Fixed point results in $\alpha - \eta$-complete metric spaces via w-distances

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Abstract. We introduce the concepts of $(\alpha, \eta, \mathfrak{F}, \psi, \phi, p)$-rational contractions via w-distances. For such type contractions, we ensure the existence and uniqueness of a fixed point in the setting of $\alpha - \eta-$complete metric spaces. An example is also given to support our given theorems. Our paper generalizes the work of Lakzian et al. [11] and several related articles in the literature.

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1. Introduction and preliminaries

Fixed point strategies have been connected in numerous fields in sciences, In particular, applied and natural sciences. Over the years, Several mathematicians have been generalized and amplified the famous contractive Banach contraction principle in many directions ([1, 2, 9, 10, 16, 3, 22, 5, 23, 24, 25, 26, 27, 28, 29, 30, 32, 33, 34, 35, 36]). In 2012, Samet et al. [6] established several fixed point results for $\alpha - \psi$-contractive mappings in complete metric spaces. The concept of a w-distance metric space was presented by Kada et al [7] to generalize some important results in nonconvex minimizations. In 2012, Imdad and Rouzkard [8] used the concept of a w-distance to proved fixed point results in ordered metric spaces. Recently, Lakzian et al. [11] proved some fixed point theorems for $\alpha - \psi$-contractions involving w-distances.

Throughout this article, we denote by $\mathbb{N}$ the set of all positive integers. Also, we denote by Ψ the family of all functions $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions:

(ψ_1) ψ is nondecreasing;

(ψ_2) $\sum_{n=1}^{\infty} \psi^n(t) < \infty$, where ψ^n is the nth iterate of ψ.

These functions are known in the literature as (c)-comparison functions or Bianchini-Grandolfi gauge functions (see [31]).

Remark 1.1. For each $\psi \in \Psi$, we have:

(ψ_3) $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$;

(ψ_4) $\psi(t) < t$ for all $t > 0$;

(ψ_5) $\psi(0) = 0$.

Denote by Φ the set of all continuous functions $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(t) > 0$ for all $t > 0$ and $\phi(0) = 0$.

Definition 1.1 ([6]). *Let $f : Y \to Y$ and $\alpha : Y \times Y \to [0, \infty)$ be two given mappings. The mapping f is called α-admissible if*

$$y, z \in Y,\ \alpha(y, z) \geq 1 \ \Rightarrow\ \alpha(fy, fz) \geq 1.$$

Definition 1.2 ([6]). *Let (Y, d) be a metric space and let $f : Y \to Y$ be a given mapping. f is said an $\alpha - \psi$-contraction if there exist two functions $\alpha : Y \times Y \to [0, \infty)$ and $\psi \in \Psi$ such that*

$$\alpha(y, z)d(fy, fz) \leq \psi(d(y, z))$$

for all $y, z \in Y$.

Afterward, many articles appeared as extensions and generalizations of Definition 1.2 (see also [13, 14, 15, 17]).

Salimi et al. [20] presented the concept of α-admissible mappings with respect to η.

Definition 1.3. *Let $f : Y \to Y$ and $\alpha, \eta : Y \times Y \to [0, \infty)$. The mapping f is α-admissible with respect to η if*

$$y, z \in Y,\ \alpha(y, z) \geq \eta(y, z) \Rightarrow \alpha(fy, fz) \geq \eta(fy, fz). \tag{1.1}$$

If we take $\eta(y, z) = 1$, Definition 1.3 reduces to Definition 1.1. Also, if we take $\alpha(y, z) = 1$, then we say that f is an η-subadmissible mapping.

Hussain et al. [19] presented the concept of $\alpha-\eta$-complete metric spaces and $\alpha-\eta$-continuous functions.

Definition 1.4. *Let (Y, d) be a metric space and $\alpha, \eta : Y \times Y \to [0, \infty)$. The metric space Y is said to be $\alpha-\eta$-complete if every Cauchy sequence $\{y_n\}$ in Y with $\alpha(y_n, y_{n+1}) \geq \eta(y_n, y_{n+1})$ for all $n \in \mathbb{N}$ converges in Y. Y is said an α-complete metric space if $\eta(y, z) = 1$ for all $y, z \in Y$ and one says that (Y, d) is an η-complete metric space if $\alpha(y, z) = 1$ for all $y, z \in Y$.*

Example 1.1 ([19])**.** Let $Y = (0, +\infty)$ and $d(y, z) =\mid y - z \mid$ for all $y, z \in Y$. Note that Y is not complete. Let Y be a closed subset of Y. Define $\alpha, \eta : Y \times Y \to [0, +\infty)$ by

$$\alpha(y, z) = \begin{cases} (y + z)^2, & \text{if } y, z \in Y, \\ 5, & \text{otherwise,} \end{cases}$$

and

$$\eta(y, z) = 2yz.$$

Then (Y, d) is an $\alpha-\eta$-complete metric space.

Definition 1.5 ([19])**.** Let (Y, d) be a metric space and $\alpha, \eta : Y \times Y \to [0, \infty)$. A mapping $f : Y \to Y$ is said to be $\alpha-\eta$-continuous on (Y, d), if for each sequence $\{y_n\}$ in Y with $y_n \to y$ as $n \to \infty$ and $\alpha(y_n, y_{n+1}) \geq \eta(y_n, y_{n+1})$ for all $n \in \mathbb{N}$ implies $f(y_n) \to f(y)$ as $n \to \infty$.

Example 1.2 ([19])**.** Let $Y = [0, +\infty)$ be equipped by the metric $d(y, z) =\mid y - z \mid$ for all $y, z \in Y$. Define $f : Y \to Y$ and $\alpha, \eta : Y \times Y \to [0, +\infty)$ by

$$fy = \begin{cases} y^5, & \text{if } y \in [0, 1], \\ \sin \pi y + 2, & \text{if } y \in (0, +\infty), \end{cases}$$

$$\alpha(y, z) = \begin{cases} y^2 + z^2 + 1, & \text{if } y, z \in [0, 1], \\ 0, & \text{otherwise,} \end{cases}$$

$$\eta(y, z) = y^2.$$

Clearly, f is not continuous, but f is $\alpha-\eta$-continuous on (Y, d). Indeed, if $y_n \to y$ as $n \to \infty$ and $\alpha(y_n, y_{n+1}) \geq \eta(y_n, y_{n+1})$, then $y_n \in [0, 1]$ and so $\lim_{n\to\infty} fy_n = \lim_{n\to\infty} y_n^5 = y^5 = fy$.

In 2014, Ansari [18] introduced the concept of $\mathcal{C}$-class functions as follows:

Definition 1.6. *A continuous function $\mathfrak{F} : [0,\infty)^2 \to \mathbb{R}$ is called a C-class function if for any $s,t \in [0,\infty)$, the following conditions hold:*
(i) $\mathfrak{F}(s,t) \le s$;
(ii) $\mathfrak{F}(s,t) = s$ implies that either $s = 0$ or $t = 0$.

As examples of *C-class* functions, we state:

1. $\mathfrak{F}(r,t) = r - t$ for all $r,t \in [0,\infty)$;
2. $\mathfrak{F}(r,t) = mr$ for all $r,t \in [0,\infty)$ where $0<m<1$;
3. $\mathfrak{F}(r,t) = \frac{r}{(1+t)^\rho}$ for all $r,t \in [0,\infty)$ where $\rho \in (0,\infty)$;
4. $\mathfrak{F}(r,t) = (r+l)^{\frac{1}{(1+t)^\rho}} - l$ for all $r,t \in [,\infty)$ where $l > 1$ and $\rho \in (0,\infty)$;
5. $\mathfrak{F}(r,t) = r \log_{t+a} a$ for all $r,t \in [0,\infty)$ where $a > 1$;
6. $\mathfrak{F}(r,t) = r - (\frac{1+r}{2+r})(\frac{t}{1+t})$ for all $r,t \in [0,\infty)$;
7. $\mathfrak{F}(r,t) = r\beta(r)$ for all $r,t \in [0,\infty)$ where $\beta : [0,\infty) \to [0,1)$ and is continuous;
8. $\mathfrak{F}(r,t) = r - \varphi(r)$ where $\varphi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(t) = 0 \Leftrightarrow t = 0$;
9. $\mathfrak{F}(r,t) = sh(r,t)$ where $h : [0,\infty)\times[0,\infty) \to [0,\infty)$is a continuous function such that $h(t,r) < 1$ for all $t,r > 0$;
10. $\mathfrak{F}(r,t) = r - (\frac{2+t}{1+t})t$ for all $r,t \in [0,\infty)$;
11. $\mathfrak{F}(r,t) = \sqrt[n]{\ln(1+r^n)}$ for all $r,t \in [0,\infty)$.

Definition 1.7 ([7])**.** Let (Y,d) be a metric space. A function $p : Y\times Y \to [0,\infty)$ is called a w-distance on Y if it satisfies the following properties:

(p_1) $p(y,v) \le p(y,z) + p(z,v)$ for any $y,v,z \in Y$;

(p_2) p is lower semicontinuous in its second variable; i.e, if $y \in Y$ and $z_n \to z \in Y$, then $p(y,z) \le \liminf_{n\to 0} p(y,z_n)$;

(p_3) for any $\epsilon > 0$, there exists $\delta > 0$ such that $p(v,y) \le \delta$ and $p(v,z) \le \delta$ imply $d(y,z) \le \epsilon$.

Lakzian et al. [11] proved some fixed point theorems for $\alpha - \psi$-contractions involving w-distances. For other results dealing with w-distances, see [4, 12].

Definition 1.8 ([11]). *Let (Y, d) be a metric space with a w-distance p. The mapping $f : Y \to Y$ is said an (α, ψ, p)-contraction if there exist two functions $\alpha : Y \times Y \to [0, \infty)$ and $\psi \in \Psi$ such that*

$$\alpha(y, z)p(fy, fz) \leq \psi(p(y, z)) \tag{1.2}$$

for all $y, z \in Y$.

Theorem 1.1 ([11]). *Let (Y, d) be a complete metric space equipped with a w-distance p and let $f : Y \to Y$ be an $(\alpha, \psi, p)-$contractive mapping. Assume that the following conditions hold:*

1. *f is an α-admissible mapping;*
2. *there exists $y_0 \in Y$ such that $\alpha(y_0, fy_0) \geq 1$;*
3. *Either f is continuous, or for any sequence $\{y_n\}$ in Y if $\alpha(y_n, y_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $y_n \to y \in Y$ as $n \to \infty$, then $\alpha(y_n, y) \geq 1$.*

Then there exists $u \in Y$ such that $fu = u$. Moreover, if $\alpha(u, u) \geq 1$, then $p(u, u) = 0$.

We need the following Lemma to prove our main results.

Lemma 1.1 ([7]). *Let (Y, d) be a metric space and let p be a w-distance on Y. Assume that $\{y_n\}$ and $\{z_n\}$ are sequences in Y, $\{\gamma_n\}$ and $\{\lambda_n\}$ are sequences in $[0, \infty)$ converging to 0, and let $y, t, z \in Y$. Then the following assertions hold:*

1. *If $p(y_n, z) \leq \gamma_n$ and $p(y_n, t) \leq \lambda_n$ for all $n \in \mathbb{N}$, then $z = t$. In particular, if $p(y, z) = p(y, t) = 0$, then $z = t$.*
2. *If $p(y_n, z_n) \leq \gamma_n$ and $p(y_n, z) \leq \lambda_n$ for all $n \in \mathbb{N}$, then $\{z_n\}$ converges to z;*
3. *If $p(y_n, y_m) \leq \gamma_n$ for all $n, m \in \mathbb{N}$, then $\{y_n\}$ is a Cauchy sequence.*

In this article, motivated by the idea of $\mathcal{C}$-class function and w-distance due to Kada et al [7], we present the class of $(\alpha, \eta, \mathfrak{F}, \psi, \phi, p)$-rational contractive mappings via w-distances and prove some fixed point results for this class of mappings. We also present an example to support our work.

2. Main results

Definition 2.1. *Let (Y, d) be a metric space with a w-distance p and $f : Y \to Y$ be a given mapping. We say that f is an $(\alpha, \eta, \mathfrak{F}, \psi, \phi, p)-$rational contractive mapping if there exist $\alpha, \eta : Y \times Y \to [0, \infty)$, $\psi \in \Psi$, $\mathfrak{F} \in \mathcal{C}$ and $\phi \in \Phi$ such that*

$$\alpha(y, z) \geq \eta(y, z) \Rightarrow p(fy, fz) \leq \mathfrak{F}(\psi(M(y, z)), \phi(M(y, z))), \tag{2.1}$$

where

$$M(y, z) = \max\{p(y, z), \frac{p(y, fy)}{1 + p(y, fy)}, \frac{p(z, fz)}{1 + p(z, fz)}\}$$

for all $y, z \in Y$.

Theorem 2.1. *Let (Y, d) be a metric space equipped with a w-distance p and let $f : Y \to Y$ be an $(\alpha, \eta, \mathfrak{F}, \psi, \varphi, p)-$rational contractive mapping. Assume that the following conditions hold:*

1. *(Y, d) is an $\alpha - \eta-$complete metric space;*
2. *f is α-admissible with respect to η;*
3. *there exists $y_0 \in Y$ such that $\alpha(y_0, fy_0) \geq \eta(y_0, fy_0)$;*
4. *Either f is $\alpha-\eta-$continuous, or for any sequence $\{y_n\}$ in Y if $\alpha(y_n, y_{n+1}) \geq \eta(y_n, y_{n+1})$ for all $n \in \mathbb{N}$ and $y_n \to y \in Y$ as $n \to \infty$, then $\alpha(y_n, y) \geq \eta(y_n, y)$.*

Then there exists $u \in Y$ such that $fu = u$. Moreover, if $\alpha(u, u) \geq \eta(u, u)$, then $p(u, u) = 0$.

Proof. From condition(3), there exists $y_0 \in Y$ such that

$$\alpha(y_0, fy_0) \geq \eta(y_0, fy_0).$$

Define the sequence $\{y_n\}$ in Y by

$$y_{n+1} = fy_n$$

for all $n \in \mathbb{N}$. If $y_n = y_{n+1}$ for some $n \in \mathbb{N}$, then f has a fixed point. From now on, we assume that $y_n \neq y_{n+1}$ for all $n \in \mathbb{N}$. Since f is α-admissible with respect to η and $\alpha(y_0, fy_0) \geq \eta(y_0, fy_0)$, we obtain that

$$\alpha(y_1, y_2) = \alpha(fy_0, fy_1) \geq \eta(fy_0, fy_1) = \eta(y_1, y_2). \tag{2.2}$$

By continuing this process, we have

$$\alpha(y_n, y_{n+1}) \geq \eta(y_n, y_{n+1}) \tag{2.3}$$

for all $n \in \mathbb{N}$. Since f is an $(\alpha, \eta, \mathfrak{F}, \psi, \phi)$-rational contractive mapping, we get

$$\begin{aligned} p(y_n, y_{n+1}) &= p(fy_{n-1}, fy_n) \\ &\leq \mathfrak{F}(\psi(M(y_{n-1}, y_n)), \phi(M(y_{n-1}, y_n)), \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} M(y_{n-1}, y_n) &= \max\{p(y_{n-1}, y_n), \frac{p(y_{n-1}, fy_{n-1})}{1 + p(y_{n-1}, fy_{n-1})}, \frac{p(y_n, fy_n)}{1 + p(y_n, fy_n)}\} \\ &= \max\{p(y_{n-1}, y_n), \frac{p(y_{n-1}, y_n)}{1 + p(y_{n-1}, y_n)}, \frac{p(y_n, y_{n+1})}{1 + p(y_n, y_{n+1})}, \} \\ &\leq \max\{p(y_{n-1}, y_n), p(y_n, y_{n+1})\}. \end{aligned} \tag{2.5}$$

Now, if $M(y_{n-1},y_n)=p(y_n,y_{n+1})$ for some n, then from (2.4)

$$\begin{aligned} p(y_n,y_{n+1}) &\leq \mathfrak{F}(\psi(p(y_n,y_{n+1})),\phi(p(y_n,y_{n+1})) \\ &< \psi(p(y_n,y_{n+1}) < p(y_n,y_{n+1}), \end{aligned} \tag{2.6}$$

which is a contradiction. Thus $M(y_{n-1},y_n)=p(y_{n-1},y_n)$ all $n\in\mathbb{N}$. Hence

$$\begin{aligned} p(y_n,y_{n+1}) &\leq \mathfrak{F}(\psi(p(y_{n-1},y_n)),\phi(p(y_{n-1},y_n)) \\ &< \psi(p(y_{n-1},y_n). \end{aligned}$$

Iteratively, we find that

$$p(y_n,y_{n+1}) < \psi^n(p(y_0,y_1)). \tag{2.7}$$

From (ψ_3), we can conclude that $\lim_{n\to\infty}\psi^n(p(y_0,y_1))=0$. This implies that

$$\lim_{n\to\infty} p(y_n,y_{n+1})=0. \tag{2.8}$$

We want to show that $\lim_{n\to\infty} p(y_n,y_m)=0$. Assume that with $n,m\in\mathbb{N}$ with $m>n$. Then

$$p(y_n,y_m) \leq \sum_{l=n}^{m-1} p(y_l,y_{l+1}) \leq \sum_{l=n}^{\infty}\psi^l(p(y_0,y_1)). \tag{2.9}$$

Since the properties of ψ, we have $\lim_{n\to\infty}\sum_{l=n}^{\infty}\psi^l(p(y_0,y_1)) = 0$. Letting $n,m\to\infty$ in 2.9, we get

$$\lim_{n\to\infty} p(y_n,y_m)=0, \tag{2.10}$$

that is, $\{y_n\}$ is a Cauchy sequence in the $\alpha-\eta$-complete metric space, so there exists $u\in Y$ such that $\lim_{n\to\infty} y_n=u$. Using the fact that f is $\alpha-\eta$-continuous, we obtain

$$u=\lim_{n\to\infty} y_{n+1}=\lim_{n\to\infty} fy_n = f\lim_{n\to\infty} y_n = fu.$$

That is, f has a fixed point.

Otherwise, by condition (4), we have

$$\alpha(y_n,u)\geq\eta(y_n,u) \tag{2.11}$$

for each $n\in\mathbb{N}$. From (p_2), we get

$$p(y_n,u)\leq \liminf_{n\to 0} p(y_n,y_m) := \delta_n \;\; (say)$$

for all $n\in\mathbb{N}$. By using (2.10), we have $\lim_{n\to\infty}\delta_n=0$. So

$$\lim_{n\to\infty} p(y_n,u)=0. \tag{2.12}$$

Now, from (2.1) and (2.11), we get

$$\begin{aligned} p(y_{n+1}, fu) &= p(fy_n, fu) \\ &\leq \mathfrak{F}(\psi(M(y_n,u)), \phi(M(y_n,u)), \end{aligned} \tag{2.13}$$

where

$$\begin{aligned} M(y_n,u) &= \max\{p(y_n,u), \frac{p(y_n, fy_n)}{1+p(y_n, fy_n)}, \frac{p(u,fu)}{1+p(u,fu)}\} \\ &= \max\{p(y_n,u), \frac{p(y_n, y_{n+1})}{1+p(y_n, y_{n+1})}, \frac{p(u,u)}{1+p(u,u)}\}. \end{aligned} \tag{2.14}$$

By using (2.12) and (2.8), we have

$$M(y_n,u) = \max\{0, \frac{0}{1+0}, \frac{0}{1+0}\} = 0. \tag{2.15}$$

Then (2.13) becomes

$$\lim_{n\to 0} p(y_{n+1}, fu) \leq \mathfrak{F}(\psi(0), \phi(0)),$$

which implies that $\lim_{n\to 0} p(y_{n+1}, fu) = 0$. By the triangular inequality, we obtain

$$p(y_n, fu) \leq p(y_n, y_{n+1}) + p(y_{n+1}, fu). \tag{2.16}$$

Thus,

$$\lim_{n\to 0} p(y_n, fu) = 0. \tag{2.17}$$

From Lemma 1.1, (2.12) and (2.17), we deduce that $fu = u$. Now, we suppose that $\alpha(u,u) \geq \eta(u,u)$. If $p(u,u) > 0$, from (2.4), we have

$$\begin{aligned} p(u,u) &= p(fu,fu) \\ &\leq \mathfrak{F}(\psi(M(u,u)), \phi(M(u,u)), \end{aligned} \tag{2.18}$$

where

$$\begin{aligned} M(u,u) &= \max\{p(u,u), \frac{p(u,fu)}{1+p(u,fu)}, \frac{p(u,fu)}{1+p(u,fu)}\} \\ &= \max\{p(u,u), \frac{p(u,u)}{1+p(u,u)}, \frac{p(u,u)}{1+p(u,u)}\} \\ &< \max\{p(u,u), p(u,u)\} = p(u,u). \end{aligned} \tag{2.19}$$

Therefore,

$$\begin{aligned} p(u,u)) &= p(fu,fu)) \\ &\leq \mathfrak{F}(\psi(p(u,u)), \phi(p(u,u)) \\ &< \psi(p(u,u)) < p(u,u), \end{aligned}$$

which is impossible. Thus $p(u,u) = 0$. ☐

Theorem 2.2. *Let (Y, d) be a metric space equipped with a w-distance p and let $f : Y \to Y$ be an $(\alpha, \eta, \mathfrak{F}, \psi, \varphi, p)$-rational contractive mapping. Assume that the following conditions hold:*

1. *(Y, d) is an $\alpha - \eta-$complete metric space;*
2. *f is α-admissible with respect to η;*
3. *there exists $y_0 \in Y$ such that $\alpha(y_0, fy_0) \geq \eta(y_0, fy_0)$;*
4. *if for $z \in Y$ with $z \neq fz$, $\inf\{p(y, z) + p(y, fy) : y \in Y\} > 0$.*

Then there exists $u \in Y$ such that $fu = u$. Moreover, if $\alpha(u, u) \geq \eta(u, u)$, then $p(u, u) = 0$.

Proof. Following the proof of Theorem 2.1, $\{y_n\}$ is a Cauchy sequence in the $\alpha - \eta$-complete metric space (Y, d). Then there exists $u \in Y$ such that $y_n \to u$ as $n \to \infty$. Assume that $u \neq fu$. Then

$$\begin{aligned} 0 &< \inf\{p(y, u) + p(y, fy) : y \in Y\} \\ &\leq \inf\{p(y_n, u) + p(y_n, y_{n+1}) : y \in Y\} \\ &= 0, \end{aligned}$$

which is a contradiction. Thus $u = fu$. The last step of the proof follows as in Theorem 2.1 and in order to avoid duplication, the details are skipped. □

If we take $\eta(y, z) = 1$, we obtain the following two corollaries.

Corollary 2.1. *Let (Y, d) be a metric space equipped with a w-distance p and let $f : Y \to Y$ be an $(\alpha, \mathfrak{F}, \psi, \phi)-$rational contractive mapping. Assume that the following conditions hold:*

1. *(Y, d) is an $\alpha-$complete metric space;*
2. *f is an α-admissible mapping;*
3. *there exists $y_0 \in Y$ such that $\alpha(y_0, fy_0) \geq 1$;*
4. *Either f is $\alpha-$continuous, or for any sequence $\{y_n\}$ in Y if $\alpha(y_n, y_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $y_n \to y \in Y$ as $n \to \infty$, then $\alpha(y_n, y) \geq 1$.*

Then there exists $u \in Y$ such that $fu = u$. Moreover, if $\alpha(u, u) \geq 1$, then $p(u, u) = 0$.

Corollary 2.2. *Let (Y, d) be a metric space equipped with a w-distance p and let $f : Y \to Y$ be an $(\alpha, \mathfrak{F}, \psi, \phi, p)-$rational contractive mapping. Assume that the following conditions hold:*

1. *(Y, d) is an $\alpha-$complete metric space;*

2. *f is an α-admissible mapping;*

3. *there exists $y_0 \in Y$ such that $\alpha(y_0, fy_0) \geq 1$;*

4. *if for $z \in Y$ with $z \neq fz$, $\inf\{p(y,z) + p(y, fy) : y \in Y\} > 0$.*

Then there exists $u \in Y$ such that $fu = u$. Moreover, if $\alpha(u,u) \geq 1$, then $p(u,u) = 0$.

If we take $M(y,z) = p(y,z)$, we get the following results.

Corollary 2.3. *Let (Y,d) be a metric space equipped with a w-distance p and let $f : Y \to Y$ be such that*

$$\alpha(y,z) \geq \eta(y,z) \Rightarrow p(fy, fz) \leq \mathfrak{F}(\psi(p(y,z)), \phi(p(y,z))), \tag{2.20}$$

for all $y, z \in Y$, where $\mathfrak{F} \in \mathcal{C}$, $\psi \in \Psi$ and $\phi \in \Phi$. Assume that

1. *(Y,d) is an $\alpha - \eta-$complete metric space;*

2. *f is an α-admissible with respect to η;*

3. *there exists $y_0 \in Y$ such that $\alpha(y_0, fy_0) \geq \eta(y_0, fy_0)$;*

4. *Either f is $\alpha-\eta-$continuous, or for any sequence $\{y_n\}$ in Y if $\alpha(y_n, y_{n+1}) \geq \eta(y_n, y_{n+1})$ for all $n \in \mathbb{N}$ and $y_n \to y \in Y$ as $n \to \infty$, then $\alpha(y_n, y) \geq \eta(y_n, y)$.*

Then there exists $u \in Y$ such that $fu = u$. Moreover, if $\alpha(u,u) \geq \eta(u,u)$, then $p(u,u) = 0$.

Corollary 2.4. *Let (Y,d) be a metric space equipped with a w-distance p and let $f : Y \to Y$ be defined as in (2.20) and satisfy the following conditions:*

1. *(Y,d) is an $\alpha-$complete metric space;*

2. *f is α-admissible with respect to η;*

3. *there exists $y_0 \in Y$ such that $\alpha(y_0, fy_0) \geq \eta(y_0, fy_0)$;*

4. *if for $z \in Y$ with $z \neq fz$, $\inf\{p(y,z) + p(y, fy) : y \in Y\} > 0$.*

Then there exists $u \in Y$ such that $fu = u$. Moreover, if $\alpha(u,u) \geq \eta(u,u)$, then $p(u,u) = 0$.

Corollary 2.5. *Let (Y,d) be a complete metric space equipped with a w-distance p and let $f : Y \to Y$ be a mapping. Assume that there exists $\tau \in [0,1)$ such that, for all $y, z \in Y$,*

$$p(fy, fz) \leq \tau \mathfrak{F}(p(y,z), p(y,z)),$$

where $\mathfrak{F} \in \mathcal{C}$. Then there exists $u \in Y$ such that $fu = u$. Moreover, $p(u,u) = 0$.

Proof. Take in Corollary 2.3, $\alpha(y,z)=\eta(y,z)=1$, $\psi(k)=\tau k$ and $\phi(k)=k$ where $\tau\in[0,1)$. □

Example 2.1. Let $Y=[0,\infty)$ and $d(y,z)=\mid y-z\mid$ for all $y,z\in Y$. The function p defined by $p(y,z)=y$ is a w-distance on (Y,d). Take $\psi(k)=k,\phi(k)=k^2$ and $\mathfrak{F}(k,r)=\frac{k}{2+k}$ for all $k,r\in[0,\infty)$. Consider $f:Y\to Y$ as

$$f(y)=\begin{cases}\frac{1}{4}y & \text{if } 0\le y\le 1,\\ 2y, & \text{if y} > 1.\end{cases}$$

Given $\alpha,\eta:Y\times Y\to[0,\infty)$ by

$$\alpha(y,z)=\begin{cases}1 & \text{if } 0\le y,z\le 1,\\ 3, & \text{otherwise},\end{cases}$$

$$\eta(y,z)=\begin{cases}\frac{1}{5} & \text{if } 0\le y,z\le 1,\\ 5, & \text{otherwise}.\end{cases}.$$

The following statements hold:

1. (Y,d) is an $\alpha-\eta-$complete metric space;
2. f is α-admissible with respect to η;
3. there exists $y_0\in Y$ such that $\alpha(y_0,fy_0)\ge\eta(y_0,fy_0)$;
4. f is $\alpha-\eta-$continuous;
5. f satisfies the condition (2.20).

1. We will show that (Y,d) is an $\alpha-\eta$-complete metric space. Let $\{y_n\}$ be a Cauchy sequence in Y such that $\alpha(y_n,y_{n+1})\ge\eta(y_n,y_{n+1})$ for all $n\in\mathbb{N}$, then $\{y_n\}\subseteq[0.1]$. Since $([0,1],d)$ is a complete metric space, then $\{y_n\}$ converges in $[0,1]$.
2. Let $y,z\in Y$ be such that $\alpha(y,z)\ge\eta(y,z)$. Then $y,z\in[0,1]$. Since $fb\in[0,1]$ for each $b\in[0,1]$, one writes $\alpha(fy,fz)\ge\eta(fy,fz)$. Thus f is α-admissible with respect to η.
3. Take $y_0=1$. Here,
$$\alpha(1,f1)=\alpha(1,\frac{1}{4})=1\ge\frac{1}{5}=\eta(1,\frac{1}{4})=\alpha(1,f1).$$
4. Let $\{y_n\}$ be in Y such that $y_n\to y$ as $n\to\infty$ and $\alpha(y_n,y_{n+1})\ge\eta(y_n,y_{n+1})$ for all $n\in\mathbb{N}$. Recall that $\{y_n\}\subseteq[0,1]$ for all $n\in\mathbb{N}$. Also, $\lim_{n\to\infty}fy_n=\lim_{n\to\infty}\frac{1}{4}y_n=\frac{1}{4}y=fy$. That is, f is $\alpha-\eta-$continuous.

5. We will show that f satisfies (2.20). Let $y, z \in Y$ be such that $\alpha(y,z) \geq \eta(y,z)$. Then $y, z \in [0,1]$. In this case, we have

$$\begin{aligned}\mathfrak{F}(\psi(p(y,z)), \phi(p(y,z))) - p(fy, fz) &= \mathfrak{F}(y, y^2) - \frac{1}{4}y \\ &= \frac{y}{2+y} - \frac{1}{4}y \\ &= \frac{2y - y^2}{8+4y} \\ &= \frac{y(2-y)}{8+4y} \\ &\geq 0.\end{aligned}$$

Then $p(fy, fz) \leq \mathfrak{F}(\psi(p(y,z)), \phi(p(y,z)))$. Thus all assumptions of Corollary 2.3 are satisfied, and so f has a fixed point, which is, $u = 0$.

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