New independent paracompact spaces

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Abstract. The purpose of the present paper is to introduce a new type of paracompactness which is called $\omega_3$- paracompact and to obtain some results of paracompact spaces, one of which is the image of $\omega_3$- paracompact is paracompact under $\omega_3$- continuous surjection which maps $\omega_3$- open sets onto open set. We give an example shows that this type of paracompact is independent with standard paracompact space.

Keywords: $\omega_3$-open set, $\omega_3$-paracompact Space, $\omega_3$-paracompact subset.

1. Introduction

As defined by J. Dieudonne [1], a space $X$ is said to be paracompact if each open covering has locally finite open refinement. C. H. Dowker [2] generalize this concept and introduced the class of countably paracompact spaces. A space $X$ is said to be countably paracompact if each countable open cover of $X$ has a locally finite open refinement. Generalizing the concept of paracompact spaces, K. Y. Al Zoubi [7] and M. K. Singal and Shashi Prabha Arya [8]. A space $X$ is said to be $S$- Paracompact if each open cover has a locally finite semi open refinement and a space $X$ is said to be $R$- Paracompact if each open covering of $X$ of cardinality $R$ has a locally finite open refinement. Since, then a lot of work has been done of $S$- Paracompact spaces and many interesting results have been obtained [10, 11]. This type of paracompact space is on the set that is different of open set and other types of sets like as $\omega_p$- open sets [4]. The

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object of the present paper is to present some results of new independent type of paracompact spaces.

2. Preliminaries

Throughout this paper a space will always mean a topological space in which no separation axioms is assumed unless explicitly stated. A subset $G$ of a space $X$ is called $\delta$-open [8], if for each $x \in G$, there exists an open set $U$ containing $x$ such that $\text{Int}ClU \subseteq G$. For a subset $A$ of a space $X$, the $\text{Int}_{\delta}(A)$, $\text{Cl}_{\delta}(A)$ will be denoted the $\delta$-interior and $\delta$-closure of $A$ respectively. A space $X$ is said to be locally-countable [5] if each point of $X$ has a countable open neighborhood. Let $(X, \tau)$ be a space, a subset $A$ of $X$ is said to be $\omega_{\delta}$-open set [3] if for each $x \in U$, there exists an open set $G$ containing $x$ such that $G \cap \text{Int}(A)$ is countable. The complement of $\omega_{\delta}$-open set is called $\omega_{\delta}$-closed set. If $A$ is a subset of a space $X$, then the $\omega_{\delta}$-Interior ($\omega_{\delta}\text{Int}(A)$) of $A$ is a union of all $\omega_{\delta}$-open sets of $X$ which contained in $A$ and the $\omega_{\delta}$-Closure ($\omega_{\delta}\text{Cl}(A)$) of $A$ is the intersection of all $\omega_{\delta}$-closed sets which containing $A$.

3. $\omega_{\delta}$-paracompact spaces

The main purpose of this section is to define $\omega_{\delta}$-Paracompact spaces, and obtain some characterizations, properties, and relationships.

A family $\{A_\lambda : \lambda \in \Lambda\}$ of subsets of a space $(X, \tau)$ is called $\omega_{\delta}$-locally finite [4], if for each $x \in X$, there exist an $\omega_{\delta}$-open set $G$ containing $x$ such that $\{\lambda \in \Lambda : G \cap A_\lambda \neq \emptyset\}$ is finite.

Definition 3.1. A space $X$ is called an $\omega_{\delta}$-Paracompact space, if each $\omega_{\delta}$-open covering of $X$ has an $\omega_{\delta}$-locally finite $\omega_{\delta}$-open reffinement.

Proposition 3.2. A topological space $(X, \tau)$ is $\omega_{\delta}$- Paracompact if and only if the topological space $(X, \tau_{\omega_{\delta}})$ is paracompact.

From [Proposition 3.8, 3], we get the following result:

Proposition 3.3. If a topological space $(X, \tau)$ is locally countable, then $(X, \tau_{\omega_{\delta}})$ is paracompact.

Lemma 3.4. If a covering $\{U_\lambda\}_{\lambda \in \Lambda}$ of a space $X$ has $\omega_{\delta}$-locally finite $\omega_{\delta}$-open refinement, then there exist an $\omega_{\delta}$-locally finite $\omega_{\delta}$-open covering $\{G_\lambda\}_{\lambda \in \Lambda}$ of $X$ such that $G_\lambda \subseteq U_\lambda$, for all $\lambda \in \Lambda$.

Proof. Let $\{V_\gamma\}_{\gamma \in \Gamma}$ be the $\omega_{\delta}$-locally finite $\omega_{\delta}$-open reffinement of $\{U_\lambda\}_{\lambda \in \Lambda}$. Therefore, there exists a function $\beta : \Gamma \to \Lambda$ such that $V_\gamma \subseteq U_{\beta(\gamma)}$, for each $\gamma \in \Gamma$. Let $G_\lambda = \bigcup_{\gamma \in \Gamma, \beta(\gamma) = \lambda} V_\gamma$, then the family $\{G_\lambda\}_{\lambda \in \Lambda}$ is an $\omega_{\delta}$-open covering of $X$ with the property that $G_\lambda \subseteq U_\lambda$, for each $\lambda \in \Lambda$. Also, $\{U_\lambda\}_{\lambda \in \Lambda}$ is $\omega_{\delta}$-locally finite. If $x \in X$, then there is an $\omega_{\delta}$-open set $W$ containing $x$ such that the set $\Gamma_0 = \{\gamma \in \Gamma : W \cap V_\gamma \neq \emptyset\}$ is finite, since $W \cap G_\lambda \neq \emptyset$ if and only if $\lambda = \beta(\gamma)$,
for some $\gamma \in \Gamma_0$, so the set $\{\lambda \in \Lambda : W \cap G_{\lambda} \neq \emptyset\}$ is finite. Hence, the proof is complete.

\[\Box\]

**Corollary 3.5.** A space $X$ is $\omega_\delta$-paracompact if and only if for every $\omega_\delta$-open covering $\{U_{\lambda}\}_{\lambda \in \Lambda}$ of $X$, there exists an $\omega_\delta$-locally finite $\omega_\delta$-open covering $\{V_{\lambda}\}_{\lambda \in \Lambda}$ of $X$ such that $V_{\lambda} \subseteq U_{\lambda}$, for each $\lambda \in \Lambda$.

The $\omega_\delta$-boundary of a subset $A$ of a space $X(\omega_\delta b(A))$ is the difference between $\omega_\delta Cl(A)$ and $\omega_\delta Int(A)$.

**Corollary 3.6.** If $\{A_{\lambda} : \lambda \in \Lambda\}$ is an $\omega_\delta$-locally finite family of subsets of $X$, then:

1) $\{\omega_\delta Cl(A_{\lambda}) : \lambda \in \Lambda\}$ is also $\omega_\delta$-locally finite and $\omega_\delta Cl(\cup \{A_{\lambda} : \lambda \in \Lambda\}) = \cup (\omega_\delta Cl(A_{\lambda}) : \lambda \in \Lambda)$.

2) $\omega_\delta b(\bigcup_{\lambda \in \Lambda} A_{\lambda}) \subseteq \bigcup_{\lambda \in \Lambda} \omega_\delta b(A_{\lambda})$.

**Proof.**

1) Let $x \in X$. Since, $\{A_{\lambda} : \lambda \in \Lambda\}$ is $\omega_\delta$-locally finite, there exists an $\omega_\delta$-open set $G$ containing $x$ such that the set $\{\lambda : \lambda \in \Lambda : G \cap A_{\lambda} \neq \emptyset\}$ is finite. Since, $G \cap A_{\lambda} = \emptyset$ if and only if $G \cap \omega_\delta Cl(A_{\lambda}) = \emptyset$, so $\{\lambda : \lambda \in \Lambda : G \cap \omega_\delta Cl(A_{\lambda}) \neq \emptyset\}$ is finite. Hence, $\{\omega_\delta Cl(A_{\lambda}) : \lambda \in \Lambda\}$ is $\omega_\delta$-locally finite. Since, $\bigcup (\omega_\delta Cl(A_{\lambda}) : \lambda \in \Lambda) \subseteq \omega_\delta Cl(\bigcup \{A_{\lambda} : \lambda \in \Lambda\})$. To prove $\omega_\delta Cl(\bigcup \{A_{\lambda} : \lambda \in \Lambda\}) \subseteq \bigcup (\omega_\delta Cl(A_{\lambda}) : \lambda \in \Lambda)$, let $x \notin \bigcup (\omega_\delta Cl(A_{\lambda}) : \lambda \in \Lambda)$. Since by what we have proved above $\{\omega_\delta Cl(A_{\lambda}) : \lambda \in \Lambda\}$ is $\omega_\delta$-locally finite, there exist an $\omega_\delta$-open set $U$ containing $x$ such that $A_0 = \{\lambda \in \Lambda : U \cup \omega_\delta Cl(A_{\lambda}) \neq \emptyset\}$ is finite. Set $V = U \cup (\{X - \omega_\delta Cl(A_{\lambda}) : \lambda \in \Lambda\})$ is $\omega_\delta$-open subsets of $X$ containing $x$ such that $V \cap (\bigcup \{A_{\lambda} : \lambda \in \Lambda\}) = \emptyset$. Thus $x \notin \omega_\delta Cl(\bigcup \{A_{\lambda} : \lambda \in \Lambda\})$, hence $\omega_\delta Cl(\bigcup \{A_{\lambda} : \lambda \in \Lambda\}) \subseteq \bigcup (\omega_\delta Cl(A_{\lambda}) : \lambda \in \Lambda)$. Therefore, $\omega_\delta Cl(\bigcup \{A_{\lambda} : \lambda \in \Lambda\}) = \bigcup (\omega_\delta Cl(A_{\lambda}) : \lambda \in \Lambda)$.

2) Since $\omega_\delta b(\bigcup_{\lambda \in \Lambda} A_{\lambda}) = \omega_\delta Cl(\bigcup_{\lambda \in \Lambda} A_{\lambda}) \cap \omega_\delta Cl(X - \bigcup_{\lambda \in \Lambda} A_{\lambda}) = \bigcup_{\lambda \in \Lambda} (\omega_\delta Cl(A_{\lambda}) \cap \omega_\delta Cl(X - \bigcup_{\lambda \in \Lambda} A_{\lambda}) \subseteq \bigcup_{\lambda \in \Lambda} (\omega_\delta Cl(A_{\lambda}) \cap \omega_\delta Cl(X - A_{\lambda})) = \bigcup_{\lambda \in \Lambda} \omega_\delta b(A_{\lambda})$.

\[\Box\]

**Corollary 3.7.** Let $X$ be an $\omega_\delta$-paracompact space, and let $H$ and $F$ be two subsets in which $F$ is an $\omega_\delta$-closed subset of $X$ which is disjoint from $H$. If for every $x \in F$, there exist disjoint $\omega_\delta$-open sets $U_x$ and $V_x$ containing $x$ and $H$, respectively. Then, there are disjoint $\omega_\delta$-open sets $U$ and $V$ containing $F$ and $H$, respectively.

**Proof.** Consider the $\omega_\delta$-open covering $\{U_x\}_{x \in F} \cup \{X - F\}$ of an $\omega_\delta$-paracompact space $X$. Then by Corollary 3.5, there exists an $\omega_\delta$-locally finite $\omega_\delta$-open covering $\{G_x\}_{x \in F} \cup \{G\}$ of $X$ such that $G \subseteq X - F$ and $G_x \subseteq U_x$, for each $x \in F$. Since $U_x \cap V_x = \emptyset$, then $G_x \cap V_x = \emptyset$, so $\omega_\delta Cl(G_x) \cap V_x = \emptyset$, for each $x \in F$. 350 HARDI N. AZIZ, HALGWRD M. DARWESH and ADIL K. JABAR
Then by Proposition 3.6, the sets $U = \bigcup_{x \in F} G_x$ and $V = X - \bigcup_{x \in F} \omega_{5}\text{Cl}(G_x)$ are the required $\omega_5$-open sets of $X$. Thus, completes the proof.

**Definition 3.8** ([4]). A space $X$ is said to be:

1) $\omega_5-T_2$ space, if for each distinct points $x$ and $y$ of $X$, there exists disjoint $\omega_5$-open sets $U$ and $V$ containing $x$ and $y$, respectively.

2) $\omega_5-T_1$ space, if for each pair of distinct points of $X$, there exist $\omega_5$-open sets $U$ and $V$ such that $x \in U$, $y \notin U$ and $y \in V, y \notin V$.

3) $\omega_5$-regular space, if each $\omega_5$-closed subset $H$ of $X$ and a point $x$ in $X$ such that $x \notin H$, there exist disjoint $\omega_5$-open sets $U$ and $V$ containing $x$ and $H$, respectively.

4) $\omega_5$-normal space, if for each pair of disjoint $\omega_5$-closed sets $A$ and $B$ in $X$, there exist disjoint $\omega_5$-open sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$.

**Proposition 3.9.** Each $\omega_5$-paracompact $\omega_5-T_2$ ($\omega_5$-regular) space is an $\omega_5$-normal space.

**Proof.** Let $X$ be an $\omega_5$-paracompact $\omega_5-T_2$ space and let $x_0$ be any point in $X$, which is not in arbitrary $\omega_5$-closed subset $F$ of $X$. Therefore, for each $x \in F$, there are disjoint $\omega_5$-open sets $U_x$ and $V_x$ containing $x$ and $\{x_0\}$, so by Proposition 3.7, there exist disjoint $\omega_5$-open sets $U$ and $V$ containing $F$ and $x_0$. This shows that $X$ is $\omega_5$-regular. Thus, $X$ is $\omega_5$-paracompact $\omega_5$-regular space. Let $F$ and $H$ be any two disjoint $\omega_5$-closed subsets of $X$. Since, $H$ is $\omega_5$-closed, so by $\omega_5$-regularity of $X$, for each $x \in F$, there exist disjoint $\omega_5$-open sets $U_x$ and $V_x$ containing $x$ and $H$. Therefore, by Proposition 3.7, there exist disjoint $\omega_5$-open sets $U$ and $V$ containing $F$ and $H$. Thus, $X$ is an $\omega_5$-normal space.

**Example 3.10.** Since the usual space on $\mathbb{R}$ is an $\omega_5-T_2$ but not $\omega_5$-normal space, so by **Proposition 3.9**, it is not $\omega_5$-paracompact.

**Proposition 3.11.** Let $\{A_{\lambda} : \lambda \in \Lambda\}$ be a family of subsets of a space $X$ and $\{B_{\gamma} : \gamma \in \Gamma\}$ be an $\omega_5$-locally finite $\omega_5$-closed cover of $X$ such that for each $\gamma \in \Gamma$, the set $\{\lambda \in \Lambda : B_{\gamma} \cap A_{\lambda} \neq \emptyset\}$ is finite. Then there exists an $\omega_5$-locally finite family $\{G_{\lambda} : \lambda \in \Lambda\}$ of $\omega_5$-open sets of $X$ such that $A_{\lambda} \subseteq G_{\lambda}$, for each $\lambda \in \Lambda$.

**Proof.** For each $\lambda$, let $G_{\lambda} = X - (\bigcup \{B_{\gamma} : B_{\gamma} \cap A_{\lambda} = \emptyset\})$. Then $A_{\lambda} \subseteq G_{\lambda}$, and since $\{B_{\gamma} : \gamma \in \Gamma\}$ is $\omega_5$-locally finite, so by Proposition 3.6, $\omega_5\text{Cl}(\bigcup_{\gamma \in \Gamma} B_{\gamma}) = \bigcup_{\gamma \in \Gamma} \omega_5\text{Cl}(B_{\gamma})$, this implies that $G_{\lambda}$ is an $\omega_5$-open set, for each $\lambda \in \Lambda$. Let $x \in X$. Since, $\{B_{\gamma} : \gamma \in \Gamma\}$ is $\omega_5$-locally finite, so there is an $\omega_5$-open set $U$ containing $x$ such that the set $\gamma_0 = \{\gamma \in \Gamma : U \cap B_{\gamma} \neq \emptyset\}$ is finite.
Thus, $U \cap B_\gamma = \emptyset$, for each $\gamma \notin \Gamma_0$. Therefore, $U \subseteq \bigcup \{B_\gamma : \gamma \in \Gamma_0\}$. Also, since for each $\gamma \in \Gamma_0$, $G_\lambda \cap B_\gamma = \emptyset$ if and only if $A_\lambda \cap B_\gamma = \emptyset$. The finiteness of $\{\lambda \in \Lambda : B_\gamma \cap A_\lambda \neq \emptyset\}$ implies the finiteness of $\{\lambda \in \Lambda : U \cap G_\lambda \neq \emptyset\}$. Thus, \{\{G_\lambda : \lambda \in \Lambda\}\text{ is }\omega_5\text{-}locally finite of }\omega_5\text{-}open subsets of }X.

\begin{proposition}
\textbf{Theorem 3.12.} A space $X$ is $\omega_5\text{-}paracompact }\omega_5\text{-}normal if and only if every $\omega_5\text{-}open covering of }X\text{ has an }\omega_5\text{-}locally finite }\omega_5\text{-}closed refinement.
\end{proposition}

\textbf{Proof.} Necessity. Let \{\{U_\lambda\}_{\lambda \in \Lambda}\} be an $\omega_5\text{-}open covering of an }\omega_5\text{-}paracompact }\omega_5\text{-}normal space }X. \text{ So by Corollary 3.5, there exists an }\omega_5\text{-}locally finite }\omega_5\text{-}open covering \{\{V_\lambda\}_{\lambda \in \Lambda}\} of }X\text{ such that }V_\lambda \subseteq U_\lambda\text{, for all }\lambda \in \Lambda. \text{ Since, }X\text{ is an }\omega_5\text{-}normal space, then by [Theorem 5.27, 4], there exists an }\omega_5\text{-}locally finite }\omega_5\text{-}closed refinement of }\{\{V_\lambda\}_{\lambda \in \Lambda}\}\text{ which also covers }X.

\textbf{Sufficiency.} Let }X\text{ be a space with the property that every }\omega_5\text{-}open covering of }X\text{ it has an }\omega_5\text{-}locally finite }\omega_5\text{-}closed refinement. \text{ Thus, by [Theorem 5.27, 4], }X\text{ is }\omega_5\text{-normal space. It remains only to show that }X\text{ is }\omega_5\text{-paracompact.}

\textbf{Let }\{\{W_\lambda\}_{\lambda \in \Lambda}\} be an }\omega_5\text{-open covering of }X\text{ and }\{\{F_\gamma\}_{\gamma \in \Gamma}\} be an }\omega_5\text{-locally finite }\omega_5\text{-closed refinement of }\{\{W_\lambda\}_{\lambda \in \Lambda}\}. \text{ Therefore, for each }x \in \Lambda, \text{ there exists an }\omega_5\text{-open set }U_x\text{ containing }x\text{ such that the set }\{\gamma \in \Gamma : U_x \cap F_\gamma \neq \emptyset\}\text{ is finite.}

\textbf{Consider }\{\{E_\nu\}_{\nu \in \varnothing}\}\text{ is an }\omega_5\text{-locally finite }\omega_5\text{-closed refinement of the }\omega_5\text{-open covering }\{\{U_x\}_{x \in X}\} \text{ of }X\text{, then for each }\nu \in \varnothing, \text{ the set }\{\gamma \in \Gamma : F_\gamma \cap E_\nu \neq \emptyset\}\text{ is finite. So by Proposition 3.11, there exists an }\omega_5\text{-locally finite family }\{\{G_\gamma\}_{\gamma \in \Gamma}\} \text{ of }\omega_5\text{-open sets of }X\text{ such that }F_\gamma \subseteq G_\gamma\text{, for each }\gamma \in \Gamma, \text{ which is also covers }X.

\textbf{Since, }\{\{F_\gamma\}_{\gamma \in \Gamma}\}\text{ is refinement of }\{\{W_\lambda\}_{\lambda \in \Lambda}\}, \text{ so for each }\gamma \in \Gamma, \text{ there is }\lambda(\gamma) \in \Gamma \text{ such that }F_\gamma \subseteq W_{\lambda(\gamma)}\text{. Therefore, }\{\{G_\gamma \cap W_{\lambda(\gamma)}\}_{\gamma \in \Gamma}\}\text{ is an }\omega_5\text{-locally finite }\omega_5\text{-open refinement of }\{\{W_\lambda\}_{\lambda \in \Lambda}\}. \text{ Hence, }X\text{ is }\omega_5\text{-paracompact space.}

\textbf{4. }\omega_5\text{-paracompact subset}

\textbf{In this section, we study some results of paracompactness on a subspace.}

\textbf{Proposition 4.1.} Every }\omega_5\text{-paracompact, }\delta\text{-open subset of a space }X\text{ is an }\omega_5\text{-paracompact subspace.}

\textbf{Proof.} Let }A\text{ be an }\omega_5\text{-paracompact, }\delta\text{-open subset of a space }X\text{ and let }\{\{U_\lambda\}_{\lambda \in \Lambda}\}\text{ is a covering of }A\text{ by }\omega_5\text{-open subsets of }A. \text{ Then by [Theorem 3.4.4], }\{\{U_\lambda\}_{\lambda \in \Lambda}\}\text{ is a covering of }A\text{ by }\omega_5\text{-open subsets of }X. \text{ By hypothesis, there exists an }\omega_5\text{-locally finite }\omega_5\text{-open refinement }\{\{W_\gamma\}_{\gamma \in \Gamma}\}\text{ of the family }\{\{V_\lambda\}_{\lambda \in \Lambda}\}\text{ which covers }A\text{ also. Therefore, by [Proposition 3.6, 4], }\{\{W_\gamma \cap A\}_{\gamma \in \Gamma}\}\text{ is an }\omega_5\text{-locally finite }\omega_5\text{-open refinement of }\{\{U_\lambda\}_{\lambda \in \Lambda}\} \text{ in }A. \text{ Thus, }A\text{ is an }\omega_5\text{-paracompact subspace of }X.

\textbf{Proposition 4.2.} An }\omega_5\text{-closed subset of an }\omega_5\text{-paracompact space is an }\omega_5\text{-paracompact subset.}
Let $F$ be an $\omega_\beta$-closed subset of an $\omega_\beta$-paracompact space $X$ and let $(U_\lambda)_{\lambda \in \Lambda}$ be a covering of $F$ by $\omega_\beta$-open sets of $X$. Then $(U_\lambda)_{\lambda \in \Lambda} \cup \{X - F\}$ is an $\omega_\beta$-open covering of $X$. By hypothesis and by Corollary 3.5, there exists an $\omega_\beta$-locally finite $\omega_\beta$-open covering $(W_\lambda)_{\lambda \in \Lambda} \cup \{W\}$ of $X$ such that $W \subseteq X - F$ and $W_\lambda \subseteq U_\lambda$, for each $\lambda \in \Lambda$. Therefore, $(W_\lambda)_{\lambda \in \Lambda}$ is an $\omega_\beta$-locally finite $\omega_\beta$-open refinement of $(U_\lambda)_{\lambda \in \Lambda}$ which covers $F$. This shows that $F$ is an $\omega_\beta$-paracompact relative to $X$. \hfill $\Box$

**Proposition 4.3.** If a space $X$ is $\omega_\beta - T_2$ space and has a subset $F$ which is $\omega_\beta$-paracompact relative to $X$, then for each $x \in X - F$, there exist two disjoint $\omega_\beta$-open sets of $X$ containing $x$ and $F$.

**Proof.** Let $F$ be an $\omega_\beta$-paracompact subset of an $\omega_\beta - T_2$ space $X$ and let $x$ be any point of $X - F$. Then for each $y \in F$, there exist $\omega_\beta$-open sets $U_y$ and $V_y$ such that $y \in U_y$, $x \in V_y$ and $U_y \cap V_y = \emptyset$. This implies that $\omega_\beta Cl(U_y) \cap V_y = \emptyset$. Hence, $x \notin \omega_\beta Cl(U_y)$, for each $y \in F$. Now, $(U_y)_{y \in F}$ is a cover of $F$ by $\omega_\beta$-open subsets of $X$. Thus, by hypothesis and by Corollary 3.5, there exists an $\omega_\beta$-locally finite covering $(W_y)_{y \in F}$ of $F$ by $\omega_\beta$-open subsets of $X$ such that for each $y \in F$, $W_y \subseteq U_y$. Therefore, $x \notin \omega_\beta Cl(W_y)$, for each $y \in F$. Hence, by Proposition 3.6, $U = \bigcup_{y \in F} W_y$ and $V = X - \bigcup_{y \in F} \omega_\beta Cl(W_y)$, which are $\omega_\beta$-open sets which containing $F$ and $x$ respectively. \hfill $\Box$

From Proposition 4.3, we get the following result:

**Corollary 4.4.** Every $\omega_\beta$-paracompact subset of an $\omega_\beta - T_2$ space is an $\omega_\beta$-closed.

**Corollary 4.5.** Every $\omega_\beta$-regular, $\omega_\beta - T_1$ space is an $\omega_\beta - T_2$ space.

**Proposition 4.6.** If $X$ is an $\omega_\beta$-regular $\omega_\beta - T_1$ space and $F$ is a subset of $X$ which is $\omega_\beta$-paracompact relative to $X$, then for each $\omega_\beta$-open set $U$ in $X$ containing $F$ in $X$, there exists an $\omega_\beta$-closed set $H$ in $X$ containing $F$ and it is contained in $U$.

**Proof.** Since $X$ is $\omega_\beta$-regular $\omega_\beta - T_1$ space, so by Corollary 4.5, and Corollary 4.4, $F$ is $\omega_\beta$-closed subset of $X$. Therefore, by Theorem 5.4, 4, for each $x \in F$, there is $\omega_\beta$-open set $U_x$ such that $x \in U_x \subseteq \omega_\beta Cl(U_x) \subseteq U$. Since $F$ is $\omega_\beta$-paracompact relative to $X$, so there exists an $\omega_\beta$-locally finite family $(V_\gamma)_{\gamma \in \Gamma}$ of $F$ by $\omega_\beta$-open sets of $X$ which refines $(U_x)_{x \in F}$ and covers $F$. Therefore, by Proposition 3.6, $H = \bigcup_{\gamma \in \Gamma} \omega_\beta Cl(V_\gamma)$ is required $\omega_\beta$-closed set. \hfill $\Box$

**Definition 4.7.** A topological space $(X, \tau)$ is called $\omega_\beta$-connected space, if $X$ is not a union of two nonempty disjoint $\omega_\beta$-open sets, otherwise it is $\omega_\beta$-disconnected. Obviously from Definition 4.7, we get the following result:

**Theorem 4.8.** A space $X$ is $\omega_\beta$-disconnected if and only if there exists a nonempty proper subset of $X$ which is both $\omega_\beta$-open and $\omega_\beta$-closed in $X$. 

Corollary 4.9. A space $X$ is $\omega_\delta$-connected if and only if the only nonempty subset of $X$ which is both $\omega_\delta$-open and $\omega_\delta$-closed is $X$ itself.

Theorem 4.10. Let $X$ be $\omega_\delta$-disconnected space, then the following statements are equivalent:

1) $X$ is an $\omega_\delta$-paracompact space.

2) Every proper $\omega_\delta$-closed subset of $X$ is $\omega_\delta$-paracompact relative to $X$.

3) Every proper $\delta$-open, $\omega_\delta$-closed subset of $X$ is $\omega_\delta$-paracompact subspace.

4) Every proper $\delta$-open, $\omega_\delta$-clopen subset of $X$ is $\omega_\delta$-paracompact subspace.

5) There exists a proper $\delta$-open, $\omega_\delta$-clopen subset $F$ of $X$ such that both $F$ and $X - F$ are $\omega_\delta$-paracompact subspaces of $X$.

Proof. (1 $\rightarrow$ 2) and (2 $\rightarrow$ 3), Follows from Proposition 4.2 and Proposition 4.1, respectively. 

(3 $\rightarrow$ 4) and (4 $\rightarrow$ 5) are obvious. We prove if (5), we get (1). Let $X$ be a space that contains a proper $\delta$-open, $\omega_\delta$-clopen subset $F$ in which both $F$ and $X - F$ are $\omega_\delta$-paracompact, and let $\{G_\lambda\}_{\lambda \in A}$ be any $\omega_\delta$-open cover of $X$. Then $\{F \cap G_\lambda\}_{\lambda \in A}$ and $\{(X - F) \cap G_\lambda\}_{\lambda \in A}$ are covers of $X$. For each $\lambda$, $F \cap G_\lambda$ and $(X - F) \cap G_\lambda$ are $\omega_\delta$-open subset of $F$ and $X - F$, respectively. Therefore, there exist $\omega_\delta$-locally finite refinements $\{V_\gamma\}_{\gamma \in \Gamma}$ and $\{V_\nu\}_{\nu \in \vartheta}$ of $\{F \cap G_\lambda\}_{\lambda \in A}$ and $\{(X - F) \cap G_\lambda\}_{\lambda \in A}$ covering $F$ and $X - F$, respectively such that each $V_\gamma$ is $\omega_\delta$-open in $F$, for each $\gamma \in \Gamma$, $V_\nu$ is $\omega_\delta$-open in $X - F$, for each $\nu \in \vartheta$. By [Theorem 3.4, 4], both $V_\gamma$ and $V_\nu$ are $\omega_\delta$-open sets in $X$. Therefore, $\{V_\gamma\}_{\gamma \in \Gamma}$ is an $\omega_\delta$-locally finite $\omega_\delta$-open refinement of $\{G_\lambda\}_{\lambda \in A}$ which covers $X$. Hence, $X$ is $\omega_\delta$-paracompact space. 

Remark 4.11. From Theorem 4.10, we notice that, if $X$ is $\omega_\delta$-connected, then by Corollary 4.9, the only $\omega_\delta$-clopent subsets of $X$ are empty set and $X$ itself. So the condition that $X$ is $\omega_\delta$-disconnected is essential.

A function $f : X \rightarrow Y$ is said to be an $\omega_\delta$-continuous [3], if the inverse image of each open subset of $Y$ is an $\omega_\delta$-open subset in $X$.

Proposition 4.12. Let $f : X \rightarrow Y$ be an $\omega_\delta$-continuous surjection which maps $\omega_\delta$-open sets onto open sets. If $K$ is $\omega_\delta$-paracompact relative to $X$, then $f(K)$ is paracompact relative to $Y$.

Proof. Let $\{G_\lambda\}_{\lambda \in A}$ be any covering of $f(K)$ by open sets of $Y$. Since, $f$ is $\omega_\delta$-continuous surjection function, then $\{f^{-1}(G_\lambda)\}$ is a covering of $f(K)$ by $\omega_\delta$-open subsets of $X$. But, $K$ is $\omega_\delta$-paracompact relative to $X$, thus, there exists an $\omega_\delta$-locally finite $\omega_\delta$-open family $\{V_\gamma\}_{\gamma \in \Gamma}$ of subsets of $X$ which refines $\{f^{-1}(G_\lambda)\}_{\lambda \in A}$ and covers $K$, so by hypothesis, $\{f(V_\gamma)\}_{\gamma \in \Gamma}$ is a locally finite family of open subsets of $Y$ which refines $\{G_\lambda\}_{\lambda \in A}$ and covers $f(K)$. Therefore, $f(K)$ is paracompact relative to $Y$. 

□
The $\omega_5$-paracompactness and paracompactness are independent, as shown in the following examples:

**Example 4.13.** Consider the co-countable topology $(\mathbb{R}, \tau_{COC})$, $(\tau_{COC})_{\omega_5} = \{\emptyset, \mathbb{R}\}$, so $\mathbb{R}$ is $\omega_5$-paracompact, but it is not paracompact.

**Example 4.14.** Consider the closed unit interval $I$ of the usual topology $(\mathbb{R}, \mathfrak{A})$. Since $I$ is compact subset of $\mathbb{R}$, so $(I, \mathfrak{A}_I)$ is compact space and hence it is paracompact. Since $(I, \mathfrak{A}_I)$ is $\omega_5 - T_2$ but not $\omega_5$-normal, so by Proposition 3.9, it is not $\omega_5$-paracompact.

**Theorem 4.15.** The union of $\omega_5$-locally finite family of $\omega_5$-open $\omega_5$-paracompact subsets of a space $X$ is $\omega_5$-paracompact subset.

**Proof.** Let $\{U_\alpha : \alpha \in I\}$ be any $\omega_5$-locally finite family of $\omega_5$-open, $\omega_5$-paracompact sets and take $U = \bigcup\{U_\alpha : \alpha \in I\}$. Let $\{V_\beta : \beta \in J\}$ be any $\omega_5$-open covering of $U$, by $\omega_5$-open subsets of $X$. Then, for each $\alpha$, $\{V_\beta \cap U_\alpha : \beta \in J\}$ is a covering of $U_\alpha$ by $\omega_5$-open sets. Since $U_\alpha$ is $\omega_5$-paracompact relative to $X$, then there exist $\omega_5$-locally finite family of $\omega_5$-open sets, $\{D_\lambda : \lambda \in K^\alpha\}$ of $X$ which refines $\{V_\beta \cap U_\alpha : \beta \in J\}$ and covers $U_\alpha$, where $K$ is infinite cardinal. Consider the family $F = \{D_\lambda : \lambda \in K^\alpha, \alpha \in I\}$. Then $F$ is $\omega_5$-locally finite $\omega_5$-open refinement of $\{V_\beta : \beta \in J\}$ and hence, $U$ is $\omega_5$-paracompact relative to $X$. □

**References**


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