Metric projection in countably seminormed spaces

Nashat Faried
Department of Mathematics
Faculty of Science
Ain Shams University
11566 Abbassia, Cairo
Egypt
n_faried@hotmail.com

Hany A. El-Sharkawy
Department of Mathematics
Faculty of Science
Ain Shams University
11566 Abbassia, Cairo
Egypt
hany.elsharkawy@guc.edu.eg

Moustafa M. Zakaria*
Department of Mathematics
Faculty of Science
Ain Shams University
11566 Abbassia, Cairo
Egypt
moustafa.m.z@sci.asu.edu.eg

Abstract. We introduce a definition of metric projection in countably seminormed space. In order to do this, we have to introduce a definition of uniformly convex seminormed space, projection theorem in seminormed space, a new vision of the completion of countably seminormed space and a definition of uniformly convex countably seminormed space.

Keywords: countably seminormed space, Fréchet space, completion of countably seminormed space, uniformly convex countably seminormed space, projection theorem in countably seminormed space, metric projection.

1. Introduction

Ya. Alber introduced the notion of *metric projection* which solved some fixed point problems for a non-self contraction mapping which maps a nonempty, closed and convex subset of a Banach space E into E ([1, 2, 3]).

In his definition, he depended on the existence of the projection theorem which is well known in Hilbert space and it used, for example, in Optimization theory.

$*$. Corresponding author

Theorem 1.1 (Projection theorem in uniformly convex Banach space, [4, 5])**.** *Let K be a nonempty, closed and convex subset of a uniformly convex Banach space $(E, \| \ \|)$. For each $x \in E \setminus K$ there exists a unique $\bar{x} \in K$ such that*

$$\|x - \bar{x}\| = \inf_{y \in K} \|x - y\|.$$

The operator $P_K : E \longrightarrow K$ is called a *metric projection operator* if it assigns to each $x \in E$ its *nearest point* $\bar{x} \in K$. i.e. $P_K(x) = \bar{x}$ ([1, 5]).

Nashat Faried and Hany A. El-Sharkawy introduced definitions of uniformly convex, uniformly smooth countably normed spaces and the existence of the metric projection in a countably normed space ([6]).

2. Projection theorem in uniformly convex complete seminormed space

Definition 2.1 (Seminormed space and normed space, [7])**.** *Let E be a real vector space. A real valued function p defined on E is called a seminorm if for all $x, x_1, x_2 \in E$ and all $a \in \mathcal{R}$: $p(ax) = |a|p(x)$, $p(x_1 + x_2) \leq p(x_1) + p(x_2)$. (E, p) is the seminormed space equipped with the topology defined by seminorm p.*

If p is a seminorm and $p(x) = 0 \Rightarrow x = 0$, then p is a norm and (E, p) is a normed space.

Notation. Let L be a subspace of a topological linear space E. For a point x and a set K in E, we may write $x + L$ and $K/L = \{x + L : x \in K\}$ to denote the equivalence class $\hat{x}$ and the set of equivalence classes $\hat{K}$ belonging to the factor space E/L, respectively. So we can write $\hat{x} = x + L$ and $\hat{K} = K/L$ in $\hat{E} = E/L$.

Definition 2.2 (Normed space associated with seminormed space, [7])**.** For a seminormed space (E, p) there is a normed space $E/\ker p$ with the norm $\|x + \ker p\|_p = p(x)$ called the associated normed space with the seminormed space (E, p).

Since the norm in the associated normed space is defined by the seminorm, so the following propositions and the definition are easy to introduce and verified.

Proposition 2.3 ([8])**.** *A seminormed space is complete if and only if its associated normed space is complete.*

Proposition 2.4. *If (E, p) is seminormed space, K is non empty closed set in E, then for $x \in E \setminus K : d(x, K) = \inf_{y \in K} p(x - y) > 0$.*

Proof. Since $x \notin K$ and K is closed, then $\hat{x} \notin \hat{K}$ in the associated normed space. Hence $d(\hat{x}, \hat{K}) = \inf_{\hat{y} \in \hat{K}} \|\hat{x} - \hat{y}\|_p = \inf_{y \in K} p(x - y) > 0$. □

Definition 2.5 (Uniformly convex seminormed space, [8])**.** We call a seminormed linear space (E, p) *uniformly convex* if for any $\varepsilon \in (0, 2]$ there exists a $\delta = \delta(\varepsilon) > 0$ such that if $x, y \in E$ with $p(x) = 1$, $p(y) = 1$ and $p(x - y) \geq \varepsilon$ then $p(\frac{1}{2}\ (x + y)) \leq 1 - \delta$.

Proposition 2.6 ([8]). *A seminormed space (E,p) is uniformly convex if and only if its associated normed space is uniformly convex.*

This gives an equivalent definition of uniformly convex seminormed space.

In the following we give an example of uniformly convex seminormed space which are not normed space.

Example 2.7 ([8]). Let $\mathbb{Q}$ be the set of all rational numbers.

$\mathbb{Q}^2$ with the following norm $\|z=(x,y)\| = \sqrt{x^2+y^2}$ is uniformly convex.

Defining $\bar{\mathbb{Q}}^2 = \{\{z_n\} : \{z_n\} \textit{ is a Cauchy sequence in } \mathbb{Q}^2\}$ and a seminorm on $\bar{\mathbb{Q}}^2$ by $p(\{z_n\}) = \lim_{n\to\infty} \|z_n\|$. Since the associated normed space of $(\bar{\mathbb{Q}}^2, p)$ is $\mathbb{R}^2$ with the norm $\|(x,y)\| = \sqrt{x^2+y^2}$ and it is uniformly convex, then $(\bar{\mathbb{Q}}^2, p)$ is a uniformly convex seminormed space.

For the following theorem which we called Projection theorem in uniformly convex complete seminormed space, we give two proofs, one depending on the following lemma and the other depending on the associated normed space.

Lemma 2.8. *Let (E,p) be uniformly convex seminormed space. If $\{x_i\}$ is a sequence in E : $\lim_{i\to\infty} p(x_i) = 1$ and $\lim_{i,j\to\infty} p(\frac{x_i+x_j}{2}) = 1$. Then $\{x_i\}$ is a Cauchy sequence in (E,p).*

Proof. Suppose contrarily that the sequence $\{x_i\}$ is not a Cauchy sequence in E. In this case there exist two subsequences of natural numbers i_k, j_k such that $p(x_{i_k} - x_{j_k}) \geq \varepsilon$ for some $\varepsilon > 0$. Let $1 \leq M < 2$ be fixed, then $p(\frac{x_i}{M}) \longrightarrow \frac{1}{M} < 1$ as $i \to \infty$. Then $\exists\, n_0 \in \mathbb{N}$: $i_k, j_k \geq n_0 \implies p(\frac{x_{i_k}}{M}) \leq 1$; $p(\frac{x_{j_k}}{M}) \leq 1$, hence $p(\frac{x_{i_k}-x_{j_k}}{M}) \geq \frac{\varepsilon}{M} > \frac{\varepsilon}{2}, \forall i_k, j_k \geq n_0$. Since (E,p) is uniformly convex, then $\exists\, \delta > 0$ such that $p(\frac{x_{i_k}+x_{j_k}}{2M}) \leq 1-\delta$. Taking limit as $i_k, j_k \to \infty$, we get $\frac{1}{M} \leq 1-\delta$. Now take $M \to 1$, we get $1 \leq 1-\delta$ which is a contradiction. □

Theorem 2.9. Let (E,p) be a uniformly convex complete seminormed space. If $K \subset E$ is a nonempty closed convex set and $x \in E \setminus K$, then there exists $\bar{x} \in K$ such that $p(x-\bar{x}) = \inf_{y\in K} p(x-y)$.

Proof 1. Since $x \notin K$ and K is closed in (E,p), then $\inf_{y\in K} p(x-y) := d > 0$. There exists a sequence $\{y_i\}$ in K : $\lim_{i\to\infty} p(x-y_i) := d > 0$, and hence $\lim_{i\to\infty} p(\frac{x-y_i}{d}) = 1$,

$$\text{Taking } u_i := \frac{x-y_i}{d} \in E, \textit{ then } p(u_i) \longrightarrow 1 \text{ as } i \to \infty. \tag{1}$$

Since K is convex, then $\frac{y_i+y_j}{2} \in K, \forall\, i,j \in \mathbb{N}$. Then,

$$\begin{aligned} d &\leq p(x - \frac{y_i+y_j}{2}) = p(\frac{x-y_i}{2} + \frac{x-y_j}{2}) \leq p(\frac{x-y_i}{2}) + p(\frac{x-y_j}{2}) \\ &\implies 2 \leq p(\frac{x-y_i}{d} + \frac{x-y_j}{d}) \leq p(\frac{x-y_i}{d}) + p(\frac{x-y_j}{d}) \qquad (2) \\ &\implies 2 \leq p(u_i+u_j) \leq p(u_i) + p(u_j) \\ p(u_i+u_j) &\longrightarrow 2 \ \text{ as } i,j \to \infty. \end{aligned}$$

From equations (1) and (2), using Lemma (2.8), then $\{u_n\}$ is a Cauchy sequence in (E,p). i.e., $p(u_i - u_j) \longrightarrow 0$ that gives $p(\frac{y_i - y_j}{d}) \to 0$.

Therefore, $\{y_i\}$ is a Cauchy sequence in $K \subset (E,p)$.

Since K is closed in (E,p) and $y_i \in K, \forall\ i$, then $\bar{x} \in K$, hence $p(x - y_i) \longrightarrow p(x - \bar{x})$. Since $p(x - y_i) \longrightarrow d$, so from the uniqueness of the limit, we get $p(x - \bar{x}) = d : \bar{x} \in K$.

Proof 2. Since (E,p) is uniformly convex complete seminormed space, then $(E/\ker p, \|\ \|_p)$ is uniformly convex Banach space.

Since $K \subset E$ is a nonempty closed convex set, then $K/\ker p \subset E/\ker p$ is a nonempty closed convex set.

For $\hat{x} = x + \ker p \in (E/\ker p) \setminus (K/\ker p)$ there exist unique $\hat{\bar{x}} = \bar{x} + \ker p \in K/\ker p$ such that $\|\hat{x} - \hat{\bar{x}}\|_p = \inf_{\hat{y} \in K/\ker p} \|\hat{x} - \hat{y}\|$. Then

$$p(x - \bar{x}) = \inf_{y \in K} p(x - y).$$

Remark 2.10. *$\bar{x} \in K$ is unique up to the points $x^* \in K$ such that $\bar{x} - x^* \in \ker p$. This was expected because the seminormed space is not Hausdorff (means it has no uniqueness of limit).*

The following remarks and lemma are useful in the projection theorem in countably seminormed space.

Remark 2.11. *Let K_x be the set of all points in K which has a minimum distance from $x \notin K$. i.e. $K_x = \{x^* \in K : p(x - x^*) = \inf_{y \in K} p(x - y)\}$.*

Hence $p(\bar{x} - x^*) = 0, \forall \bar{x}, x^* \in K_x$.

Lemma 2.12. *$K_x \subset K$ is closed in (E,p).*

Proof. Let $\{x_i\}$ be a sequence in K_x such that $\{x_i\}$ converges to $x_0 \in E$. i.e. $p(x_i - x_0) \to 0$ *as* $i \to \infty$. Since K is closed, then $x_0 \in K$. And $\inf_{y \in K} p(x-y) \le p(x - x_0) \le p(x - x_i) + p(x_i - x_0), \forall i \in \mathbb{N}$. Since $p(x - x_i) = \inf_{y \in K} p(x-y), \forall i \in \mathbb{N}$ and $p(x_i - x_0) \to 0$ *as* $i \to \infty$ then $x_0 \in K_x$. □

3. Projection theorem in uniformly convex complete countably seminormed space

Definition 3.1 (Countably seminormed space, [7, 8, 9])**.** A linear space E equipped with a total countable family of seminorms $\{p_n, n \in \mathbb{N}\}$ is said to be countably seminormed space. Totality ($p_n(x) = 0, \forall n \Longrightarrow x = \theta$) guarantees that E is Hausdorff. *A complete countably seminormed space is called Fréchet space.*

Remark 3.2 ([7, 10])**.** 1-Without loss of generality (by taking the equivalent system of seminorms $\dot{p}_n(x) = \max_{m=1}^{n} p_m(x)$), one can assume that the sequence of seminorms $\{p_n; n = 1, 2, \ldots\}$ is increasing, i.e., $p_1(x) \le p_2(x) \le \cdots \le p_n(x) \le \cdots, \forall\ x \in E$.

2-Any countably seminormed space is metrizable and its metric d can be defined by $d(x,y) = \sum_{m=1}^{\infty} \frac{1}{2^m} \frac{p_m(x-y)}{1+p_m(x-y)}$.

Definition 3.3 (Compatible norms, [10]). Two norms $\| \|_1$ and $\| \|_2$ in a linear space E are said to be *compatible* if, whenever a sequence $\{x_i\}$ in E is Cauchy with respect to both norms and converges to a limit $x \in E$ with respect to one of them, it also converges to the same limit x with respect to the other norm.

Remark 3.4 ([9, 10]). If only one of the seminorms say p_{n_0} is a norm, then by adding this norm to each of the seminorms, we will get an equivalent system of increasing norms and if these norms are pair wise compatible, then E is, in fact, a countably normed space.

Since the definition of uniform convexity is always useful on complete spaces, we quoted Merkle's work ([11]) on completion of countably seminormed space while changed some notations to be suitable for our work as will as in ([8]).

For fixed seminorm p_n one can define a seminorm $\bar{p}_n$ on the space

$$E_{p_n} = \{\{x_i\} : \{x_i\} \text{ is } p_n \text{ Cauchy sequence}\}$$

as follows

$$\bar{p}_n(\{x_i\}_{i\in\mathbb{N}}) = \lim_{i\to\infty} p_n(x_i).$$

This limit exists, because $\{p_n(x_i)\}_{i\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$.

By standard arguments one can prove that $(E_{p_n}, \bar{p}_n)$ is the completion of the countably seminormed space E if equipped with only the seminorm p_n.

In fact, defining a map $\pi : E \to E_{p_n}$ by $\pi(x) = (x, x, x, \ldots)$ gives a 1-1 isometrical and isomorphical mapping of E (as a seminormed space with p_n) onto a dense linear subspace of the space $(E_{p_n}, \bar{p}_n)$.

Since for any $x \in E \quad p_n(x) \leq p_{n+1}(x), \forall n \in \mathbb{N}$, then every p_{n+1} Cauchy sequence is p_n Cauchy sequence.

So,

$$E \subset \cdots \subset E_{p_{(n+1)}} \subset E_{p_n} \subset \cdots \subset E_{p_1}.$$

Therefore, $\bigcap_{n\in\mathbb{N}} E_{p_n}$ is a countably seminormed space with

$$\bar{p}_1(x) \leq \bar{p}_2(x) \leq \cdots \leq \bar{p}_n(x) \leq \cdots, \forall\, x \in \bigcap_{n\in\mathbb{N}} E_{p_n},$$

but it is not, in general, a Hausdorff space because $\bar{p}_n(\{x_i\}_{i\in\mathbb{N}}) = 0$ for all $n \in \mathbb{N}$ does not necessarily imply that $\{x_i\}_{i\in\mathbb{N}}$ be the zero sequence. In fact $\bigcap_{n\in\mathbb{N}} \ker \bar{p}_n$ may not be the zero sequence.

On the factor space $\bigcap_{n\in\mathbb{N}} E_{p_n} / \bigcap_{i\in\mathbb{N}} \ker \bar{p}_i$ we define

$$\hat{p}_n(\hat{x}) = \hat{p}_n(\{x_i\}_{i\in\mathbb{N}} + \bigcap_{i\in\mathbb{N}} \ker \bar{p}_i) = \bar{p}_n(\{x_i\}).$$

Assume $\hat{E}_{p_n} = E_{p_n}/\bigcap_{i\in\mathbb{N}} \ker \bar{p}_i$. In this case $\bigcap_{n\in\mathbb{N}} \hat{E}_{p_n}$ equipped with the seminorms $\hat{p}_1(\hat{x}) \le \hat{p}_2(\hat{x}) \le \cdots \le \hat{p}_n(\hat{x}) \le \cdots, \forall\, \hat{x} \in \bigcap_{n\in\mathbb{N}} \hat{E}_{p_n}$ is countably seminormed Hausdorff space.

Defining $\hat{\pi} : E \to \hat{E}_{p_n}$ such that $\hat{\pi}(x) = (x, x, x, \dots) + \bigcap_{i\in\mathbb{N}} \ker \bar{p}_i$, we see that E is isomorphically isometric to a linear dense subset of $\hat{E}_{p_n}$, i.e.,

$$E \subset \cdots \subset \hat{E}_{p_n+1} \subset \hat{E}_{p_n} \subset \cdots \subset \hat{E}_{p_1}.$$

Proposition 3.5 ([8, 11])**.** *Let E be a countably seminormed space, then $\bigcap_{n\in\mathbb{N}} \hat{E}_{p_n}$ is a complete space. Moreover, there is an isometric and isomorphic, 1-1 mapping $\hat{\pi}$ of E onto a dense subspace of $\bigcap_{n\in\mathbb{N}} \hat{E}_{p_n}$. E is complete if and only if*

$$\hat{\pi}(E) = \bigcap_{n\in\mathbb{N}} \hat{E}_{p_n}.$$

We may write $E = \bigcap_{n\in\mathbb{N}} \hat{E}_{p_n}$.

Definition 3.6 (Uniformly convex countably seminormed space, [8])**.** A countably seminormed space E is said to be *uniformly convex* if $\hat{E}_{p_n}$ is uniformly convex for all n, i.e., it is associated normed space $(\bar{E}_{p_n}, \|\ \|_{\bar{p}_n})$ is uniformly convex where $\bar{E}_{p_n} = \frac{E_{p_n}}{\ker \bar{p}_n}$.

In the following we give an example of uniformly convex countably seminormed space which is not countably normed space.

Example 3.7 ([8])**.** Let $X = \mathbb{Q}^2$. Defining $X^\infty = \mathbb{Q}^2 \times \mathbb{Q}^2 \times \mathbb{Q}^2 \times \dots$ and seminorms on X^∞ by $p_n(z) = \|z_n\| = \sqrt{x_n^2 + y_n^2}$ where $z = (z_1, z_2, z_3, \dots) \in X^\infty$ and $z_n = (x_n, y_n) \in \mathbb{Q}^2$. $(X^\infty, \{p_n, n \in \mathbb{N}\})$ is a countably seminormed space.

Defining $\bar{X}^\infty_{p_n} = \{\{z^i\} : \{z^i\} \text{ is } p_n \text{ Cauchy sequence}\}$ where $z^i = (z^i_1, z^i_2, z^i_3, \dots) \in X^\infty$ and a seminorm on $\bar{X}^\infty_{p_n}$ by $\bar{p}_n(\{z^i\}) = \lim_{i\to\infty} p_n(z^i)$.

To show that $(X^\infty, \{p_n, n \in \mathbb{N}\})$ is a uniformly convex seminormed space, we must prove that $\bar{X}^\infty_{p_n}/\bigcap_{n\in\mathbb{N}} \ker \bar{p}_n$ with the seminorm $\hat{p}_n(\{z^i\} + \bigcap_{n\in\mathbb{N}} \ker \bar{p}_n) = \bar{p}_n(\{z^i\})$ is uniformly convex or its associated normed space $\bar{X}^\infty_{p_n}/\ker \bar{p}_n$ with $\|\{z^i\} + \ker \bar{p}_n\|_{p_n} = \bar{p}_n(\{z^i\})$ is uniformly convex. In the associated normed space two elements $\{z^i\}$ and $\{\grave{z}^i\}$ where $z^i = (z^i_1, z^i_2, z^i_3, \dots)$ and $\grave{z}^i = (\grave{z}^i_1, \grave{z}^i_2, \grave{z}^i_3, \dots)$ belong to the same equivalence class if both $\{z^i_n\}$ and $\{\grave{z}^i_n\}$ are p_n Cauchy sequence (i.e. it is Cauchy sequence in the n$\underline{th}$ coordinate) and the p_n limit of the difference between $\{z^i_n\}$ and $\{\grave{z}^i_n\}$ is convergent to zero. Thus the associated normed space is $\mathbb{R}^2$ with the norm $\|(x, y)\| = \sqrt{x^2 + y^2}$ which is uniformly convex.

In the following we give an example of countably normed space which is uniformly convex with respect to one seminorm but not uniformly convex countably seminormed space.

Example 3.8 ([8])**.** For $\mathbb{R}^2$ with two norms $\|(x,y)\|_2 = \sqrt{x^2+y^2}$ and $\|(x,y)\|_\infty = \max\{|x|,|y|\}$, $(\mathbb{R}^2, \|\ \|_2)$ is uniformly convex but $(\mathbb{R}^2, \|\ \|_\infty)$ is not uniformly convex.

Defining $\ell^1(\mathbb{R}^2) = \{\{z_i\} : z_i \in \mathbb{R}^2, \forall i \in \mathbb{N}, \sum_{i=1}^\infty \|z_i\|_2 < \infty\}$ and $p_0(\{z_i\}) = \sum_{i=1}^\infty \|z_i\|_2$, $p_1(\{z_i\}) = \|z_1\|_2$ and $p_n(\{z_i\}) = \|z_n\|_\infty$, $\forall n = 2,3,\ldots$.

$(\ell^1(\mathbb{R}^2), p_0)$ is not uniformly convex normed space. In fact for $\epsilon = 1$ and $z_1 = ((1,0),(0,0),(0,0),\ldots)$, $z_2 = ((0,0),(-1,0),(0,0),\ldots)$.

Clearly $p_0(z_1) = 1 = p_0(z_2)$, $p_0(z_1 - z_2) = 2 > \epsilon$. However $p_0(\frac{1}{2}(z_1+z_2)) = 1$.

So, there is no $\delta > 0$ satisfying $p_0(\frac{1}{2}(z_1+z_2)) = 1-\delta$. $(\ell^1(\mathbb{R}^2), \{p_n, n = 0,1,2,\ldots\})$ is countably normed space (see Remark(3.3)).

$(\ell^1(\mathbb{R}^2), p_1)$ is uniformly convex seminormed space. $(\ell^1(\mathbb{R}^2), p_n)$ is not uniformly convex seminormed space $\forall n = 2,3,\ldots$. Therefore, $(\ell^1(\mathbb{R}^2), \{p_n, n = 0,1,2,\ldots\})$ is not uniformly convex seminormed space.

Notation. For a complete countably seminormed space $(E, \{p_n, n \in \mathbb{N}\})$, let the completion of E with respect to the seminormed p_n be E_n and for every $x \in E \subset E_n$ the seminorm which has been defined on E_n will be p_n it self i.e. $\hat{E}_{p_n} = E_n$, $\hat{p}_n(\hat{x}) = p_n(x), \forall x \in E$.

In the following theorem, we establish one of the most important and interesting geometric properties in countably seminormed spaces.

Theorem 3.9. *Let $(E, \{p_n, n \in \mathbb{N}\})$ be a real uniformly convex complete countably seminormed space, K be a nonempty convex proper subset of E such that K is closed in each seminormed space (E_n, p_n). Then*

$$\forall\ x \in E \setminus K \quad \exists!\ \bar{x} \in K\ :\ p_n(x - \bar{x}) = \inf_{y \in K} p_i(x-y) \quad \forall\ n \in \mathbb{N}.$$

In order to proof it, we introduce the following lemmas and proposition.

Lemma 3.10. *Let (E,p) be uniformly convex seminormed space. If $\bar{x}$ and x^* in $S(x_0,d) := \{x : p(x_0 - x) = d\}$ and $\bar{x} \neq x^*$, then $p(x_0 - \frac{\bar{x}+x^*}{2}) < d$.*

Proof. Since $\bar{x}$ *and* $x^* \in S(x_0,d)$, then $p(x_0-\bar{x}) = p(x_0 - x^*) = d$.

Hence $p(\frac{x_0-\bar{x}}{d}) = p(\frac{x_0-x^*}{d}) = 1$.

Since (E,p) is uniformly convex, then $p(\frac{\frac{x_0-\bar{x}}{d}+\frac{x_0-x^*}{d}}{2}) < 1$.

Hence $p(x_0 - \frac{\bar{x}+x^*}{2}) < d$. □

Proposition 3.11 (Principle of nested sequence of closed sets)**.** *Let $(E, \{p_n, n \in \mathbb{N}\})$ be a complete countably seminormed space, K_n be a nonempty closed subset in each seminormed space (E_n,p_n), $K_{n+1} \subset K_n$ and $p_n(x-y) = 0, \forall x,y \in K_n, \forall n \in \mathbb{N}$. Then $\bigcap_{n\in\mathbb{N}} K_n$ is non empty. Moreover the intersection is only one point in E.*

Proof. Let $x_i \in K_i, \forall i \in \mathbb{N}$, we have to prove the sequence $\{x_i\}_{i=n}^\infty$ is p_n Cauchy sequence $\forall n \in \mathbb{N}$. Since $x_j - x_{j+p} \in E_j, \forall j,p \in \mathbb{N}$, then $p_n(x_j - x_{j+p}) = 0, \forall j,p \in \mathbb{N}$ and $n = 1,2,...,j$. Take the limit as $j \to \infty$, then $p_n(x_j - x_{j+p}) \to 0$

as $j \to \infty, \forall n \in \mathbb{N}$. Since E is complete, then $\{x_i\}$ has a limit $\bar{x} \in E$. Since K_i is closed $\forall i \in \mathbb{N}$, then $\bar{x} \in K_i, \forall i \in \mathbb{N}$.

$$\begin{array}{ccccccccccccc}
E = \bigcap_{i\in\mathbb{N}} E_i & \subset & \dots & \subset & E_{i+1} & \subset & E_i & \subset & \dots & \subset & E_2 & \subset & E_1 \\
p_1 \le p_2 \le \dots & & & & p_1 \le \dots \le p_{i+1} & & p_1 \le \dots \le p_i & & & & p_1 \le p_2 & & p_1 \\
\bigcap_{i\in\mathbb{N}} K_i & \subset & \dots & \subset & K_{i+1} & \subset & K_i & \subset & \dots & \subset & K_2 & \subset & K_1 \\
\bar{x} & \leftarrow & \dots & , & x_{i+1} & , & x_i & , & \dots & , & x_2 & , & x_1
\end{array}$$

Now, assume $\bar{x}, x^* \in \bigcap_{i\in\mathbb{N}} K_i$, then $p_i(\bar{x} - x^*) = 0, \forall i \in \mathbb{N}$. Hence $\bar{x} = x^*$. □

Lemma 3.12. *Let $(E, \{p_n, n \in \mathbb{N}\})$ be a countably seminormed space. If K is a nonempty closed subset in each seminormed space (E, p_n), then K is closed in $(E, \{p_n, n \in \mathbb{N}\})$.*

Let $\{x_i\}$ be a sequence in $K \subset (E, \{p_n, n \in \mathbb{N}\})$ such that $\lim_{i\to\infty} p_n(x_i - x_0) = 0, \forall n \in \mathbb{N}$. Since K is closed in each seminormed space (E, p_n), then $x_0 \in K \subset (E, p_n), \forall n \in \mathbb{N}$. Hence $x_0 \in K \subset (E, \{p_n, n \in \mathbb{N}\})$.

Proof of Theorem 3.8. Since $x \notin K$ and K is closed in (E_n, p_n), then there exists a set $K_{x,n}$ of points in K which has a minimum distance with respect to $p_n, \forall n \in \mathbb{N}$.

Since $p_n(x) \le p_{n+1}(x)$, $E_{n+1} \subset E_n, \forall n \in \mathbb{N}$ and each E_n has the seminorms $p_1, p_2, ..., p_n$, then $K_{x,n}$ are nested decreasing sequence of closed sets. i.e. $K_{x,n+1} \subset K_{x,n}\ \ n \in \mathbb{N}$. Using Proposition 3.10, $\bigcap_{n\in\mathbb{N}} K_{x,n}$ is nonempty and it has a unique point denote it by $\bar{x} \in K$.

Now we prove the uniqueness:

Assume that $x^* \in K : p_n(x - x^*) = \inf_{y\in K} p_n(x - y) := d_n, \forall n \in \mathbb{N}$ and $x^* \ne \bar{x}$. Since $\frac{\bar{x}+x^*}{2} \in K$ because of the convexity of K, then

$$d_n \le p_n(x - \frac{\bar{x} + x^*}{2}) \le p_n(\frac{x - \bar{x}}{2}) + p_n(\frac{x - x^*}{2}) = \frac{d_n}{2} + \frac{d_n}{2} = d_n,$$

i.e., $p_n(x - \frac{\bar{x}+x^*}{2}) = d_n, \forall n \in \mathbb{N}$ where $\bar{x} \ne \frac{\bar{x}+x^*}{2} \ne x^*$. This contradicts Lemma 3.9.

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