On centralizers and multiplicative generalized derivations of semiprime ring

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Abstract. In this note we gave the description of commutativity of prime and semiprime rings with the help of some identities involving multiplicative generalized derivation and multiplicative left centralizer.

Keywords: semiprime (prime) ring, multiplicative generalized derivation, multiplicative left centralizer.

1. Introduction

Throughout the paper R will be an associative ring. We shall denote by $Z(R)$ the centre of ring R. A ring R is said to be a prime (resp. semiprime) if $aRb = (0)$ implies that either $a = 0$ or $b = 0$ (resp. $aRa = (0)$ implies that $a = 0$). We shall denote for any pair of elements $x, y \in R$ the commutator $xy - yx$ and anticommutator $x \circ y = xy + yx$. We extensively use the basic commutator identities (i) $[x, yz] = [x, y]z + y[x, z]$, (ii) $[xy, z] = [x, z]y + x[y, z]$ and anticommutator identities $x \circ yz = (x \circ y)z - y[x, z]$, $xy \circ z = (x \circ z)y = x[y, z]$. By a derivation, we mean an additive mapping $d : R \longrightarrow R$ satisfying $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. A derivation d is inner if there exists $a \in R$ such

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that $d_a(x) = [a, x]$, for all $x \in R$. Following Hvala [7] generalized derivation is an additive mapping $F : R \longrightarrow R$ satisfying $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$ associated with a derivation d. Also, he calls the maps of the form $x \longmapsto ax + xb$ where a, b are fixed elements in R by the inner generalized derivations. Hence the concept of a generalized derivation covers both the concepts of a derivation and a left centralizer (i.e., an additive map g satisfying $g(xy) = g(x)y$ for all $x, y \in R$).

In [4] Daif introduced the notion of multiplicative derivation as follows: Let $d : R \longrightarrow R$ be a map. If $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$; then d is said to be multiplicative derivation. Thus the concept of multiplicative derivation involves the concept of derivation. Daif and El-Sayiad [5] introduced the multiplicative generalized derivation as follows: Let $F : R \longrightarrow R$ be an arbitrary map and $d : R \longrightarrow R$ be a derivation such that $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$, F is called a multiplicative generalized derivation associated with d. Hence the idea of multiplicative generalized derivation involves the concept of generalized derivation. A map H is said to be a multiplicative left centralizer if it satisfies $H(xy) = H(x)y$ for all $x, y \in R$, more details of such mappings can be found in [9]. In [6], the authors extend the previous definition by defining a mapping $F : R \longrightarrow R$ (not necessarily additive) is said to be a multiplicative (generalized)-derivation if $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$, where d is any mapping (neither a derivation nor an additive map). Moreover, multiplicative (generalized)-derivation with $d = 0$ covers the requirement of multiplicative centralizers (not necessarily additive). Therefore, every generalized derivation is a multiplicative (generalized)-derivation on R. But converse need not be true.

Example 1. Let K be any arbitrary ring and $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} : a, b, c \in K \right\}$, be a ring. Define the mappings $F, d : R \longrightarrow R$ such that $F \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & bc \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $d \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. It is easy to see that F is a multiplicative generalized derivation associated with d but F is not a generalized derivation.

A number of authors in literature has investigated the commutativity of prime and semiprime rings satisfying certain functional identities involving derivations generalized derivations, multiplicative generalized derivations for references [1, 2, 4, 6]. In [1], authors studied the following situations: (i)$F(xy) \in Z(R)$, (ii) $F([x, y]) = 0$, (iii) $(F(xy) \pm yx) \in Z(R)$ and (iv) $(F(xy) \mp [x, y]) \in Z(R)$; for all x, y in some nonzero left ideal of semiprime ring R, where F is a generalized derivation of R. Recently, Dhara et.al. [6] studied the commutativity in semiprime rings with left ideal and multiplicative (generalized)-derivation. Very recently authors in [3] study the following conditions: Let R be a semiprime ring,$F : R \longrightarrow R$ be a multiplicative (generalized)- derivation associated with the map d and the map $H : R \longrightarrow R$ be a multiplicative left centralizer such that i) $F(xy) \pm H(xy) = 0$, ii) $F(xy) \pm H(yx) = 0$, iii) $F(x)F(y) \pm H(xy) = 0$, iv) $F(xy) \pm H(xy) \in Z(R)$, v) $F(xy) \pm H(yx) \in Z(R)$, vi) $F(x)F(y) \pm H(xy) \in Z(R)$ for all $x, y \in R$. In this paper we investigate the commutativity of left ideal of semiprime rings by considering some identities involving multiplicative generalized derivation and multiplicative left centralizer.

2. Preliminaries

Firstly we would prefer to fix some results which will be useful in our proofs.

Lemma 2.1 ([8])**.** *Let R be a semiprime ring, then:*

(1) *The center of R contains no nonzero nilpotent elements.*

(2) *R does not contain any nonzero nilpotent left ideal.*

Lemma 2.2. *Let R be a semiprime ring and I be a nonzero left ideal of R. If F is a multiplicative generalized derivation of R with associated map d such that $F(xy) = 0$ for all $x, y \in I$, then either $F(I)I = IF(I) = Id(I) = 0$.*

Proof. Consider $F(xy) = 0$ for all $x, y \in I$. Replace y by yz to get $0 = F(xyz) = F(xy)z + xyd(z) = xyd(z)$ for all $x, y \in I$. This implies that $xyd(z) = 0$ for all $x, y, z \in I$. Replace y by ry to get $xryd(z) = 0$ for all $x, y, z \in I, r \in R$. Therefore we have $IRId(I) = 0$, that is, $Id(I)RId(I) = 0$. Semiprimeness of R yields that $Id(I) = 0$. By a simple calculation we can get $F(I)I = IF(I) = 0$.

Lemma 2.3. *Let R be a semiprime ring and I be a nonzero left ideal of R. If F is a multiplicative generalized derivation of R with associated map d such that $F([x, y]) = 0$ for all $x, y \in I$, then $I[d(x), x] = 0$ for all $x \in I$.*

Proof. Consider $F([x, y]) = 0$ for all $x, y \in I$. Replace y by yx to get $0 = F([x, y]x) = F([x, y])x + [x, y]d(x) = [x, y]d(x)$ for all $x, y \in I$. This implies that

$$[x, y]d(x) = 0 \quad \text{for all } x, y \in I. \tag{2.1}$$

substitute $d(x)y$ for y in (2.1) to find

$$[x, d(x)]yd(x) = 0 \quad \text{for all } x, y \in I. \tag{2.2}$$

Replacing y by yx in (2.2) and multiplying (2.2) by x from right and subtracting the last two resulting equation we arrive at

$$[x, d(x)]y[x, d(x)] = 0 \quad \text{for all } x, y \in I. \tag{2.3}$$

This implies that $(y[d(x), x])^2 = 0$ for all $x, y \in I$. Since R is semiprime, we have $I[x, d(x)] = 0$ for all $x \in I$ by application of Lemma 2.1. □

3. Main results

Theorem 3.1. *Let R be a semiprime ring and I be a nonzero left ideal of R. If F is a multiplicative generalized derivation of R with associated map d and H is multiplicative left centralizer such that $F([x,y]) \pm H(xy) = 0$ for all $x, y \in I$, then $I[d(x), x] = 0$ for all $x \in I$.*

Proof. If $H = 0$, then we have $F([x,y]) = 0$ for all $x, y \in I$. Hence conclusion follows from Lemma 2.3. Let us consider $H \neq 0$

$$F([x,y]) \pm H(xy) = 0 \quad \text{for all } x, y \in I. \tag{3.1}$$

Replacing y by yx we get

$$F([x,y])x + [x,y]d(x) + \pm H(xy)x = 0 \quad \text{for all } x, y \in I. \tag{3.2}$$

In view of (3.1), (3.2) reduces to

$$[x,y]d(x) = 0 \quad \text{for all } x, y \in I. \tag{3.3}$$

substitute $d(x)y$ for y in (3.4) to find

$$[x, d(x)]yd(x) = 0 \quad \text{for all } x, y \in I. \tag{3.4}$$

Replacing y by yx in (3.4) and multiplying (3.4) by x from right and subtracting the last two resulting equations we arrive at

$$[x, d(x)]y[x, d(x)] = 0 \quad \text{for all } x, y \in I. \tag{3.5}$$

Last equation yields that $(y[d(x), x])^2 = 0$ for all $x, y \in I$. Since R is semiprime, we have $I[x, d(x)] = 0$ for all $x \in I$ by application of Lemma 2.1. □

Theorem 3.2. *Let R be a semiprime ring and I be a nonzero left ideal of R. If F is a multiplicative generalized derivation of R with associated map d and H is a multiplicative left centralizer such that $F([x,y]) \pm H([x,y]) = 0$ for all $x, y \in I$, then $I[d(x), x] = 0$ for all $x \in I$.*

Proof. If $H = 0$, then we have $F([x,y]) = 0$ for all $x, y \in I$. Hence conclusion follows from Lemma 2.3. Now suppose that $H \neq 0$. Consider

$$F([x,y]) \pm H([x,y]) = 0 \quad \text{for all } x, y \in I. \tag{3.6}$$

Replacing y by yx in (3.6) we have

$$F([x,y])x + [x,y]d(x) \pm H([x,y])x = 0 \quad \text{for all } x, y \in I. \tag{3.7}$$

In view of (3.6), (3.7) yields that

$$[x,y]d(x) \quad \text{for all } x, y \in I. \tag{3.8}$$

Repeating the same arguments as in the above theorem, we get the desired result. □

Theorem 3.3. *Let R be a semiprime ring and I be a nonzero left ideal of R. If F is a multiplicative generalized derivation of R with associated map d and H is a multiplicative left centralizer such that $F(x)F(y) \pm H(xy) \in Z(R)$ for all $x, y \in I$, then $I[d(x), x] = 0$ for all $x \in I$.*

Proof. Consider the hypothesis

$$F(x)F(y) \pm H(xy) \in Z(R) \quad \text{for all } x, y \in I. \tag{3.9}$$

Replace y by yz in (3.9) to obtain

$$F(x)F(y)z + F(x)yd(z) \pm H(xy)z \in Z(R) \quad \text{for all } x, y, z \in I. \tag{3.10}$$

Commuting (3.10) with z, we find

$$[F(x)yd(z), z] = 0 \quad \text{for all } x, y, z \in I. \tag{3.11}$$

Substitute xu for x in (3.11) and use (3.11) to get

$$[xd(u)yd(z), z] = 0 \quad \text{for all } u, x, y, z \in I. \tag{3.12}$$

Replacing x by rx in (3.12), we have

$$[r, z]xd(u)yd(z) = 0 \quad \text{for all } u, x, y, z \in I, r \in R. \tag{3.13}$$

After simple computation we arrive at $[v, z]d(z)R[v, z]d(z) = 0$ for all $v, z \in I$. Since R is semiprime we have $[v, z]d(z) = 0$ for all $v, z \in I$. Repeating the same arguments as in the Theorem 3.1, we reach the conclusion. □

Theorem 3.4. *Let R be a semiprime ring and I be a nonzero left ideal of R. If F is a multiplicative generalized derivation of R with associated derivation d and H is a multiplicative left centralizer such that $F(x)F(y) \pm H([x, y]) = 0$ for all $x, y \in I$, then $xd(x^2) = 0$ for all $x \in I$.*

Proof. Consider the hypothesis

$$F(x)F(y) \pm H([x, y]) = 0 \quad \text{for all } x, y \in I. \tag{3.14}$$

Substitute yx for y in(3.14) to get

$$F(x)F(y)x + F(x)yd(x) \pm H([x, y])x = 0 \quad \text{for all } x, y \in I. \tag{3.15}$$

In view of (3.14), (3.15) reduces to

$$F(x)yd(x) = 0 \quad \text{for all } x, y \in I. \tag{3.16}$$

Replace x by xz in (3.16) and use (3.16) to find

$$F(x)zyxd(z) + xd(z)yd(x)z + xd(z)yxd(z) = 0 \quad \text{for all } x, y, z \in I. \tag{3.17}$$

In particular, take $x = z$ in above equation to obtain

$$xd(x)yd(x)x + xd(x)yxd(x) = 0 \quad \text{for all } x, y \in I. \tag{3.18}$$

Which gives that

$$xd(x)yd(x^2) = 0 \quad \text{for all } x, y \in I. \tag{3.19}$$

A simple calculation yields that $xd(x^2)Rxd(x^2) = 0$ for all $x \in I$. Since R is semiprime, we have $xd(x^2) = 0$ for all $x \in I$. □

Theorem 3.5. *Let R be a semiprime ring and I be a nonzero left ideal of R. If F is a multiplicative generalized derivation of R with associated derivation d and H is a multiplicative left centralizer such that $F(x)H(y) \pm x \circ y \in Z(R)$ for all $x, y \in I$, then $I[I, I] = 0$ for all $x \in I$.*

Proof. Let us consider

$$F(x)H(y) \pm x \circ y \in Z(R) \quad \text{for all } x, y \in I. \tag{3.20}$$

Replacing y by yz in (3.20) to find

$$F(x)H(y)z \pm (x \circ y)z - y[x, z] \in Z(R) \quad \text{for all } x, y, z \in I. \tag{3.21}$$

Commuting (3.21) with z, we have

$$[(F(x)H(y) \pm (x \circ y))z, z] - [y[x, z], z] = 0 \quad \text{for all } x, y, z \in I. \tag{3.22}$$

In view of (3.20), (3.22) reduces to

$$[y[x, z], z] = 0 \quad \text{for all } x, y, z \in I. \tag{3.23}$$

This implies that

$$y[[x, z], z] + [y, z][x, z] = 0 \quad \text{for all } x, y, z \in I. \tag{3.24}$$

Substitute ry for y in (3.24) and use (3.24) to get $[r, z]y[x, z] = 0$ for all $x, y, z \in I$, $r \in R$. This implies that $y[x, z]Ry[x, z] = 0$ for all $x, y, z \in I$, $r \in R$. Semiprimeness of R force that $y[x, z] = 0$ for all $x, y, z \in I$, i.e. $I[I, I] = 0$. □

Theorem 3.6. *Let R be a Prime ring and I be a nonzero left ideal of R. If F is a multiplicative generalized derivation of R with associated derivation d and H is a multiplicative left centralizer such that $[F(x), H(y)] = 0$ for all $x, y \in I$, then one of the conditions hold:*

(1) $H = 0$;

(2) $I[I, I] = 0$ *for all* $x \in I$;

(3) *F acts as multiplicative left centralizer.*

Proof. We have

$$[F(x), H(y)] = 0 \quad \text{for all } x, y \in I. \tag{3.25}$$

Replacing y by yz in (3.25)and using (3.25) we obtain

$$H(y)[F(x), z] = 0 \quad \text{for all } x, y, z \in I. \tag{3.26}$$

Again replace y by yu in (3.26) to get $H(y)Ru[F(x), z] = 0$ for all $x, y, z \in I,\ r \in R$. Primeness of R yield that either $H(y) = 0$ or $u[F(x), z] = 0$ for all $x, y, z \in I,\ r \in R$. If $H(y) = 0$, then $0 = H(ry) = H(r)y = H(r)Ry$ for all $y \in I,\ r \in R$. Hence $H = 0$ because I is a nonzero ideal and R is prime. Consider the later case

$$y[F(x), z] = 0 \quad \text{for all } x, y, z \in I. \tag{3.27}$$

Substitute xz for x in (3.27) and use (3.27) to find

$$yx[d(z), z] + y[x, z]d(z) = 0 \quad \text{for all } x, y, z \in I. \tag{3.28}$$

Replacing x by vx in (3.28), we have

$$yvx[d(z), z] + yv[x, z]d(z) + y[v, z]xd(z) = 0 \quad \text{for all } v, x, y, z \in I. \tag{3.29}$$

In view of (3.28), (3.29) reduces to $y[v, z]xd(z) = 0$ for all $v, x, y, z \in I$. This implies that $y[v, z]Rxd(z) = 0$ for all $v, x, y, z \in I$. Primeness of R forces that either $y[v, z] = 0$, i.e. $I[I, I] = 0$ or $xd(z) = 0$ for all $v, x, y, z \in I$. In later case we have $xd(z) = 0$, which gives that $F(xz) = F(x)z + xd(z) = F(x)z$ for all $x, z \in I$. Therefore F acts as multiplicative left centralizer. This complete the proof. □

Theorem 3.7. *Let R be a Prime ring and I be a nonzero left ideal of R. If F is a multiplicative generalized derivation of R with associated derivation d and H is a multiplicative left centralizer such that $F(x) \circ H(y) = 0$ for all $x, y \in I$, then one of the conditions hold:*

(1) $H = 0$;

(2) $I[I, I] = 0$ for all $x \in I$;

(3) F acts as multiplicative left centralizer.

Proof. We have

$$F(x) \circ H(y) = 0 \quad \text{for all } x, y \in I. \tag{3.30}$$

Replacing y by yz in (3.30) we have

$$(F(x) \circ H(y))z - H(y)[F(x), z] = 0 \quad \text{for all } x, y, z \in I. \tag{3.31}$$

By application of (3.30), (3.31) yields that $H(y)[F(x), z] = 0$ for all $x, y, z \in I$. Repeating the same steps as in the proof of Theorem 3.6, we arrive at the conclusion. □

Example 2. Let $R = \left\{ \begin{pmatrix} a & 0 & 0 \\ b & 0 & 0 \\ c & d & 0 \end{pmatrix} : a,b,c,d \in \mathbb{Z}_2 \right\}$, be a ring and $I = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ b & 0 & 0 \\ c & d & 0 \end{pmatrix} : b,c,d \in \mathbb{Z}_2 \right\}$ be an ideal of R. Define $F \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b+c & 0 & 0 \end{pmatrix}$, $d \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & 0 & 0 \end{pmatrix}$ and $H \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & c & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ ba & 0 & 0 \end{pmatrix}$. F is a multiplicative generalized derivation associated with the map d and H is a multiplicative left centralizer. R satisfies the hypothesis of Theorem 3.6 and Theorems 3.7 but $H \neq 0$. Hence Primeness is essential condition. □

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