0
χ_6	10	−2	1	0	0	0	$-i\sqrt{2}$	$i\sqrt{2}$
χ_7	10	−2	1	0	0	0	$i\sqrt{2}$	$-i\sqrt{2}$
χ_8	16	0	−2	0	0	1	0	0

We keep the information that $|\mathrm{Irr}(A_6^{\cdot}2)| = 8$ and $|\mathrm{Irr}(2 \times A_6)| = 14$ in mind and we continue to look at the third inertia factor group H_3, where later we will determine both H_2 and H_3 simultaneously.

Now the third inertia factor group H_3 has index 220 in $2 \times M_{11}$. Again from Table 2 we can see that the only indices of maximal subgroups that divides 220 are 2, 11 and 55. Therefore H_3 is either

- index 110 subgroup of M_{11},
- index 20 subgroup of $2 \times (A_6^{\cdot}2)$ or
- index 4 subgroup of $2 \times (3^2{:}QD_{16})$.

Now this requires more analysis on the structures of the subgroups of M_{11}, $2 \times (A_6^{\cdot}2)$ and $2 \times (3^2{:}QD_{16})$. We consider each case of the above:

• From the Atlas we can see that an index 110 subgroup of M_{11} must be either an index 10 subgroup of $M_{10} \cong A_6^{\cdot}2$, or an index 2 subgroup of $3^2{:}QD_{16}$. Now $A_6^{\cdot}2$ has four maximal subgroups, namely A_6, $3^2{:}Q_8$, $5{:}4$ and QD_{16} of respective orders 360, 72, 20 and 16. Thus H_3 is either a subgroup of A_6 with index 5 or is isomorphic to $3^2{:}Q_8$. However the group A_6 does not contain a subgroup of order 72 (the largest maximal subgroup of A_6 is A_5, which is of order 60). Therefore a subgroup of $A_6^{\cdot}2$ of index 10 must necessarily be isomorphic to $3^2{:}Q_8$.

From another side the group $3^2{:}QD_{16}$ has four maximal subgroups, namely $3^2{:}Q_8$, $3^2{:}8$, $(S_3 \times S_3){:}2$ and QD_{16} with respective orders 72, 72, 72 and 16. Therefore an index 2 subgroup of $3^2{:}QD_{16}$ must necessarily be isomorphic to one of the first three maximal subgroups; that is either $3^2{:}Q_8$, $3^2{:}8$ or $(S_3 \times S_3){:}2$. Now we conclude this case by saying that if H_3 is an index 110 subgroup of M_{11}, then $H_3 \in \{3^2{:}Q_8,\ 3^2{:}8,\ (S_3 \times S_3){:}2\}$.

• We now consider the case that H_3 is an index 20 subgroup of $2 \times (A_6^{\cdot}2)$. The $2 \times (A_6^{\cdot}2)$ has 6 conjugacy classes of maximal subgroups, each class represented by $A_6^{\cdot}2$ (2 isomorphic non-conjugate copies), $2 \times A_6$, $2 \times (3^2{:}Q_8)$, $2 \times (5{:}4)$ and $2 \times QD_{16}$. Because of the order, only the first four maximal subgroups ($A_6^{\cdot}2$, $A_6^{\cdot}2$, $2 \times A_6$ and $2 \times (3^2{:}Q_8)$) can contain H_3 with respective indices 10, 10, 10 and 2. As of the above case, if H_3 is an index 10 in $A_6^{\cdot}2$ (any of the non-conjugate copies), then H_3 will be isomorphic to a subgroup of the form $3^2{:}Q_8$.

The group $2 \times A_6$ has 6 conjugacy classes of maximal subgroups, each class represented by A_6, $2 \times A_5$ (2 isomorphic non-conjugate copies), $2 \times (3^2{:}4)$ and $2 \times S_4$ (2 isomorphic non-conjugate copies) with respective orders 360, 120 (twice), 72, 48 (twice). We know that A_6 has no index 5 subgroup (see the Atlas). Therefore if H_3 is a subgroup of $2 \times A_6$, then it must be isomorphic to $2 \times (3^2{:}4)$.

The group $2 \times (3^2{:}Q_8)$ has 8 conjugacy classes of maximal subgroups, each class represented by $2 \times (3^2{:}4)$ (3 isomorphic non-conjugate copies), $3^2{:}Q_8$ (4 isomorphic non-conjugate copies) and $2 \times Q_8$ with respective orders 72 (7 times) and 16. Therefore $H_3 \in \{2 \times (3^2{:}4), 3^2{:}Q_8\}$.

Now gathering all the possibilities of H_3 in this case, we conclude this case by saying that if H_3 is an index 20 subgroup of $2 \times (A_6^{\cdot}2)$, then $H_3 \in \{3^2{:}Q_8,\ 2 \times (3^2{:}4)\}$.

• Here we consider the case that H_3 is an index 4 of $2 \times (3^2{:}QD_{16})$. Firstly $2 \times (3^2{:}QD_{16})$ has 8 conjugacy classes of maximal subgroups, each class represented by $(3^2{:}8){:}2$ (4 isomorphic non-conjugate copies), $2 \times (3^2{:}Q_8)$, $2 \times ((S_3 \times S_3){:}2)$, $2 \times (3^2{:}8)$ and $2 \times QD_{16}$ with respective orders 144 (7 times) and 32. We look at the cases $(3^2{:}8){:}2$, $2 \times (3^2{:}Q_8)$, $2 \times ((S_3 \times S_3){:}2)$ and $2 \times (3^2{:}8)$.

The group $(3^2{:}8){:}2$ has 4 conjugacy classes of maximal subgroups, each class represented by $(S_3 \times S_3){:}2$, $3^2{:}Q_8$, $3^2{:}8$ and QD_6 with respective orders 72 (3 times) and 16. Thus possibilities for H_3 are $(S_3 \times S_3){:}2$, $3^2{:}Q_8$ or $3^2{:}8$.

The group $2 \times (3^2{:}Q_8)$ has 8 conjugacy classes of maximal subgroups, each class represented by $2 \times (3^2{:}4)$, $3^2{:}Q_8$ (4 times), $2 \times (3^2{:}4)$ (twice) and $2 \times Q_8$ with respective orders 72 (7 times) and 16. Thus possibilities for H_3 are $2 \times (3^2{:}4)$ or $3^2{:}Q_8$.

The group $2 \times ((S_3 \times S_3){:}2)$ has 8 conjugacy classes of maximal subgroups, each class represented by $2 \times (3^2{:}4)$, $(S_3 \times S_3){:}2$ (4 times), $2 \times S_3 \times S_3$ (twice) and $2 \times D_8$ with respective orders 72 (7 times) and 16. Thus possibilities for H_3 are $2 \times (3^2{:}4)$, $(S_3 \times S_3){:}2$ or $2 \times S_3 \times S_3$.

The group $2 \times (3^2{:}8)$ has 4 conjugacy classes of maximal subgroups, each class represented by $2 \times (3^2{:}4)$, $3^2{:}8$ (twice) and 8×2 with respective orders 72 (3 times) and 16. Thus the only possibility for H_3 is to be $3^2{:}8$.

Now we conclude this case by saying that if H_3 is an index 4 subgroup of $2\times(3^2{:}QD_{16})$, then $H_3 \in \{3^2{:}Q_8,\ 2\times(3^2{:}4),\ (S_3\times S_3){:}2,\ 3^2{:}8,\ 2\times S_3\times S_3,\ 3^2{:}8\}$.

Now combining the possibilities of H_3 in the above three cases, we deduce that

$$H_3 \in \left\{3^2{:}Q_8,\ 3^2{:}8,\ (S_3 \times S_3){:}2,\ 2 \times (3^2{:}4),\ 2 \times S_3 \times S_3\right\}. \tag{3}$$

Remark 2. Using GAP we were able to obtain that $|\mathrm{Irr}(3^2{:}Q_8)| = 6$, $|\mathrm{Irr}(3^2{:}8)| = |\mathrm{Irr}((S_3 \times S_3){:}2)| = 9$, $|\mathrm{Irr}(2 \times (3^2{:}4))| = 12$ and $|\mathrm{Irr}(2 \times S_3 \times S_3)| = 18$. Also we recall that $|\mathrm{Irr}(A_6^{\cdot}2)| = 8$ and $|\mathrm{Irr}(2 \times A_6)| = 14$. These information will help us in determining H_2 and eliminating many possibilities for H_3 as we will see in Proposition 3.1 below.

Proposition 3.1. *The group H_2 is $A_6^{\cdot}2$ while H_3 is either $3^2{:}8$ or $(S_3 \times S_3){:}2$.*

Proof. By Table 1 we can see that the number of conjugacy classes of $\overline{G}$ is 37 and thus $|\mathrm{Irr}(\overline{G})| = 37$. We have all also indicated before that since the extension $\overline{G}$ splits and the kernel 3^5 is ableian, then all the character tables of the inertia factor groups that we will use to construct the character table of $\overline{G}$ are the ordinary ones. Now by Equations (2) and (3) we have

$$\begin{aligned}(H_2, H_3) \ \in\ & \{(A_6^{\cdot}2, 3^2{:}Q_8),\ (A_6^{\cdot}2, 3^2{:}8),\ (A_6^{\cdot}2, (S_3 \times S_3){:}2),\ (A_6^{\cdot}2, 2 \times (3^2{:}4)),\\ & (A_6^{\cdot}2, 2 \times S_3 \times S_3),\ (2 \times A_6, 3^2{:}Q_8),\ (2 \times A_6, 3^2{:}8),\ (2 \times A_6,\\ & (S_3 \times S_3){:}2), (2 \times A_6, 2 \times (3^2{:}4)),\ (2 \times A_6, 2 \times S_3 \times S_3)\}\end{aligned}$$

and respectively we have

$$\begin{aligned}(|\mathrm{Irr}(H_2)|, |\mathrm{Irr}(H_3)|) \ \in\ & \{(8,6),\ (8,9),\ (8,9),\ (8,12),\ (8,18),\\ & (14,6),\ (14,9),\ (14,9),\ (14,12)),\ (14,18)\}.\end{aligned} \tag{4}$$

We know that $|\mathrm{Irr}(\overline{G})| = \sum_{i=1}^{3} |\mathrm{Irr}(H_i)|$. From Table 3 we have that $|\mathrm{Irr}(H_1)| = 20$. It follows that $|\mathrm{Irr}(H_2)|+|\mathrm{Irr}(H_3)| = |\mathrm{Irr}(\overline{G})|-|\mathrm{Irr}(H_1)| = 37-20 = 17$. Now the only possible pairs that satisfying the relation $|\mathrm{Irr}(H_2)| + |\mathrm{Irr}(H_3)| = 17$ are $(8, 9)$ and $(8, 9)$; that is $(H_2, H_3) \in \{(A_6^{\cdot}2, 3^2{:}8),\ (A_6^{\cdot}2, (S_3 \times S_3){:}2)\}$. In either case we can see that $H_2 = A_6^{\cdot}2$ and H_3 is either $3^2{:}8$ or $(S_3\times S_3){:}2$ as claimed. □

The character table supplied in Table 4 is the character table of the second inertia factor group H_2. We now proceed to determine H_3, which by Proposition 3.1 is either $3^2{:}8$ or $(S_3 \times S_3){:}2$. Using GAP we constructed the character tables of both $3^2{:}8$ and $(S_3 \times S_3){:}2$, which we show in Tables 5 and 6 respectively.

Proposition 3.2. *The third inertia factor group H_3 is $(S_3 \times S_3){:}2$.*

Table 5: The character table of $3^2{:}8$

	$1a$	$2a$	$3a$	$4a$	$4b$	$8a$	$8b$	$8c$	$8d$
$\lvert C_{3^2:8}(h)\rvert$	72	8	9	8	8	8	8	8	8
χ_1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	−1	−1	−1	−1
χ_3	1	−1	1	A	$-A$	B	$-B$	$\overline{B}$	$-\overline{B}$
χ_4	1	−1	1	$-A$	A	$-\overline{B}$	$\overline{B}$	$-B$	B
χ_5	1	−1	1	$-A$	A	$\overline{B}$	$-\overline{B}$	B	$-B$
χ_6	1	−1	1	A	$-A$	$-B$	B	$-\overline{B}$	$\overline{B}$
χ_7	1	1	1	−1	−1	A	A	$-A$	$-A$
χ_8	1	1	1	−1	−1	$-A$	$-A$	A	A
χ_9	8	0	−1	0	0	0	0	0	0

where in Table 5, $A = -i$ and $B = -E(8)$.

Table 6: The character table of $(S_3 \times S_3){:}2$

	$1a$	$2a$	$2b$	$2c$	$3a$	$3b$	$4a$	$6a$	$6b$
$\lvert C_{(S_3\times S_3):2}(h)\rvert$	72	12	12	8	18	18	4	6	6
χ_1	1	1	1	1	1	1	1	1	1
χ_2	1	−1	1	1	1	1	−1	−1	1
χ_3	1	1	−1	1	1	1	−1	1	−1
χ_4	1	−1	−1	1	1	1	1	−1	−1
χ_5	2	0	0	−2	2	2	0	0	0
χ_6	4	2	0	0	−2	1	0	−1	0
χ_7	4	−2	0	0	−2	1	0	1	0
χ_8	4	0	−2	0	1	−2	0	0	1
χ_9	4	0	2	0	1	−2	0	0	−1

Proof. By Table 1 we can see that the class $g_{10} \in [6B]_{2\times M_{11}}$ affords 2 conjugacy classes in $\overline{G}$. This implies that $c(g_{10}) = 2$ and hence the corresponding Fischer matrix $\mathcal{F}_{10}$ to this class will be of size 2. By description of Fischer matrices (for example see [2]), the first row of $\mathcal{F}_{10}$ will correspond to the first inertia factor group $H_1 = 2 \times M_{11}$. Thus the second row of $\mathcal{F}_{10}$ must correspond to either H_2 or H_3 depending on the fusions. From Tables 4 and 5 we can see that neither H_2 nor $3^2{:}8$ has an element of order 6 so that it fuses into $[6B]_{2\times M_{11}}$. This argument shows that $H_3 \neq 3^2{:}8$. Since by Proposition 3.1, $H_3 \in \{3^2{:}8, (S_3 \times S_3){:}2\}$ and we just established that $H_3 \neq 3^2{:}8$, it follows that $H_3 = (S_3 \times S_3){:}2$. Hence the result. □

We summarize that the inertia factor groups are $H_1 = G = 2 \times M_{11}$, $H_2 = A_6^{\cdot}2$ (non-split) and $H_3 = (S_3 \times S_3){:}2$.

Next we turn to determine the fusions of classes of the inertia factor groups of the extension into the classes of $G = 2 \times M_{11}$. We have used the permutation characters of G on the inertia factor groups and the centralizer sizes to determine the fusions of these inertia factors into G. We list these fusions in Tables 7 and 8.

Table 7: The fusion of H_2 into $2 \times M_{11}$

$[g]_{H_2}$	$\longrightarrow$	$[x]_{2\times M_{11}}$	$[g]_{H_2}$	$\longrightarrow$	$[x]_{2\times M_{11}}$
$1a$		$1A$	$4b$		$4B$
$2a$		$2B$	$5a$		$5A$
$3a$		$3A$	$8a$		$8A$
$4a$		$4A$	$8b$		$8B$

Table 8: The fusion of H_3 into $2 \times M_{11}$

$[g]_{H_3}$	$\longrightarrow$	$[x]_{2\times M_{11}}$	$[g]_{H_3}$	$\longrightarrow$	$[x]_{2\times M_{11}}$
$1a$		$1A$	$3b$		$3A$
$2a$		$2B$	$4a$		$4B$
$2b$		$2B$	$6a$		$6B$
$2c$		$2C$	$6b$		$6C$
$3a$		$3A$			

4. Fischer matrices of $\overline{G}$

In this section, we use the arithmetical properties of Fischer matrices, given in Proposition 3.6 of [3], to calculate some of the entries of the Fischer matrices and also to build an algebraic system of equations. To build these systems of equations, we firstly recall that we label the top and bottom of the columns of the Fischer matrix $\mathcal{F}_i$, corresponding to g_i, by the sizes of the centralizers of g_{ij}, $1 \le j \le c(g_i)$ in $\overline{G}$ and m_{ij} respectively. In Table 1 we supplied $|C_{\overline{G}}(g_{ij})|$ and m_{ij}, $1 \le i \le 20$, $1 \le j \le c(g_i)$. Also having obtained the fusions of the inertia factor groups into $2 \times M_{11}$, we are able to label the rows of the Fischer matrices as described in [2, 3].

Since the size of the Fischer matrix $\mathcal{F}_i$ is $c(g_i)$, it follows from Table 1 that the sizes of the Fischer matrices of $\overline{G}$ range between 1 and 4 for all $i \in \{1, 2, \cdots, 20\}$. Now with the help of the symbolic mathematical package Maxima [17], we were able to solve the systems of equations and hence we have computed all the Fischer matrices of $\overline{G}$, where we found that all these matrices are integer valued. Below we list these matrices.

$\mathcal{F}_1$

g_1		g_{11}	g_{12}	g_{13}
$o(g_{1j})$		1	3	3
$\lvert C_{\overline{G}}(g_{1j})\rvert$		3849120	34992	29160
(k,m)	$\lvert C_{H_k}(g_{1km})\rvert$			
$(1,1)$	15840	1	1	1
$(2,1)$	720	22	-5	4
$(3,1)$	72	220	4	-5
m_{1j}		1	110	132

$\mathcal{F}_2$

g_2		g_{21}
$o(g_{2j})$		2
$\lvert C_{\overline{G}}(g_{2j})\rvert$		15840
(k,m)	$\lvert C_{H_k}(g_{2km})\rvert$	
$(1,1)$	15840	1
m_{2j}		243

$\mathcal{F}_3$

g_3		g_{31}	g_{32}	g_{33}	g_{34}
$o(g_{3j})$		2	6	6	6
$\lvert C_{\overline{G}}(g_{3j})\rvert$		2592	432	324	216
(k,m)	$\lvert C_{H_k}(g_{3km})\rvert$				
$(1,1)$	96	1	1	1	1
$(2,1)$	16	6	3	-3	0
$(3,1)$	12	8	-4	-1	2
$(3,2)$	8	12	0	3	-3
m_{3j}		9	54	72	108

$\mathcal{F}_4$

g_4		g_{41}	g_{42}
$o(g_{4j})$		2	6
$\lvert C_{\overline{G}}(g_{4j})\rvert$		864	108
(k,m)	$\lvert C_{H_k}(g_{4km})\rvert$		
$(1,1)$	96	1	1
$(3,1)$	12	8	-1
m_{4j}		27	216

$\mathcal{F}_5$

g_5		g_{51}	g_{52}	g_{53}	g_{54}
$o(g_{5j})$		3	3	9	9
$\lvert C_{\overline{G}}(g_{5j})\rvert$		324	162	162	81
(k,m)	$\lvert C_{H_k}(g_{5km})\rvert$				
$(1,1)$	36	1	1	1	1
$(2,1)$	9	4	-2	-2	1
$(3,1)$	18	2	0	-1	-1
$(3,2)$	18	2	1	2	-1
m_{5j}		27	54	54	108

$\mathcal{F}_6$

g_6		g_{61}	g_{62}
$o(g_{6j})$		4	12
$\lvert C_{\overline{G}}(g_{6j})\rvert$		864	108
(k,m)	$\lvert C_{H_k}(g_{6km})\rvert$		
$(1,1)$	16	1	1
$(2,1)$	8	2	-1
m_{6j}		81	162

$\mathcal{F}_7$

g_7		g_{71}	g_{72}	g_{73}
$o(g_{7j})$		4	12	12
$\lvert C_{\overline{G}}(g_{7j})\rvert$		144	36	36
(k,m)	$\lvert C_{H_k}(g_{7km})\rvert$			
$(1,1)$	16	1	1	1
$(2,1)$	4	4	-2	1
$(3,1)$	4	4	1	-2
m_{7j}		27	108	108

$\mathcal{F}_8$

g_8		g_{81}	g_{82}
$o(g_{8j})$		5	15
$\lvert C_{\overline{G}}(g_{8j})\rvert$		30	15
(k,m)	$\lvert C_{H_k}(g_{8km})\rvert$		
$(1,1)$	10	1	1
$(2,1)$	5	2	-1
m_{8j}		81	162

$\mathcal{F}_9$

g_9		g_{91}
$o(g_{9j})$		6
$\lvert C_{\overline{G}}(g_{9j})\rvert$		36
(k,m)	$\lvert C_{H_k}(g_{9km})\rvert$	
$(1,1)$	36	1
m_{9j}		243

$\mathcal{F}_{10}$

g_{10}		$g_{10,1}$	$g_{10,2}$
$o(g_{10j})$		6	18
$\lvert C_{\overline{G}}(g_{10j})\rvert$		36	18
(k,m)	$\lvert C_{H_k}(g_{10km})\rvert$		
$(1,1)$	12	1	1
$(3,1)$	6	2	-1
m_{10j}		81	162

$\mathcal{F}_{11}$

g_{11}		$g_{11,1}$	$g_{11,2}$
$o(g_{11j})$		6	6
$\lvert C_{\overline{G}}(g_{11j})\rvert$		36	18
(k,m)	$\lvert C_{H_k}(g_{11km})\rvert$		
$(1,1)$	12	1	1
$(3,1)$	6	2	-1
m_{11j}		81	162

$\mathcal{F}_{12}$

g_{12}		$g_{12,1}$	$g_{12,2}$
$o(g_{12j})$		8	24
$\lvert C_{\overline{G}}(g_{12j})\rvert$		48	24
(k,m)	$\lvert C_{H_k}(g_{12km})\rvert$		
$(1,1)$	16	1	1
$(2,1)$	8	2	-1
m_{12j}		81	162

$\mathcal{F}_{13}$

g_{13}		$g_{13,1}$	$g_{13,2}$
$o(g_{13j})$		8	24
$\lvert C_{\overline{G}}(g_{13j})\rvert$		48	24
(k,m)	$\lvert C_{H_k}(g_{13km})\rvert$		
$(1,1)$	16	1	1
$(2,1)$	8	2	-1
m_{13j}		81	162

$\mathcal{F}_{14}$

g_{14}		$g_{14,1}$
$o(g_{14j})$		8
$\lvert C_{\overline{G}}(g_{14j})\rvert$		16
(k,m)	$\lvert C_{H_k}(g_{14km})\rvert$	
$(1,1)$	16	1
m_{14j}		243

$\mathcal{F}_{15}$

g_{15}		$g_{15,1}$
$o(g_{15j})$		8
$\lvert C_{\overline{G}}(g_{15j})\rvert$		16
(k,m)	$\lvert C_{H_k}(g_{15km})\rvert$	
$(1,1)$	16	1
m_{15j}		243

$\mathcal{F}_{16}$

g_{16}		$g_{16,1}$
$o(g_{16j})$		10
$\lvert C_{\overline{G}}(g_{16j})\rvert$		10
(k,m)	$\lvert C_{H_k}(g_{16km})\rvert$	
$(1,1)$	10	1
m_{16j}		243

$\mathcal{F}_{17}$

g_{17}		$g_{17,1}$
$o(g_{17j})$		11
$\lvert C_{\overline{G}}(g_{17j})\rvert$		22
(k,m)	$\lvert C_{H_k}(g_{17km})\rvert$	
$(1,1)$	22	1
m_{17j}		243

$\mathcal{F}_{18}$

g_{18}		$g_{18,1}$
$o(g_{18j})$		11
$\lvert C_{\overline{G}}(g_{18j})\rvert$		22
(k,m)	$\lvert C_{H_k}(g_{18km})\rvert$	
$(1,1)$	22	1
m_{18j}		243

$\mathcal{F}_{19}$

g_{19}		$g_{19,1}$
$o(g_{19j})$		22
$\lvert C_{\overline{G}}(g_{19j})\rvert$		22
(k,m)	$\lvert C_{H_k}(g_{19km})\rvert$	
$(1,1)$	22	1
m_{19j}		243

$\mathcal{F}_{20}$

g_{20}		$g_{20,1}$
$o(g_{20j})$		22
$\lvert C_{\overline{G}}(g_{20j})\rvert$		22
(k,m)	$\lvert C_{H_k}(g_{20km})\rvert$	
$(1,1)$	22	1
m_{20j}		243

5. The character tables of $\overline{G}$

Throughout Sections 2, 3 and 4 we have found:

- the conjugacy classes of $\overline{G}$ (Table 1),
- the inertia factor groups H_1, H_2 and H_3 of $\overline{G}$ and their character tables (Tables 3, 4 and 6). Also we obtained the fusions of classes of the inertia factors H_2 and H_3 of $\overline{G}$ into $2 \times M_{11}$ (Tables 7 and 8),
- the Fischer matrices of $\overline{G}$ (Section 4).

It follows by [2, 3] that the full character table of $\overline{G}$ can be constructed easily in the format of Clifford-Fischer theory. This table will be partitioned into 60 parts corresponding to the 20 cosets and the three inertia factor groups. The full character table of $\overline{G}$ is 37×37 $\mathbb{C}$-valued matrix. In Table 9, we supply the character table of $\overline{G}$ in the format of Clifford-Fischer Theory. In this table we have also included the fusions of the conjugacy classes of $\overline{G}$ into the conjugacy classes of the Conway group Co_3, where the classes of Co_3 as in the Atlas. Finally we would like to remark that the accuracy of this character table has been tested using GAP.

Table 9: The character table of $\overline{G}$

$[g_i]_G$	$1A$			$2A$	$2B$				$2C$		$3A$				$4A$		$4B$		
$[g_{ij}]_{\overline{G}}$	$1a$	$3a$	$3b$	$2a$	$2b$	$6a$	$6b$	$6c$	$2c$	$6d$	$3c$	$3d$	$9a$	$9b$	$4a$	$12a$	$4b$	$12b$	$12c$
$\lvert C_{\overline{G}}(g_{ij})\rvert$	3849120	34992	29160	15840	2592	432	324	216	1184	108	324	162	162	81	48	24	144	36	36
$\hookrightarrow Co_3$	$1A$	$3A$	$3B$	$2B$	$2A$	$6A$	$6B$	$6C$	$2B$	$6D$	$3B$	$3C$	$9A$	$9B$	$4B$	$12B$	$4A$	$12A$	$12C$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	−1	1	1	1	1	−1	−1	1	1	1	1	1	1	−1	−1	−1
χ_3	10	10	10	−10	2	2	2	2	−2	−2	1	1	1	1	2	2	−2	−2	−2
χ_4	10	10	10	10	2	2	2	2	2	2	1	1	1	1	2	2	2	2	2
χ_5	10	10	10	−10	−2	−2	−2	−2	2	2	1	1	1	1	0	0	0	0	0
χ_6	10	10	10	−10	−2	−2	−2	−2	2	2	1	1	1	1	0	0	0	0	0
χ_7	10	10	10	10	−2	−2	−2	−2	−2	−2	1	1	1	1	0	0	0	0	0
χ_8	10	10	10	10	−2	−2	−2	−2	−2	−2	1	1	1	1	0	0	0	0	0
χ_9	11	11	11	−11	3	3	3	3	−3	−3	2	2	2	2	−1	−1	1	1	1
χ_{10}	11	11	11	11	3	3	3	3	3	3	2	2	2	2	−1	−1	−1	−1	−1
χ_{11}	16	16	16	−16	0	0	0	0	0	0	−2	−2	−2	−2	0	0	0	0	0
χ_{12}	16	16	16	−16	0	0	0	0	0	0	−2	−2	−2	−2	0	0	0	0	0
χ_{13}	16	16	16	16	0	0	0	0	0	0	−2	−2	−2	−2	0	0	0	0	0
χ_{14}	16	16	16	16	0	0	0	0	0	0	−2	−2	−2	−2	0	0	0	0	0
χ_{15}	44	44	44	−44	4	4	4	4	−4	−4	−1	−1	−1	−1	0	0	0	0	0
χ_{16}	44	44	44	44	4	4	4	4	4	4	−1	−1	−1	−1	0	0	0	0	0
χ_{17}	45	45	45	−45	−3	−3	−3	−3	3	3	0	0	0	0	1	1	−1	−1	−1
χ_{18}	45	45	45	45	−3	−3	−3	−3	−3	−3	0	0	0	0	1	1	1	1	1
χ_{19}	55	55	55	−55	−1	−1	−1	−1	1	1	1	1	1	1	−1	−1	1	1	1
χ_{20}	55	55	55	55	−1	−1	−1	−1	−1	−1	1	1	1	1	−1	−1	−1	−1	−1
χ_{21}	22	−5	4	0	6	3	−3	0	0	0	4	−2	−2	1	2	−1	4	−2	1
χ_{22}	22	−5	4	0	6	3	−3	0	0	0	4	−2	−2	1	2	−1	−4	2	−1
χ_{23}	198	−45	36	0	6	3	−3	0	0	0	0	0	0	0	2	−1	4	−2	1
χ_{24}	198	−45	36	0	6	3	−3	0	0	0	0	0	0	0	2	−1	−4	2	−1
χ_{25}	220	−50	40	0	−12	−6	6	0	0	0	4	−2	−2	1	0	0	0	0	0
χ_{26}	220	−50	40	0	−12	−6	6	0	0	0	4	−2	−2	1	0	0	0	0	0
χ_{27}	220	−50	40	0	12	6	−6	0	0	0	4	−2	−2	1	−4	2	0	0	0
χ_{28}	352	−80	64	0	0	0	0	0	0	0	−8	4	4	−2	0	0	0	0	0
χ_{29}	220	4	−5	0	20	−4	2	−1	−8	1	4	1	1	−2	0	0	−4	−1	2
χ_{30}	220	4	−5	0	20	−4	2	−1	8	−1	4	1	1	−2	0	0	4	1	−2
χ_{31}	220	4	−5	0	4	4	4	−5	8	−1	4	1	1	−2	0	0	−4	−1	2
χ_{32}	220	4	−5	0	4	4	4	−5	−8	1	4	1	1	−2	0	0	4	1	−2
χ_{33}	440	8	−10	0	−24	0	−6	6	0	0	8	2	2	−4	0	0	0	0	0
χ_{34}	880	16	−20	0	−16	8	2	−4	0	0	−2	−2	−5	1	0	0	0	0	0
χ_{35}	880	16	−20	0	0	0	0	0	−16	2	−2	−5	4	1	0	0	0	0	0
χ_{36}	880	16	−20	0	0	0	0	0	16	−2	−2	−5	4	1	0	0	0	0	0
χ_{37}	880	16	−20	0	16	−8	−2	4	0	0	−2	4	−5	1	0	0	0	0	0

$[g_i]_G$	$5A$		$6A$	$6B$		$6C$		$8A$		$8B$		$8C$	$8D$	$10A$	$11A$	$11B$	$22A$	$22B$
$[g_{ij}]_{\overline{G}}$	$5a$	$15a$	$6e$	$6f$	$18a$	$6g$	$6h$	$8a$	$24a$	$8b$	$24b$	$8c$	$8d$	$10a$	$11a$	$11b$	$22a$	$22b$
$\lvert C_{\overline{G}}(g_{ij})\rvert$	30	15	36	36	18	18	36	48	24	48	24	16	16	10	22	22	22	22
$\hookrightarrow Co_3$	$5B$	$15B$	$6D$	$6C$	$18A$	$6E$	$6D$	$8A$	$24A$	$8A$	$24A$	$8C$	$8C$	$10B$	$11A$	$11B$	$22B$	$22B$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	−1	1	1	−1	−1	−1	−1	−1	−1	1	1	−1	1	1	−1	−1
χ_3	0	0	−1	−1	−1	1	1	0	0	0	0	0	0	0	−1	−1	1	1
χ_4	0	0	1	−1	−1	−1	−1	0	0	0	0	0	0	0	−1	−1	−1	−1
χ_5	0	0	−1	1	1	−1	−1	A	A	$-A$	$-A$	$-A$	A	0	−1	−1	1	1
χ_6	0	0	−1	1	1	−1	−1	$-A$	$-A$	A	A	A	$-A$	0	−1	−1	1	1
χ_7	0	0	1	1	1	1	1	A	A	$-A$	$-A$	A	$-A$	0	−1	−1	−1	−1
χ_8	0	0	1	1	1	1	1	$-A$	$-A$	A	A	$-A$	A	0	−1	−1	−1	−1
χ_9	1	1	−2	0	0	0	0	1	1	1	1	−1	−1	−1	0	0	0	0
χ_{10}	1	1	2	0	0	0	0	−1	−1	−1	−1	−1	−1	1	0	0	0	0
χ_{11}	1	1	2	0	0	0	0	0	0	0	0	0	0	−1	C	$\overline{C}$	$-C$	$-\overline{C}$
χ_{12}	1	1	2	0	0	0	0	0	0	0	0	0	0	−1	$\overline{C}$	C	$-\overline{C}$	$-C$
χ_{13}	1	1	−2	0	0	0	0	0	0	0	0	0	0	1	C	$\overline{C}$	C	$\overline{C}$
χ_{14}	1	1	−2	0	0	0	0	0	0	0	0	0	0	1	$\overline{C}$	C	$\overline{C}$	C
χ_{15}	−1	−1	1	1	1	−1	−1	0	0	0	0	0	0	1	0	0	0	0
χ_{16}	−1	−1	−1	1	1	1	1	0	0	0	0	0	0	−1	0	0	0	0
χ_{17}	0	0	0	0	0	0	0	1	1	1	1	−1	−1	0	1	1	−1	−1
χ_{18}	0	0	0	0	0	0	0	−1	−1	−1	−1	−1	−1	0	1	1	1	1
χ_{19}	0	0	−1	−1	−1	1	1	−1	−1	−1	−1	1	1	0	0	0	0	0
χ_{20}	0	0	1	−1	−1	−1	−1	1	1	1	1	1	1	0	0	0	0	0
χ_{21}	2	−1	0	0	0	0	0	2	−1	2	−1	0	0	0	0	0	0	0
χ_{22}	2	−1	0	0	0	0	0	−2	1	−2	1	0	0	0	0	0	0	0
χ_{23}	−2	1	0	0	0	0	0	−2	1	−2	1	0	0	0	0	0	0	0
χ_{24}	−2	1	0	0	0	0	0	2	−1	2	−1	0	0	0	0	0	0	0
χ_{25}	0	0	0	0	0	0	0	B	A	$-B$	$-A$	0	0	0	0	0	0	0
χ_{26}	0	0	0	0	0	0	0	$-B$	$-A$	B	A	0	0	0	0	0	0	0
χ_{27}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{28}	2	−1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{29}	0	0	0	2	−1	1	−2	0	0	0	0	0	0	0	0	0	0	0
χ_{30}	0	0	0	2	−1	−1	2	0	0	0	0	0	0	0	0	0	0	0
χ_{31}	0	0	0	−2	1	−1	2	0	0	0	0	0	0	0	0	0	0	0
χ_{32}	0	0	0	−2	1	1	−2	0	0	0	0	0	0	0	0	0	0	0
χ_{33}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{34}	0	0	0	2	−1	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{35}	0	0	0	0	0	−1	2	0	0	0	0	0	0	0	0	0	0	0
χ_{36}	0	0	0	0	0	1	−2	0	0	0	0	0	0	0	0	0	0	0
χ_{37}	0	0	0	−2	1	0	0	0	0	0	0	0	0	0	0	0	0	0

where in Table 9, $A = -i\sqrt{2}$, $B = 2i\sqrt{2}$ and $C = \frac{1}{2} - i\frac{\sqrt{11}}{2} = b11$.

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References

[1] M. L. Alotaibi, *On character tables of split extensions*, MSc Project, Al Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh, 2018.

[2] A. B. M. Basheer, *Clifford-Fischer theory applied to certain groups associated with symplectic, unitary and Thompson groups*, PhD Thesis, University of KwaZulu-Natal, Pietermaitzburg, 2012.

[3] A. B. M. Basheer, J. Moori, *Fischer matrices of Dempwolff group* $2^{5\cdot}GL(5,2)$, International Journal of Group Theory, 4 (2012), 43-63.

[4] A. B. M. Basheer, J. Moori, *On the non-split extension group* $2^{6\cdot}Sp(6,2)$, Bulletin of the Iranian Mathematical Society, 39 (2013), 1189-1212.

[5] A. B. M. Basheer, J. Moori, *On the non-split extension* $2^{2n\cdot}Sp(2n,2)$, Bulletin of the Iranian Mathematical Society, 41 (2015), 499-518.

[6] A. B. M. Basheer, J. Moori, *On a maximal subgroup of the Thompson simple group*, Mathematical Communications, 20 (2015), 201-218.

[7] A. B. M. Basheer, J. Moori, *A survey on Clifford-Fischer theory*, London Mathematical Society Lecture Notes Series, Cambridge University Press, 422 (2015), 160-172.

[8] A. B. M. Basheer, J. Moori, *On a group of the form* $3^7{:}Sp(6,2)$, International Journal of Group Theory, 5 (2016), 41-59.

[9] A. B. M. Basheer, T. T. Seretlo, *On a group of the form* $2^{14}{:}Sp(6,2)$, Quaestiones Mathematicae, 39 (2016), 45-57.

[10] A. B. M. Basheer, J. Moori, *On two groups of the form* $2^8{:}A_9$, Afrika Matematika, 28 (2017), 1011-1032.

[11] A. B. M. Basheer, J. Moori, *On a group of the form* $2^{10}{:}(U_5(2){:}2)$, Italian Journal of Pure and Applied Mathematics, 37 (2017), 645-658.

[12] A. B. M. Basheer, J. Moori, *Clifford-Fischer theory applied to a group of the form* $2_-^{1+6}{:}((3^{1+2}{:}8){:}2)$, Bulletin of the Iranian Mathematical Society, 43 (2017), 41-52.

[13] A. B. M. Basheer, J. Moori, *On a maximal subgroup of the affine general linear group* $AGL(6,2)$, submitted.

[14] W. Bosma, J. J. Cannon, *Handbook of magma functions*, Department of Mathematics, University of Sydeny, November 1994.

[15] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, R. A. Wilson, *Atlas of Finite Groups*, Clarendon Press, Oxford, 1985.

[16] The GAP Group, *GAP-Groups, Algorithms and Programming*, Version 4.4.10, 2007. (http://www.gap-system.org)

[17] Maxima, *A computer algebra system*, Version 5.18.1, 2009. (http://maxima.sourceforge.net)

[18] J. Moori, *On the Groups* G^+ *and* $\overline{G}$ *of the form* $2^{10}{:}M_{22}$ *and* $2^{10}{:}\overline{M}_{22}$, PhD Thesis, University of Birmingham, 1975.

[19] J. Moori, *On certain groups associated with the smallest Fischer group*, J. London Math. Soc., 2 (1981), 61-67.

[20] J. Moori, T. T. Seretlo, *On two non-split extension groups associated with HS and HS:2*, Turkish Journal of Maths, 37 (2013), 60-78.

[21] J. Moori, T. T. Seretlo, *On the Fischer-Clifford matrices of a maximal subgroup of the Lyons Group Ly*, Bulletin of the Iranian Maths Soc., 39 (2013), 1037-1052.

[22] Z. E. Mpono, *Fischer Clifford theory and character tables of group extensions*, PhD Thesis, University of Natal, Pietermaritzburg, 1998.

[23] T. T. Seretlo, *Fischer Clifford matrices and character tables of certain groups associated with simple groups* $O_{10}^{+}(2)$, HS and Ly, PhD thesis, University of KwaZulu-Natal, Pietermaritzburg, 2012.

[24] N. S. Whitely, *Fischer matrices and character tables of group extensions* , MSc Thesis, University of Natal, Pietermaritzburg, 1993.

[25] R. A. Wilson et al., *Atlas of finite group representations*, (http://brauer.maths.qmul.ac.uk/Atlas/v3/)

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