Nowhere-zero 3-flows in 4-connected simple graphs with independence number 3

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Abstract. It is mainly proved in this paper that every 4-connected simple graph with $\alpha(G) = 3$ admits a nowhere-zero 3-flow, which is a partial result to Tutte's 3-flow conjecture and generalizes the result by Luo *et al.* [Graphs and Combin., (29)(2013)1899–1907] that characterized all graphs with independence number at most 2.

Keywords: nowhere zero 3-flow, independence number.

1. Introduction

Graphs considered in this paper are simple, finite without loops. Terminology and notations not defined here can be seen in the book [1] by Bondy and Murty.

For a graph G, a set S of vertices in G is *independent* if no two vertices of S are adjacent in G. The maximum cardinality of an independent set of vertices of G is called the *independence number* of G and denoted by $\alpha(G)$. If V is a vertex subset of G, we denote by $G - V$ the graph obtained from G by deleting all vertices in V and all incident edges. For two subsets $A, B \subseteq V(G)$, $e(A, B)$ denotes the number of edges with one endpoint in A and the other endpoint in B. For simplicity, if $A = \{v\}$, we write $e(v, B)$ to mean $e(\{v\}, B)$ and if H_1 and H_2 are two subgraphs of G, we write $e(H_1, H_2)$ to mean $e(V(H_1), V(H_2))$. The set of neighbors of x in G is denoted by $N_G(x)$, or simply $N(x)$. In addition, $N[x] = N(x) \cup \{x\}$.

The complete graph on n vertices is denoted by K_n. Let K_n^- denote the graph obtained from K_n by deleting an edge. The *wheel* W_k ($k \geq 2$) is the graph obtained from a k-cycle by adding a new vertex, called the *center* of the

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wheel, which is adjacent to every vertex of the k-cycle. We define W_k to be *odd* (even) if k is odd (or even, respectively).

Let D be an orientation of a graph G, and f a function from $E(G)$ to Z with $-k < f(e) < k$ for every $e \in E(G)$. Then the pair (D, f) is a *k-flow* if it satisfies the Kirchhoff condition $\sum_{e \in E^+(v)} f(e) = \sum_{e \in E^-(v)} f(e)$ for every vertex $v \in V(G)$, where $E^+(v)$ and $E^-(v)$ denote the sets of outgoing and incoming edges of v with respect to D. A k-flow (D, f) is *nowhere-zero* if $f(e) \neq 0$ for every $e \in E(G)$.

Nowhere-zero k-flow was first introduced by Tutte [15]. We shall restrict our attention to the case that $k = 3$ and this paper is mainly motivated by the well-known 3-flow conjecture of Tutte.

Conjecture 1.1 (Tutte [15]). *Every 4-edge-connected graph admits a nowhere-zero 3-flow.*

Let Z_k denote the cyclic group of order k. A graph G is *Z_k-connected* if G has an orientation D such that for any function $b : V(G) \to Z_k$ with $\sum_{v \in V(G)} b(v) = 0$, there is a function $f : E(G) \to Z_k - \{0\}$ such that $b(v) = \sum_{e \in E^+(v)} f(e) - \sum_{e \in E^-(v)} f(e)$ for any $v \in V(G)$, where "$\sum$" refers to the addition in Z_k.

Group connectivity was first introduced by Jaeger *et al.* [9] as a generalization of nowhere-zero flows. Clearly, if G is Z_3-connected, then G admits a nowhere-zero 3-flow.

Many authors are involved in the study of nowhere-zero 3-flows. In particular, Fan and Zhou [6, 7] used degree sum condition and Ore-condition to guarantee the existence of nowhere-zero 3-flows. Recently, Thomassen [14] first made a breakthrough in the 3-flow conjecture. He verified the following theorem.

Theorem 1.2. *Every 8-edge-connected graph admits a nowhere-zero 3-flow.*

This result was soon further improved by Lovasz *et al.* [13].

Theorem 1.3. *Every 6-edge-connected graph admits a nowhere-zero 3-flow.*

So far, Conjecture 1.1 is still open. Chvátal and Erdös [2] proved a classical result: for a 2-connected simple graph G, if $k(G) \geq \alpha(G)$, then G is hamiltonian. Recently, Han, Lai, Xiong and Yan [8] studied whether a graph with a slightly weaker Chvátal and Erdös condition is supereulerian. It is well-known that every hamiltonian graph is supereulerian which has a nowhere-zero 4-flow. In view of this, we hope to explore that which hamiltonian graphs admit a nowhere-zero 3-flow. Luo *et al.*[10] characterized all the graphs with independence number at most 2 that admit a nowhere-zero 3-flow. However, it is difficult to have a complete characterization for the case of independence number 3. In this paper, we study a slightly stronger Chvátal-Erdös condition (that is $k(G) \geq \alpha(G) + 1$) and nowhere zero 3-flows and show that Conjecture 1.1 holds for a family of graphs, as follows.

Theorem 1.4. *Every 4-connected simple graph with independence number 3 admits a nowhere-zero 3-flow.*

2. Preliminaries

In this section, we establish several lemmas. Some results in [3, 4, 5, 9, 11, 12] on group connectivity are summarized as follows.

Lemma 2.1. *Let A be an abelian group with $|A| \geq 3$. The following results are known.*

(1) *K_n and K_n^- are A-connected if $n \geq 5$.*

(2) *C_n is A-connected if and only if $|A| \geq n+1$.*

(3) *W_{2k} is Z_3-connected and W_{2k+1} is not Z_3-connected, where k is a positive integer.*

(4) *L_3^+ in Fig. 1 is Z_3-connected.*

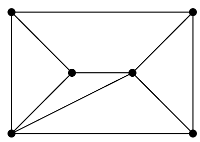


Fig. 1 The graph L_3^+

Let H be a subgraph of G. The *contraction* G/H denotes the graph obtained from G by identifying all vertices in H and then deleting all loops generating in this process.

Lemma 2.2 ([3, 4])**.** *Let H be a Z_3-connected subgraph of G. The following results hold.*

(1) *If G/H admits a nowhere-zero 3-flow, then G admits a nowhere-zero 3-flow.*

(2) *If G/H is Z_3-connected, then G is Z_3-connected.*

Lemma 2.3 ([16])**.** *Let G be a 2-connected simple graph with $\delta(G) \geq 4$. If $\alpha(G) \leq 2$, then G is Z_3-connected.*

Let G be a graph and let $v \in V(G)$ be a vertex of degree $m \geq 4$. Let $N(v) = \{v_1, v_2, \cdots, v_m\}$. The graph $G_{[vv_1, vv_2]}$ is obtained from G by deleting two edges vv_1 and vv_2 and adding a new edge v_1v_2. If $m = 2k$ is even, let

$$M =< \{v_1, v_{k+1}\}, \{v_2, v_{k+2}\}, \cdots, \{v_k, v_{2k}\} >$$

be a way to pair the vertices in $N(v)$. $G_{(v,M)}$ denotes the graph obtained from $G - v$ by adding k new edges v_iv_{k+i}, where $1 \leq i \leq k$.

Lemma 2.4. *Let G be a graph and let $v \in V(G)$ be a vertex of degree $m \geq 4$. Let $N(v) = \{v_1, v_2, \cdots, v_m\}$. Each of the following statements holds.*

(1) *(Lai, [11]) If $G_{[vv_1,vv_2]}$ is Z_3-connected, then G is Z_3-connected.*

(2) *If $m = 2k$ is even and $G_{(v,M)}$ admits a nowhere-zero 3-flow, then G admits a nowhere-zero 3-flow.*

Proof. (2) Let f^* be a nowhere-zero 3-flow of $G_{(v,M)}$. If the newly added edge v_iv_{k+i} is oriented from v_i to v_{k+i}, then the two edges vv_i and vv_{k+i} are oriented from v_i to v and from v to v_{k+i}, respectively.

Define $f : E(G) \to \{1, 2\}$ as

$$f(e) = \begin{cases} f^*(e), & \text{if } e \in E(G) - \{v_1v_{k+1}, v_2v_{k+2}, \cdots, v_kv_{2k}\} \\ f^*(v_iv_{k+i}), & \text{if } e \in \{vv_i, vv_{k+i}\}(1 \leq i \leq k). \end{cases}$$

Then f is a nowhere-zero 3-flow of G. □

Lemma 2.5. *Let G be a simple connected graph with $\delta(G) = 3$ and $\alpha(G) = 2$. If G is not Z_3-connected, then G is one of 19 graphs shown in Fig. 2 or G can be Z_3-contracted to one graph in $\{K_4^{'}, K_4^{''}\}$, where $K_4^{'}$ and $K_4^{''}$ denote the graphs obtained by connecting one edge and two edges between K_1 and K_4, respectively.*

Proof. Let G be a simple graph with $\delta(G) = 3$. Then $\kappa^{'}(G) \leq \delta(G) = 3$. If $\kappa^{'}(G) = 3$, by [Theorem 1.3, 16], $G = H_i$, where $1 \leq i \leq 19$ but $i \notin \{15, 16\}$. If $\kappa^{'}(G) = 1$, let e be a cut edge of G, then $G - e$ has two connected components which are complete graphs since $\alpha(G) = 2$. So $G = H_{15}$ or G can be Z_3-contracted to $K_4^{'}$. Similarly we can prove that if $\kappa^{'}(G) = 2$, then $G = H_{16}$ or G can be Z_3-contracted to $K_4^{''}$. □

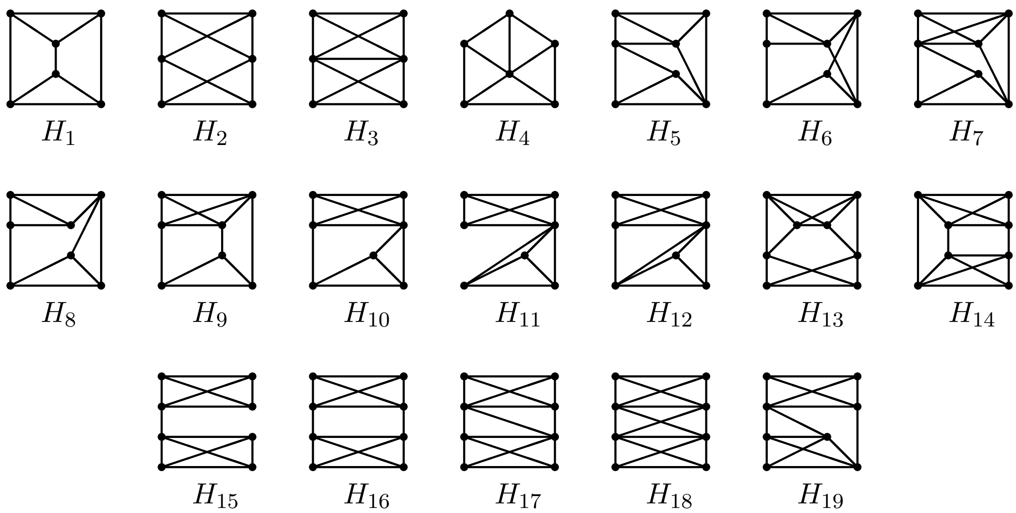


Fig. 2 19 graphs for Lemma 2.5

Lemma 2.6. *Let G be a 4-edge-connected simple graph with $\alpha(G) \leq 3$. If G contains a maximal Z_3-connected subgraph H and $\alpha(G-H) = 2$, then G admits a nowhere-zero 3-flow.*

Proof. Let $G^* = G/H$. Then G^* is also a 4-edge-connected simple graph. If G^* is Z_3-connected, then G admits a nowhere-zero 3-flow follows from Lemma 2.2. Next we assume that G^* is not Z_3-connected. By Lemma 2.3, $\alpha(G^*) = 3$. Let v^* be a vertex into which H is contracted. Then $d(v^*) \geq 4$ and $\alpha(G^* - v^*) = \alpha(G - H) = 2$. If $e(v^*, G^* - v^*) = 4$, we consider the graph $G_{(v^*,M)}$, which may contains 2-cycles, where M is any way to pair the vertices in $N(v^*)$. We contract all generating 2-cycles, the resulting graph is denoted by $G^*_{(v^*,M)}$. It is clear that $G^*_{(v^*,M)}$ is a 4-edge connected simple graph and $\alpha(G^*_{(v^*,M)}) \leq 2$. By Lemma 2.3, $G^*_{(v^*,M)}$ is Z_3-connected. It follows from Lemmas 2.1 and 2.2 that $G_{(v^*,M)}$ is also Z_3-connected. Hence G admits a nowhere-zero 3-flow follows from Lemma 2.4. Next we suppose that $e(v^*, G^* - v^*) \geq 5$. Note that $\alpha(G^*) = 3$. There must be two vertices in $G^* - v^*$, say v_1 and v_2, such that $\{v^*, v_1, v_2\}$ is an independent set of G^*. This implies that $|V(G^* - v^*)| \geq 7$ and $d(v_i) \geq 4$, where $i \in \{1, 2\}$. If $G^* - v^*$ is not connected, then $G^* - v^*$ has just two connected components which are complete graphs with order more than 4 since $\alpha(G^* - v^*) = 2$. Let G_1 and G_2 be two connected components of $G^* - v^*$. Then $e(v^*, G_1) \geq 2$ or $e(v^*, G_2) \geq 2$. It is clear that $G_1 + v^*$ or $G_2 + v^*$ is Z_3-connected which is contrary to the maximum of H. Hence $G^* - v^*$ is connected and $\delta(G^* - v^*) \geq 3$. It follows from Lemma 2.5 that $G^* - v^* \in \{H_5, H_7, H_{13}, H_{14}, H_{16}, H_{17}, H_{19}\}$ or can be Z_3-contracted to one graph in $\{K_4^{'}, K_4^{''}\}$. It is a routine work to prove that G^* is Z_3-connected, a contradiction. Therefore this lemma holds. □

3. Proof of the main theorem

Proof of Theorem 1.4. Theorem 1.3 tells us that if $\kappa^{'}(G) \geq 6$, then G admits a nowhere-zero 3-flow. So we only need to show that this theorem holds for $4 = \kappa(G) \leq \kappa^{'}(G) \leq 5$. Let A be a maximal independent set of G and $B = V(G) - A$. Then $|A| = \alpha(G) = 3$ and $|B| \geq 4$ since G is a 4-connected simple graph. It follows that $|V(G)| \geq (3 + 4) = 7$. Next we proceed our proof by induction on $|V(G)|$.

Let $|V(G)| = 7$. Then $|B| = 4$ and $N(x) = B$ for any vertex $x \in A$. Hence G contains $K_{3,4}$ as a spanning subgraph. Furthermore, we get that $G[B]$ is connected. Otherwise A is a cut of G, contrary to that $\kappa(G) = 4$. So there exist two adjacent edges uv and vw in $G[B]$. Let $x, y \in A$. Then $G[\{x, y, u, v, w\}]$ contains W_4 (center at v) as a subgraph. Contracting W_4 and all 2-cycles generating in this process, we can get a K_1. It follows from Lemma 2.1 that G is Z_3-connected, of course, admits a nowhere-zero 3-flow.

Suppose that the theorem is true for any graph $G^{'}$ with $|V(G^{'})| < |V(G)|$. Then we consider the graph G.

Case 1. There exist one vertex $x \in V(G)$ such that $d(x) = 4$.

In this case, we concentrate our attention on the graph $G_{(x,M)}$, where M is a way to pair the vertices in $N(x)$. By Lemma 2.4, if $G_{(x,M)}$ admits a nowhere-zero 3-flow, then G admits a nowhere-zero 3-flow. Therefore we suppose that there is no such M that $G_{(x,M)}$ admits a nowhere-zero 3-flow.

Claim 1. $\alpha(G_{(x,M)}) = 3$.

Proof of Claim 1. Note that $\delta(G_{(x,M)}) \geq \delta(G) = 4$ for any way to pair the vertices in $N(x)$. It follows from Lemma 2.3 that if $\alpha(G_{(x,M)}) \leq 2$, then $G_{(x,M)}$ admits a nowhere-zero 3-flow, a contradiction. Hence $3 \leq \alpha(G_{(x,M)}) \leq \alpha(G) = 3$, that is $\alpha(G_{(x,M)}) = 3$.

Note that $4 = \kappa(G) \leq \kappa'(G) \leq \delta(G) = 4$. We have that $\kappa'(G) = 4$. Let S be a minimal vertex cut of G which contains x. Then $|S| \geq 4$. Let G_1 and G_2 be two components(some component maybe not connected) of $G - S$ and $N(x) = \{x_1, x_2, x_3, x_4\}$. Denote by $N_1 = N(x) \cap V(G_1)$, $N_2 = N(x) \cap V(G_2)$ and $N_s = N(x) \cap S$. We assume, without loss of generality, that $|N_1| = 0$. Then $|N_2| \neq 0$, $|N_s| \neq 0$ and $|S| \geq 5$. If $|N_s| \leq 2$, there must be a way M which pairs the vertices of $N(x)$ such that $G_{(x,M)}$ (or $G^*_{(x,M)}$) is simple and $\kappa(G_{(x,M)}) = 4$ (or $\kappa(G^*_{(x,M)}) = 4$), where $G^*_{(x,M)}$ denotes the graph obtained from $G_{(x,M)}$ by contracting some 2-cycles. By Claim 1 and the induction hypothesis, $G_{(x,M)}$ (or $G^*_{(x,M)}$) admits a nowhere-zero 3-flow, a contradiction. Hence $|N_s| = 3$ and $|N_2| = 1$. If $G[N_s] \neq K_3$, as the above, we are done. Next we suppose that $G[N_s] = K_3$. If $e(N_2, N_s) \geq 1$, then $G_{(x,M)}[N(x)]$ is Z_3-connected. Note that $\alpha(G - N[x]) \leq 2$. It follows from Lemma 2.6, $G_{(x,M)}$ admits a nowhere-zero 3-flow, a contradiction. So $e(N_2, N_s) = 0$. Let $N_s = \{x_1, x_2, x_3\}$ and $N_2 = \{x_4\}$. Choose $M = \langle\{x_1, x_2\}, \{x_3, x_4\}\rangle$, then $G_{(x,M)}[N_s]$ is Z_3-connected. Let H be the maximal Z_3-connected subgraph which contains $G_{(x,M)}[N_s]$. Next we shall prove that $\alpha(G_{(x,M)} - H) \leq 2$. Note that $\alpha(G - N[x]) = 2$. Let $v_1, v_2 \in V(G - N[x])$ and $v_1v_2 \notin E(G)$. If $\{v_1, v_2, x_4\}$ is an independent set of three vertices, then $N_s \subseteq N(v_1) \cup N(v_2)$. This implies that $v_1 \in H$ or $v_2 \in H$. Therefore $\alpha(G_{(x,M)} - H) \leq 2$. By Lemma 2.6, $G_{(x,M)}$ admits a nowhere-zero 3-flow, a contradiction. Next we suppose that $|N_1| \geq 1$ and $|N_2| \geq 1$. It follows that $|N_s| \leq 2$. We discuss it in three cases.

Subcase 1.1. $|N_s| = 0$.

If $|N_1| = 1$, $|N_2| = 3$ or $|N_1| = 3$, $|N_2| = 1$, as the above, we are done. So we suppose that $N_1 = \{x_1, x_3\}$, $N_2 = \{x_2, x_4\}$. Choose $M = \langle\{x_1, x_2\}, \{x_3, x_4\}\rangle$, then $G_{(x,M)}$ is simple and $\kappa(G_{(x,M)}) = 4$. By Claim 1 and the induction hypothesis, $G_{(x,M)}$ admits a nowhere-zero 3-flow, a contradiction.

Subcase 1.2. $|N_s| = 1$.

We assume, without loss of generality, that $N_1 = \{x_1, x_3\}$, $N_2 = \{x_2\}$ and $N_s = \{x_4\}$. If $x_1x_4 \notin E(G)$ or $x_3x_4 \notin E(G)$, choose $M = \langle\{x_1, x_4\}, \{x_2, x_3\}\rangle$ or $M = \langle\{x_3, x_4\}, \{x_1, x_2\}\rangle$, then $G_{(x,M)}$ is simple and $\kappa(G_{(x,M)}) = 4$. By

Claim 1 and the induction hypothesis, $G_{(x,M)}$ admits a nowhere-zero 3-flow, a contradiction. Next we suppose that $x_1x_4 \in E(G)$ and $x_3x_4 \in E(G)$.

If $x_1x_3 \in E(G)$, we consider the graph $G_{(x,M)}$, where M is any way to pair the vertices in $N(x)$. It is clear that $G_{(x,M)}[\{x_1, x_3, x_4\}]$ is Z_3-connected. Let H be the maximal Z_3-connected subgraph which contains $G_{(x,M)}[\{x_1, x_3, x_4\}]$. If $\alpha(G_{(x,M)} - H) \leq 2$, by Lemma 2.6, $G_{(x,M)}$ admits a nowhere-zero 3-flow, a contradiction. Hence $\alpha(G_{(x,M)} - H) = 3$. At first, we claim that $x_2x_4 \notin E(G)$. For otherwise, $x_2 \in V(H)$. Thus $\alpha(G_{(x,M)} - H) \leq \alpha(G_{(x,M)} - N(x)) = \alpha(G - N[x]) \leq 2$, a contradiction. If G_1 is a complete graph, then $G_{(x,M)}[V(G_1) \cup \{x_4\}] \subset H$. If $y \in S - V(H)$, then $e(y, G_1) \leq 1$. Note that $\alpha(G_2) \leq 2$. Then $\alpha(G_2 + y) \leq 2$ since $\alpha(G) = 3$. Therefore $\alpha(G_{(x,M)} - H) \leq 2$, a contradiction. Next we assume that G_1 is not a complete graph. Then $\alpha(G_1) = 2$ and $\alpha(G_1 - x_1x_3) = \alpha(G_2) = 1$. Furthermore, there must be $u \in V(G_1 - x_1x_3)$ and $v \in S - \{x, x_4\}$ such that $\{u, v, x_2\}$ is an independent set of three vertices. Note that $e(x_2, \{x_1, x_3, x_4\}) = 0$. Then $\{x_1, x_3, x_4\} \subset N(u) \cup N(v)$. So $u \in H$ or $v \in H$. This yields that $\alpha(G_{(x,M)} - H) \leq 2$, a contradiction.

Next we suppose that $x_1x_3 \notin E(G)$. Then G_2 is a complete graph. If $x_2x_4 \in E(G)$, choose $M = \langle \{x_1, x_3\}, \{x_2, x_4\} \rangle$, we consider the graph $G_{(x,M)}$. If there exist one vertex $y' \in S - \{x, x_4\}$ such that $y' \in N(x_2) \cap N(x_4)$ and $y'x_3 \in E(G)$ or $y'x_1 \in E(G)$, then $G[\{x, x_1, x_2, x_4, y'\}] = W_4$ or $G[\{x, x_2, x_3, x_4, y'\}] = W_4$ which is Z_3-connected. So $G[N[x] \cup \{y'\}]$ is Z_3-connected. Note that $\alpha(G - N[x]) \leq 2$. It follows from Lemma 2.6, G is Z_3-connected. Next we assume that $y'x_3 \notin E(G)$ and $y'x_1 \notin E(G)$. Then $G_2 + y$ is also a complete graph since $\alpha(G) = 3$. Choose $M = \langle \{x_1, x_3\}, \{x_2, x_4\} \rangle$, we consider the graph $G_{(x,M)}$. It is clear that $G_{(x,M)}[V(G_2) \cup \{x_4, y'\}]$ is Z_3-connected. Let H' be the maximal Z_3-connected subgraph which contains $G_{(x,M)}[V(G_2) \cup \{x_4, y'\}]$. If $|V(G_2)| \geq 2$, then $\alpha(G_{(x,M)} - H) \leq 2$. By Lemma 2.6, $G_{(x,M)}$ is Z_3-connected, a contradiction. Hence $|V(G_2)| = 1$. Denote by $G^*_{(x,M)} = G_{(x,M)}/H'$. We find that $G^*_{(x,M)}$ is a 4-connected simple graph with $\alpha(G^*_{(x,M)}) \leq 3$. By Lemma 2.3 or the induction hypothesis, $G^*_{(x,M)}$ admits a nowhere-zero 3-flow. It follows from Lemma 2.2 that $G_{(x,M)}$ also admits a nowhere-zero 3-flow, a contradiction. Hence we suppose that $N(x_2) \cap N(x_4) \cap S = \{x\}$. Similarly if $N(x_2) \cap N(x_4) \cap V(G_2) \neq \emptyset$, we can also get that $G_{(x,M)}$ is Z_3-connected, a contradiction. So $N(x_2) \cap N(x_4) \cap V(G_2) = \emptyset$. Note that $\kappa(G) = 4$. Then $|S| \geq 5$. Thus $G_{(x,M)}$ is also a 4-connected simple graph. By induction hypothesis, $G_{(x,M)}$ admits a nowhere-zero 3-flow, a contradiction. Therefore $x_2x_4 \notin E(G)$. If $|S| \geq 5$, as the above, we are done. Hence we assume that $|S| = 4$. If $N(x_1) \cap N(x_4) = \{x\}$ or $N(x_3) \cap N(x_4) = \{x\}$, choose $M = \langle \{x_1, x_4\}, \{x_2, x_3\} \rangle$ or $M = \langle \{x_1, x_2\}, \{x_3, x_4\} \rangle$, then $G_{(x,M)}$ contains only one 2-cycle. Contracting the 2-cycle, the resulting graph is denoted by $G^*_{(x,M)}$. It is easy to prove that $G^*_{(x,M)}$ is also a 4-connected simple with $\alpha(G) = 3$. By induction hypothesis, $G^*_{(x,M)}$ admits a nowhere-zero 3-flow. It follows from Lemma 2.2 that $G_{(x,M)}$, a contradiction. Hence $|N(x_1) \cap N(x_4)| \geq 2$ and $|N(x_3) \cap N(x_4)| \geq 2$. Choose $M = \langle \{x_1, x_4\}, \{x_2, x_3\} \rangle$

or $M = \langle\{x_1, x_2\}, \{x_3, x_4\}\rangle$. It is a routine work to prove that there exist a vertex subset $x_4 \in A \subseteq S$ such that $G_{(x,M)}[V(G_1) \cup A]$ is Z_3-connected and $\alpha(G_{(x,M)} - G_{(x,M)}[V(G_1) \cup A]) \leq 2$. It follows from Lemma 2.6 that $G_{(x,M)}$ is Z_3-connected, a contradiction.

Subcase 1.3. $|N_s| = 2$.

If $|N_s| = 2$, then $|N_1| = 1$ and $|N_2| = 1$. Let $N_1 = \{x_1\}$, $N_2 = \{x_2\}$ and $N_s = \{x_3, x_4\}$.

Claim 2. $x_3x_4 \in E(G)$.

Proof of Claim 2. If $x_3x_4 \notin E(G)$, choose $M = \langle\{x_1, x_2\}, \{x_3, x_4\}\rangle$, then $G_{(x,M)}$ is simple and $\kappa(G_{(x,M)}) = \kappa(G) = 4$. By the induction hypothesis, $G_{(x,M)}$ admits a nowhere-zero 3-flow, a contradiction.

It is clear that $\alpha(G_1) \leq 1$ or $\alpha(G_2) \leq 1$ since $\alpha(G_1) + \alpha(G_2) \leq 3$. We assume, without loss of generality that $G_1 = K_m$.

Claim 3. $m \geq 2$.

Proof of Claim 4. If $m = 1$, then $N(x_1) = S$. Choose $M = \langle\{x_1, x_3\}, \{x_2, x_4\}\rangle$ or $M = \langle\{x_1, x_4\}, \{x_2, x_3\}\rangle$, we consider the graph $G_{(x,M)}$. It is clear that $G_{(x,M)}[\{x_1, x_3, x_4\}]$ is Z_3-connected. Let H be the maximal Z_3-connected subgraph which contains $G_{(x,M)}[\{x_1, x_3, x_4\}]$. If $\alpha(G_{(x,M)} - H) \leq 2$, by Lemma 2.6, $G_{(x,M)}$ is Z_3-connected, a contradiction. So $\alpha(G_{(x,M)} - H) = 3$. If $x_2x_3 \in E(G)$ or $x_2x_4 \in E(G)$, then $x_2 \in H$. Note that $\alpha(G_{(x,M)} - H) \leq \alpha(G - G[N[x]]) = 2$, a contradiction. Hence $x_2x_3 \notin E(G)$ and $x_2x_4 \notin E(G)$. Let $y \in S - H - \{x\}$. Then $yx_3 \notin E(G)$ and $yx_4 \notin E(G)$. If there exist $z \in V(G_2 - x_2)$ such that $\{y, z, x_2\}$ is an independent set of three vertices, then $x_3z \in E(G)$ and $x_4z \in E(G)$. If not, $\{y, z, x_2, x_3\}$ or $\{y, z, x_2, x_4\}$ is an independent set of four vertices, contrary to that $\alpha(G) = 3$. Thus $z \in H$. This implies $\alpha(G_{(x,M)} - H) \leq 2$, a contradiction. Therefore $m \geq 2$.

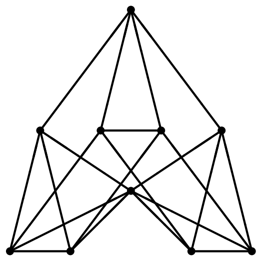
Claim 4. $|S| = 4$.

Proof of Claim 3. If $|S| \geq 5$, by the minimum property of S, then $d(x_i) \geq 5$, where $i \in \{1, 2, 3, 4\}$. Firstly, we assume that $x_1x_3 \in E(G)$ and $x_1x_4 \in E(G)$. If $x_2x_3 \in E(G)$ or $x_2x_4 \in E(G)$, choose $M = \langle\{x_1, x_3\}, \{x_2, x_4\}\rangle$, then $G_{(x,M)}[N(x)]$ is Z_3-connected. Note that $\alpha(G - N[x]) \leq 2$. By Lemma 2.6, $G_{(x,M)}$ is Z_3-connected, a contradiction. So $x_2x_3 \notin E(G)$ and $x_2x_4 \notin E(G)$. Let $V_1 = \{v | v \in S - \{x, x_3, x_4\}$ and $G[V(G_1) \cup \{v\}]$ is a complete graph $\}$. If there exist one vertex $u \in V(G_1 - x_1) \cup V_1$ such that $ux_3 \in E(G)$ or $ux_4 \in E(G)$, choose $M = \langle\{x_1, x_3\}, \{x_2, x_4\}\rangle$, then $G_{(x,M)}[V(G_1) \cup V_1 \cup \{x_3, x_4\}]$ is Z_3-connected and $\alpha(G_{(x,M)} - G_{(x,M)}[V(G_1) \cup V_1 \cup \{x_3, x_4\}]) \leq 2$. By Lemma 2.6, $G_{(x,M)}$ is Z_3-connected, a contradiction. Hence $vx_3 \notin E(G)$ and $vx_4 \notin E(G)$ for each $v \in V(G_1 - x_1) \cup V_1$. Let $V_1' = G_2 + S - V_1 - \{x, x_2, x_3, x_4\}$. If there exist one vertex $w \in V_1'$ such that $\{v, w, x_2\}$ is an independent set, then $wx_3 \in E(G)$ and $wx_4 \in E(G)$. For otherwise, $\{v, x_3, x_2, w\}$ or $\{v, x_4, x_2, w\}$ is an independent set of four vertices, contrary to that $\alpha(G) = 3$. Let $V_2 = \{w|w \in V_1'$ and $\{v, w, x_2\}$ is an independent set$\}$. Then $G_{(x,M)}[V_2 \cup \{x_1, x_3, x_4\}]$ is Z_3-connected. Note that $\alpha(G_{(x,M)} - G_{(x,M)}[V_2 \cup \{x_1, x_3, x_4\}]) \leq 2$. By Lemma 2.6, $G_{(x,M)}$ is Z_3-connected, a contradiction. Secondly, we assume, without loss of generality, that $x_1x_3 \in E(G)$, but $x_1x_4 \notin E(G)$. If $x_2x_3 \notin E(G)$, choose $M = \langle\{x_1, x_4\}, \{x_2, x_3\}\rangle$, then $\kappa(G_{(x,M)}) = \kappa(G) = 4$. By the induction hypothesis, $G_{(x,M)}$ admits a nowhere-zero 3-flow, a contradiction. Hence $x_2x_3 \in E(G)$. If $N(x_1) \cap N(x_3) \cap S = \{x\}$, choose $M = \langle\{x_1, x_3\}, \{x_2, x_4\}\rangle$, we can prove that $G_{(x,M)}[V(G_1) \cup \{x_3, x_4\}]$ is Z_3-connected or $G_{(x,M)}$ is a 4-connected simple graph with $\alpha(G_{(x,M)}) = 3$ since $|S| \geq 5$. By Lemma 2.6 or the induction hypothesis, $G_{(x,M)}$ admits a nowhere-zero 3-flow, a contradiction. So $|N(x_1) \cap N(x_3) \cap S| \geq 2$. Similarly $|N(x_3) \cap N(x_4) \cap S| \geq 2$ and $|N(x_2) \cap N(x_3) \cap S| \geq 2$. Let $y_1 \in N(x_1) \cap N(x_3)$, $y_2 \in N(x_3) \cap N(x_4)$ and $y_3 \in N(x_2) \cap N(x_3)$. If $y_1 = y_2$, then $G[\{x, x_1, x_3, x_4, y_1\}] = K_4$ (center at x_4) which is Z_3-connected. It follows that $G[N[x] \cup \{y_1\}]$ is Z_3-connected. Denote by $G^* = G/G[N[x] \cup \{y_1\}]$. Then $\alpha(G^*) \leq 2$ and $\delta(G^*) \geq 4$. By Lemmas 2.3 and 2.2, G is Z_3-connected. Similarly if $y_1 = y_3$ or $y_2 = y_3$, G is also Z_3-connected. So we suppose that $y_1 \neq y_2 \neq y_3$. Note that $\alpha(G) = 3$. Then $G[\{y_1, y_2, y_3\}]$ contains at least one edge. We assume, without loss of generality, that $y_1y_2 \in E(G)$. Then $G[N[x] \cup \{y_1, y_2, y_3\}]$ is a triangularly connected graph. It is easy to prove that $G_{(x,M)}[N[x] \cup \{y_1, y_2, y_3\}]$ is Z_3-connected, where M is any way to pair the vertices of $N(x)$. As the above, we can prove that $G_{(x,M)}$ is Z_3-connected, a contradiction. Thirdly, we assume that $x_1x_3 \notin E(G)$ and $x_1x_4 \notin E(G)$. If $x_2x_3 \notin E(G)$ or $x_2x_4 \notin E(G)$, choose $M = \langle\{x_1, x_4\}, \{x_2, x_3\}\rangle$ or $M = \langle\{x_1, x_3\}, \{x_2, x_4\}\rangle$, then $G_{(x,M)}$ is a 4-connected simple graph with $\alpha(G_{(x,M)}) = 3$ since $|S| \geq 5$. As the above, we are done. Next we suppose that $x_2x_3 \in E(G)$ and $x_2x_4 \in E(G)$. Choose $M = \langle\{x_1, x_3\}, \{x_2, x_4\}\rangle$. It is clear that $G_{(x,M)}[\{x_2, x_3, x_4\}]$ is Z_3-connected. Let H be the maximal Z_3-connected subgraph which contains $G_{(x,M)}[\{x_2, x_3, x_4\}]$. Next we shall prove that $\alpha(G_{(x,M)} - H) \leq 2$. On the one hand, $\alpha(G - N[x]) = 2$. On the other hand, if there exist two vertices $u_1, u_2 \in G_{(x,M)} - H$ such that $\{x_1, u_1, u_2\}$ is an independent set, then there must be one vertex in $\{x_2, x_3, x_4\}$, say x_3 such that $\{x_1, x_3, u_1, u_2\}$ is an independent set of four vertices, contrary to that $\alpha(G) = 3$. It follows from Lemma 2.6 that $G_{(x,M)}$ is Z_3-connected, a contradiction. Therefore $|S| = 4$.

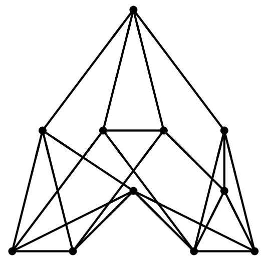
Note that $\kappa(G) = 4$. Then $e(x_i, G_1 - x_1) \geq 1$, where $i \in \{3, 4\}$. If $x_1x_3 \in E(G)$ or $x_1x_4 \in E(G)$, choose $M=\langle\{x_1, x_3\}, \{x_2, x_4\}\rangle$ or $M=\langle\{x_1, x_4\}, \{x_2, x_3\}\rangle$, respectively. It is easy to prove that $G_{(x,M)}[V(G_1) \cup \{x_1, x_3, x_4\}]$ is Z_3-connected. If there exist one vertex, say $v \in V(G_1)$ such that $vy \notin E(G)$, then $\alpha(G_2+y) \leq 2$. If $vy \in E(G)$ for each $v \in V(G_1)$, then $G_{(x,M)}[V(G_1) \cup \{x_1, x_3, x_4, y\}]$ is Z_3-connected. By Lemma 2.6, in either case, $G_{(x,M)}$ is Z_3-connected, a contradiction. So $x_1x_3 \notin E(G)$ and $x_1x_4 \notin E(G)$. Thus $m \geq 3$. If $e(\{x_3, x_4\}, G_1) \geq 3$, choose $M = \langle\{x_1, x_2\}, \{x_3, x_4\}\rangle$. It is clear that $G[V(G_1) \cup \{x_3, x_4\}]$ is Z_3-connected. By Lemma 2.6, $G_{(x,M)}$ is Z_3-connected, a contradiction. So $e(\{x_3, x_4\}, G_1) = 2$. Thus $\alpha(G[V(G_2) \cup \{x, x_3, x_4\}]) \leq 2$ and $\alpha(G[V(G_2) \cup$

$\{x, y, x_3, x_4\}]) \leq 2$ if $e(y, G_1) < |V(G_1)|$. When $m \geq 5$, G_1 is Z_3-connected follows from Lemma 2.1. Choose $M = \langle\{x_1, x_2\}, \{x_3, x_4\}\rangle$, then $G[V(G_1) \cup \{x_3, x_4\}]$ is Z_3-connected, we are done. Therefore we suppose that $3 \leq m \leq 4$. If $G_2 = K_n$, by the symmetry, $3 \leq n \leq 4$, $x_2x_i \notin E(G)$ and $e(\{x_3, x_4\}, G_2) = 2$, were $i \in \{3, 4\}$. If $x_3y \in E(G)$ or $x_4y \in E(G)$, choose $M = \langle\{x_1, x_2\}, \{x_3, x_4\}\rangle$, then $G[V(G_1) \cup \{y, x_3, x_4\}]$ is Z_3-connected. It follows from Lemmas 2.6 that $G_{(x,M)}$ is Z_3-connected, a contradiction. Hence we assume that $x_3y \notin E(G)$ and $x_4y \notin E(G)$. Let $G_1 = G_2 = K_3$. Then $G[V(G_1 \cup \{y\})] = K_4$ and $G[V(G_2 \cup \{y\})] = K_4$. Thus G is the graph (a) showed in Figure 3 which is even. So G admits a nowhere-zero 3-flow. Let $G_1 = G_2 = K_4$. If there exist $w_1 \in (V(G_1) - x_1)$ and $w_2 \in (V(G_2) - x_2)$ such that $w_1y \notin E(G)$ and $w_2y \notin E(G)$, then $\{x, y, w_1, w_2\}$ is an independent set of four vertices, contrary to that $\alpha(G) = 3$. Hence, $e(y, G_1) = 4$ or $e(y, G_2) = 4$. Thus, G contains Z_3-connected subgraph W_4. By Lemma 2.6, G admits a nowhere-zero 3-flow. At last, we consider that $G_1 = K_3$ and $G_2 = K_4$. Then G is even or the graph (b) showed in Figure 3, which has a $\{C_3, C_4, C_3 \oplus C_4\}$-composition. So G also admits a nowhere-zero 3-flow. Next we suppose that $G_2 \neq K_n$. It is easy to check that $G_2 - x_2 = K_l$ since $\alpha(G) = 3$, where $l \geq 2$. Note that $\kappa(G) = 4$ and $\alpha(G_2) = 2$. We have that $1 \leq e(x_2, K_l) \leq l - 1$. Note that $\alpha(G_1 + y) \leq 2$. If there exist some way M which pairs the vertices of $N(x)$ such that $G_{(x,M)}[V(G_2) \cup \{x_3, x_4\}]$ is Z_3-connected, then $G_{(x,M)}$ is Z_3-connected follows from Lemma 2.6, a contradiction. Next we suppose that there is no such M that $G_{(x,M)}[V(G_2) \cup \{x_3, x_4\}]$ is Z_3-connected. If $x_2x_3 \in E(G)$ and $x_2x_4 \in E(G)$, choose $M = \langle\{x_1, x_2\}, \{x_3, x_4\}\rangle$, then $G_{(x,M)}[\{x_2, x_3, x_4\}]$ is Z_3-connected. Note that $\kappa(G) = 4$. Then $e(\{x_2, x_3, x_4\}, K_l) \geq 3$. It is clear that if $l \geq 4$ or $l = 2$, then $G_{(x,M)}[V(G_2) \cup \{x_3, x_4\}]$ is Z_3-connected, a contradiction. Hence $l = 3$. If $e(\{x_2, x_3, x_4\}, K_l) \geq 4$, as the above, we are done. Next we assume that $e(\{x_2, x_3, x_4\}, K_l) = 3$. In this case, $G[V(K_l) \cup \{y\}] = K_4$. Thus $G_{(x,M)}[V(G_2) \cup \{x_3, x_4, y\}]$ is Z_3-connected, a contradiction. Therefore $x_2x_3 \notin E(G)$ or $x_2x_4 \notin E(G)$. As the above, if $e(\{x_3, x_4\}, K_l) \geq 3$, we need only consider the case of $l = 3$ and $e(\{x_3, x_4\}, K_l) = 3$. Choose $M = \langle\{x_1, x_2\}, \{x_3, x_4\}\rangle$, it is easy to check that $G_{(x,M)}[V(K_l) \cup \{x_3, x_4\}]$ is Z_3-connected or can be contracted to K_4. For the former, we are done; for the letter, $e(x_2, V(K_l) \cup \{x_3, x_4) \geq 3$ if $x_2y \notin E(G)$ and $e(x_2y, V(K_l) \cup \{x_3, x_4) \geq 4$ if $x_2y \in E(G)$. We get that $G_{(x,M)}[V(G_2) \cup \{x_3, x_4, y\}]$ is Z_3-connected follows from Lemmas 2.1 and 2.2, a contradiction. Hence $e(\{x_3, x_4\}, K_l) = 2$. If $m = 2$, then $G_2 = K_3$, a contradiction. If $m \geq 5$, it is clear that $G_{(x,M)}[V(G_2) \cup \{x_3, x_4\}]$ is Z_3-connected, a contradiction. Therefore $3 \leq m \leq 4$. Note that $\kappa(G) = 4$. Then x_3 and x_4 have no common neighbor vertex in K_l. For $l = 3$, let $G_2 - x_2 = G[\{u_1, u_2, u_3\}]$, then $1 \leq e(x_2, \{u_1, u_2, u_3\}) \leq 2$. We assume, without loss of generality, that $x_3u_2 \in E(G)$, $x_4u_3 \in E(G)$ and $x_2u_1 \in E(G)$. If $x_2x_3 \notin E(G)$, then $x_2u_3 \in E(G)$. If not, $\{x_1, x_2, x_3, u_3\}$ constructs an independent set of four vertices, a contradiction. Similarly, if $x_2x_4 \notin E(G)$, then $x_2u_2 \in E(G)$. Note that $\alpha(G_2) = 2$. Then $x_2x_3 \in E(G)$ or $x_2x_4 \in E(G)$. Choose $M = \langle\{x_1, x_3\}, \{x_2, x_4\}\rangle$ or $M =$

$\langle\{x_1, x_4\}, \{x_2, x_3\}\rangle$, we also get that $G_{(x,M)}$ is Z_3-connected, a contradiction. For $l = 4$, let $G_2 - x_2 = G[\{u_1, u_2, u_3, u_4\}]$, then $1 \le e(x_2, \{u_1, u_2, u_3, u_4\}) \le 3$. If $e(x_2, \{u_1, u_2, u_3, u_4\}) = 3$, then $G[\{x_2, u_1, u_2, u_3, u_4\}]$ contains W_4 as a spanning subgraph. By Lemma 2.1, $G[\{x_2, u_1, u_2, u_3, u_4\}]$ is Z_3-connected. Choose $M = \langle\{x_1, x_2\}, \{x_3, x_4\}\rangle$, it is clear that $G_{(x,M)}[V(G_2) \cup \{x_3, x_4\}]$ is Z_3-connected, a contradiction. Hence $1 \le e(x_2, \{u_1, u_2, u_3, u_4\}) \le 2$. Thus there must be $x_2x_3 \in E(G)$ and $x_2x_4 \in E(G)$ since $\alpha(G) = 3$, a contradiction.



(a)



(b)

Fig. 3 Two graphs which admits a nowhere-zero 3-flow.

Case 2. $d(v) \ge 5$ for any vertex $v \in V(G)$.

We first consider that $\kappa'(G) = 4$. There must be a nontrivial minimal edge cut E_c such that $|E_c| = 4$. Let G_1 and G_2 be two components of $G - E_c$.

Subcase 2.1. Either G_1 or G_2 is Z_3-connected.

We assume, without loss of generality, that G_1 is Z_3-connected. Denote by $G_1^\star = G/G_1$. By Lemma 2.2, it is sufficient to show that $G_1^\star$ admits a nowhere-zero 3-flow. Clearly $\alpha(G_1^\star) \le \alpha(G)$. Note that $\delta(G_1^\star) \ge \kappa'(G_1^\star) \ge 4$. If $\alpha(G_1^\star) \le 2$, by Lemma 2.3, $G_1^\star$ admits a nowhere-zero 3-flow. Next we assume that $\alpha(G_1^\star) = 3$. If $\kappa(G_1^\star) \ge \kappa(G) = 4$, by the induction hypothesis, $G_1^\star$ also admits a nowhere-zero 3-flow. Therefore it is crucial to consider that $\kappa(G_1^\star) < \kappa(G)$.

Let v^* be the new vertex into which G_1 is contracted and let V^* be a nontrivial minimal vertex cut of $G_1^\star$. Then $v^* \in V^*$. For otherwise, V^* is also a nontrivial minimum vertex cut of G, contrary to that $\kappa(G_1^\star) < \kappa(G)$. Let $V_1^* = V^* - v^*$. Then V_1^* is a nontrivial minimal vertex cut of G_2. If $|V_1^*| = 1$, we can get $\kappa(G) \le 3$, a contradiction. If $|V_1^*| \ge 3$, then $|V^*| \ge 4$. It follows that $\kappa(G_1^\star) \ge \kappa(G) = 4$, contrary to that $\kappa(G_1^\star) < \kappa(G)$. Hence $|V_1^*| = 2$. Let G_{21} and G_{22} be two connected components of $G_2 - V_1^*$. If $\alpha(G_{21}) = 2$ or $\alpha(G_{22}) = 2$, we can get that $\alpha(G) = 4$, contrary to that $\alpha(G) = 3$. Therefore both G_{21}

and G_{22} are complete graphs. We assume, without loss of generality, that $E_c = \{x_1y_1, x_2y_2, x_3y_3, x_4y_4\}$. Then $x_i \neq x_j$ and $y_i \neq y_j$, where $1 \leq i < j \leq 4$. Let $X = \{x_1, x_2, x_3, x_4\} \subset V(G_1)$ and $Y = \{y_1, y_2, y_3, y_4\} \subset V(G_2)$. Then $e(X, G_{21}) = 2$ and $e(X, G_{22}) = 2$ since $\kappa(G) = 4$. Let $G_{21} = K_m$ and $G_{22} = K_n$. Then $m \geq 4$ and $n \geq 4$ since $d(v) \geq 5$ for any vertex $v \in V(G)$. If $m \geq 5$, by Lemma 2.1(1), G_{21} is Z_3-connected. If $m = 4$, then $e(V_1^*, G_{21}) \geq 6$. So there must be one vertex, say $u^* \in V_1^*$ such that $e(u^*, G_{21}) \geq 3$. It is clear that $G_{21} + u^*$ contains W_4 as a spanning subgraph. It follows from Lemma 2.1 that $G_{21} + u^*$ is Z_3-connected. Therefore it is easy to prove that $G_1^\star$ admits a nowhere-zero 3-flow.

Subcase 2.2. Neither G_1 nor G_2 are Z_3-connected.

If neither G_1 nor G_2 are Z_3-connected, we can immediately get the following claim.

Claim 5. G_i contains no Z_3-connected subgraph H_i with $|V(G_i - H_i)| \leq 3$, where $i \in \{1, 2\}$.

Proof of Claim 5. We prove by contradiction and suppose that G_i contains Z_3-connected subgraph H_i with $|V(G_i - H_i)| \leq 3$. Let $G_i^* = G_i/H_i$ and w_i^* be the new vertex into which H_i is contracted. Then $d(w_i^*) \geq 4$ since $\kappa^{'}(G) \geq 4$. Note that $|V(G_i - H_i)| \leq 3$. So we have that $|V(G_i^*)| \leq 4$ and G_i^* contains 2-cycles. It is easy to prove that G_i^* is Z_3-connected. By Lemma 2.2, G_i is Z_3-connected, a contradiction. Hence, this claim holds.

We have that $|V(G_1)| \geq 6$ and $|V(G_2)| \geq 6$ since $\delta(G) \geq 5$. It follows that $\alpha(G_1) = \alpha(G_2) = 2$. Let A be a maximal independent set of G_1 and $B = \{y | yx \notin E(G), y \in V(G_2), x \in A\}$. Then there are at most two vertices in B whose neighbors in A. Hence $|B| \geq (6-2) = 4$ and $G[B]$ is a complete graph since $\alpha(G) = 3$. If $|B| \geq 5$, by Lemma 2.1, $G[B]$ is Z_3-connected, contrary to Claim 5. So $|B| = 4$. It is just that $|A \cap \{x_1, x_2, x_3, x_4\}| = 2$ and $|V(G_2)| = 6$. We assume, without loss of generality, that $A = \{x_1, x_2\}$. Then for $1 \leq i \leq 2$, $e(y_i, B) \geq 2$ since $\delta(G) \geq 4$. Note that $G[B] = K_4$. If there exist some y_i such that $e(y_i, B) \geq 3$, then $G[B \cup \{y_i\}]$ contains Z_3-connected spanning subgraph W_4, contrary to Claim 5. Hence $e(y_1, B) = 2$ and $e(y_2, B) = 2$ while $y_1y_2 \in E(G)$. Let $B = \{y_3, y_4, y_5, y_6\}$. If $|N(y_1) \cap N(y_2)| \leq 1$, then G_2 contains L_3^+ or W_4 as a subgraph which is Z_3-connected follows from Lemma 2.1(3)(4), contrary to Claim 5. Therefore we assume that $|N(y_1) \cap N(y_2)| = 2$. Then $N(y_1) \cap N(y_2) = \{y_5, y_6\}$. If $|V(G_1)| \geq 7$, then G_1 contains K_m as a subgraph, contrary to Claim 5, where $m \geq |V(G_1)| - 2 \geq 5$. So $|V(G_1)| = 6$. Let $V(G_1) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$. Then $G[\{x_2, x_3, x_5, x_6\}] = K_4$ since $y_1y_4 \notin E(G)$ and $G[\{x_2, x_4, x_5, x_6\}] = K_4$ since $y_1y_3 \notin E(G)$. It is clear that G_1 contains K_5^- as a subgraph, contrary to Claim 5.

Next we consider that $\kappa^{'}(G) = 5$. Let $S = \{v_1, v_2, v_3, v_4\}$ be a minimal cut of G. Then $G[S]$ contains at least one edge, say v_1v_2. Let G_1 and G_2 be two components of $G - S$. Note that $\alpha(G) = 3$. Then one of G_1 and G_2 must be a complete graph. We assume, without loss of generality, that $G_1 = K_m$. Thus $\alpha(G_2) \leq 2$. Let $A \subseteq S$ and $A \neq \emptyset$. We have the following claim.

Claim 6. If $G[V(G_1) \cup A]$ is Z_3-connected, then G is also Z_3-connected.

Proof of Claim 6. Let $G^* = G/G[V(G_1) \cup A]$. Note that $\alpha(G_2) \leq 2$. If $A = S$, by Lemma 2.6, G^* is Z_3-connected. If $A \subset S$, then $e(v, G_1) = 1$, for any vertex $v \in S - A$. This yields that $\alpha(G_2 + v) \leq 2$ since $\alpha(G) = 3$. Hence $\alpha(G_2 + S - A) \leq 2$. By Lemma 2.6, G^* is Z_3-connected. Therefore if $G[V(G_1) \cup A]$ is Z_3-connected, G is also Z_3-connected follows from Lemma 2.2.

Note that $\delta(G) \geq \kappa'(G) = 5$. So $m \geq 2$. If $m \geq 5$, by Lemma 2.1 G_1 is Z_3-connected. Note that $e(V(G_1), S) \geq 5$. Then there must be one vertex $v \in S$ such that $e(v, V(G_1)) \geq 2$. Thus $G[V(G_1) \cup \{v\}]$ is Z_3-connected. By Claim 6, G is Z_3-connected. If $m = 4$, then $e(V(G_1), S) \geq 8$. If there exist one vertex $v \in S$ such that $e(v, V(G_1)) \geq 3$, then $G[V(G_1) \cup \{v\}] = K_5$ or K_5^- which is Z_3-connected follows from Lemma 2.1. By Claim 6, G is Z_3-connected. Next we suppose that $e(v, V(G_1)) = 2$ for each vertex $v \in S$. It follows that $e(\{v_1, v_2\}, V(G_1)) = 4$. If $|N_{G_1}(v_1) \cap N_{G_1}(v_2)| \leq 1$, then $G[V(G_1) \cup \{v_1, v_2\}]$ contains L_3^+ or W_4 as a subgraph. It follows from Lemma 2.1 that $G[V(G_1) \cup \{v_1, v_2\}]$ is Z_3-connected. By Claim 6, G is Z_3-connected. Next we assume that $|N_{G_1}(v_1) \cap N_{G_1}(v_2)| = 2$. Then $|N_{G_1}(v_3) \cap N_{G_1}(v_4)| = 2$. If $v_iv_j \in E(G)$, then $G[V(G_1) \cup S]$ is Z_3-connected, where $i \in \{1, 2\}, j \in \{3, 4\}$. By Lemma 2.6, G is Z_3-connected. Next we assume that $v_iv_j \notin E(G)$, where $i \in \{1, 2\}, j \in \{3, 4\}$. Let $N_{G_1}(v_3) = \{x, y\}$. We consider the graph $G_{[v_3x, v_3y]}$ which contains a 2-cycle xyx. Denote by $H_1 = G_1 + S - v_3 + xy$. It is clear that H_1 is a Z_3-connected subgraph of $G_{[v_3x, v_3y]}$. Note that $e(S - v_3, G_2) \geq 7$ and $\kappa'(G) = 5$. Then $G_{[v_3x, v_3y]} - v_3$ is a 4-edge-connected graph. By Lemma 2.6, $G_{[v_3x, v_3y]} - v_3$ is Z_3-connected. So is $G_{[v_3x, v_3y]}$ since $e(v_3, G_2) \geq 3$. It follows from Lemma 2.4 that G is Z_3-connected. If $m = 3$, then $e(V(G_1), S) \geq 9$. Let $D_3 = \{v | v \in S$ and $e(v, G_1) = 3\}$. Then $|D_3| \geq 1$. If $|D_3| \geq 2$, then G contains Z_3-connected subgraph K_5^-. By Lemma 2.6, G is Z_3-connected. Hence we suppose that $|D_3| = 1$. Then $e(v, G_1) \geq 2$ for each vertex $v \in S$. So $4 \leq e(\{v_1, v_2\}, V(G_1)) \leq 5$. Let $A = \{v_1, v_2\}$. We shall show that $G[V(G_1) \cup A]$ is Z_3-connected. If $e(\{v_1, v_2\}, V(G_1)) = 5$, then $G[V(G_1) \cup A]$ contains K_5^- as a spanning subgraph. If $e(\{v_1, v_2\}, V(G_1)) = 4$, then $e(v_i, G_1) = 2$, where $i \in \{1, 2\}$. Note that $e(u, S) \geq 3$ for each vertex $u \in V(G_1)$. Then $|N_{G_1}(v_1) \cap N_{G_1}(v_2)| \leq 1$. Thus $G[V(G_1) \cup A]$ contains W_4 as a spanning subgraph. Therefore $G[V(G_1) \cup A]$ is Z_3-connected follows from Lemma 2.1. By Claim 6, G is Z_3-connected. At last, we assume that $m = 2$. Let $G_1 = u_1u_2$. Then $N(u_1) = \{u_2, v_1, v_2, v_3, v_4\}$ and $N(u_2) = \{u_1, v_1, v_2, v_3, v_4\}$. We consider the graph $G_{[u_1v_1, u_1v_2]}$ which contains a 2-cycle $v_1v_2v_1$. Let H_2 be the maximal Z_3-connected subgraph of $G_{[u_1v_1, u_1v_2]}[V(G_1) \cup S]$. If $v_iv_j \in E(G)$, then $H_2 = G_{[u_1v_1, u_1v_2]}[V(G_1) \cup S]$, where $i \in \{1, 2\}, j \in \{3, 4\}$. By Lemma 2.6, G^* is Z_3-connected. It follows from Lemmas 2.1 and 2.4 that G is Z_3-connected. Next we assume that $v_iv_j \notin E(G)$, where $i \in \{1, 2\}, j \in \{3, 4\}$. Thus $H_2 = v_1u_2v_2 + v_1v_2$ and $v^*v_j \in E(G^*)$, where

$j \in \{3,4\}$. Note that $d_{G^*}(v_3) \geq 5$. Next we consider the graph $G^*_{[v_3u_1, v_3v^*]}$ which contains a 2-cycle u_1v^*. Let H^* be the maximal Z_3-connected subgraph which contains a 2-cycle u_1v^* of $G^*_{[v_3u_1, v_3v^*]}$ and v^{**} be the vertex into which H^* is contracted. Then $\alpha(G_2 + v^{**}) \leq 2$. For otherwise, there are two vertices in G_2, say x, y such that $\{v^{**}, x, y\}$ is an independent set of three vertices. Note that $v_1v_4 \notin E(G)$. Thus $\{v_1, v_4, x, y\}$ is an independent set of four vertices, contrary to that $\alpha(G) = 3$. Hence $\alpha(G_2 + v^{**} - v_3) \leq 2$. Denote by $G^{**} = G^*_{[v_3u_1, v_3v^*]}/H^*$. Then $G^{**} = G_2 + v^{**}$. Note that $\delta(G^{**} - v_3) \geq 4$. By Lemma 2.3, $G^{**} - v_3$ is Z_3-connected. So is G^{**} since $d_{G^{**}}(v_3) \geq 3$. It follows from Lemmas 2.1 and 2.4 that both G^* and G are Z_3-connected, a contradiction.

Therefore we conclude that if $\alpha(G) = 3$, then G admits a nowhere-zero 3-flow, this completes the proof.

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