Associativity of max-min composition of three fuzzy relations

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Abstract. In this paper, we introduce some concepts and definitions related to the Max-Min composition of fuzzy relations. We also prove the associativity of Max-Min composition of three fuzzy relations.
Keywords: fuzzy sets, fuzzy relations, composition of fuzzy relations.

1. Introduction

Fuzzy relations play an essential role in fuzzy diagnosis, fuzzy control, and fuzzy modeling [1]. They have applications in fields such as Sociology, Economics, Psychology, and Medicine. They are also an appropriate tool for characterizing correspondences between objects. The utilization of fuzzy relations started from the fact that real-life objects can be related to each other to a certain degree [2]. However, in many cases, we can consider that a certain object X is in relation R with another object Y to a certain degree, but it is possible that we are not so sure about it [3]. In other words, it may be an uncertainty about the degree that is allocated to the relationship between X and Y. In fuzzy set theory, there is no means to incorporate that uncertainty in the membership degrees. Atanassov in 1983 suggested to use intuitionistic fuzzy sets to solve this.

The authors in [4] studied the utilization of fuzzy composition to solve the problem using matrix which is an efficient solution when the particular data base is given. As a case study, they have explained the technique to find the good resistor for long life with respect to resistor size, among the given set of resistors. The definition of rules is based on sampled data for each resistor. The sampled data for each resistor is expressed in sq.mm. The number of rules is two, the first rule is used for the relation between size and power dissipation, and the

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second rule is used for the relationship between power dissipation and lifetime of a particular resistor. The better resistor can be suggested using composition of fuzzy relation in accord with his size for longer life time.

Shakhatreh M.A and Hayajenh M.A. They gave new definition for the fuzzy partition which is called SH fuzzy partition [5] using fuzzy equivalence classes [6, 7], these concepts will help us to use the associativity of Max-Min composition of three fuzzy relations and in general n fuzzy relation to build new concepts related to the title of this paper.

In this paper, we study the associativity of Max-Min composition of three fuzzy relations. To prove the *Associativity of Max-Min composition of three fuzzy relations*, we should know in advance the definitions of the fuzzy set, the equality of two fuzzy sets, the fuzzy relations and the Max-Min composition of two fuzzy relations. At the end of the paper, we present the proof of Associativity of max-min composition of three fuzzy relations.

2. Definitions and preliminaries

First, we present some definitions and preliminaries related to fuzzy sets and compositions of fuzzy relations.

Definition 2.1 (Fuzzy Sets, see [8]). *Let X be a set of objects characterized by a membership (characteristic) function $f_{\underset{\sim}{\mathbf{A}}}(x)$ which associates with each object in X a real number in the interval $[0,1]$, with the value of $f_{\underset{\sim}{\mathbf{A}}}(x)$ at x representing the grade of membership of x in $\underset{\sim}{\mathbf{A}}$.*

Definition 2.2 (Equality of Two Fuzzy Sets, see [9]). *Let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be two fuzzy sets in $X \neq \emptyset$. Then $\underset{\sim}{\mathbf{A}} = \underset{\sim}{\mathbf{B}}$ iff $\mu_{\underset{\sim}{\mathbf{A}}}(x) = \mu_{\underset{\sim}{\mathbf{B}}}(x)\ \forall x \in X$.*

Definition 2.3 (Fuzzy Relations, see [10]). *Let X and Y be nonempty sets. Then $\underset{\sim}{\mathbf{R}} = \{((x,y), \mu_{\underset{\sim}{\mathbf{R}}}(x,y)) : (x,y) \in X \times Y\}$ is called a fuzzy relation in $X \times Y$.*

Definition 2.4 (Max-Min Composition of Two Fuzzy Relations, see [8]). *Let X, Y and Z be nonempty sets, $\underset{\sim}{\mathbf{R}}_1$ be a fuzzy relation in $X \times Y$, and $\underset{\sim}{\mathbf{R}}_2$ be a fuzzy relation in $Y \times Z$. Then*

$$\underset{\sim}{\mathbf{R}}_2 \circ \underset{\sim}{\mathbf{R}}_1 = \left\{ \left((x,z), \max_y \left\{ \min \left\{ \mu_{\underset{\sim}{\mathbf{R}}_1}(x,y), \mu_{\underset{\sim}{\mathbf{R}}_2}(y,z) \right\} \right\} \right) : x \in X, y \in Y, z \in Z \right\}$$

is called Max–Min Composition of $\underset{\sim}{\mathbf{R}}_1$ and $\underset{\sim}{\mathbf{R}}_2$.

3. Associativity of max-min composition of three fuzzy relations

In Theorem 3.1, we prove the associativity of Max-Min composition of three fuzzy relations.

Theorem 3.1 (Associativity of Max-Min Composition of Three Fuzzy Relations)**.** *Let* $\underset{\sim}{\mathbf{R}}_1$ *be a fuzzy relation in* $X \times Y$*,* $\underset{\sim}{\mathbf{R}}_2$ *be a fuzzy relation in* $Y \times Z$*, and* $\underset{\sim}{\mathbf{R}}_3$ *be a fuzzy relation in* $Z \times W$*, then:*

$$\underset{\sim}{\mathbf{R}}_3 \circ (\underset{\sim}{\mathbf{R}}_2 \circ \underset{\sim}{\mathbf{R}}_1) = (\underset{\sim}{\mathbf{R}}_3 \circ \underset{\sim}{\mathbf{R}}_2) \circ \underset{\sim}{\mathbf{R}}_1.$$

Proof. It is sufficient to prove that:

$$\mu_{\underset{\sim}{\mathbf{R}}_3 \circ (\underset{\sim}{\mathbf{R}}_2 \circ \underset{\sim}{\mathbf{R}}_1)}(x,w) = \mu_{(\underset{\sim}{\mathbf{R}}_3 \circ \underset{\sim}{\mathbf{R}}_2) \circ \underset{\sim}{\mathbf{R}}_1}(x,w) \quad \forall (x,w) \in X \times W.$$

We have:

$$\begin{aligned} \mu_{\underset{\sim}{\mathbf{R}}_2 \circ \underset{\sim}{\mathbf{R}}_1}(x,\ z) &= \max_y \left\{ \min \left\{ \mu_{\underset{\sim}{\mathbf{R}}_1}(x,y), \mu_{\underset{\sim}{\mathbf{R}}_2}(y,z) \right\} \right\}, \\ \mu_{\underset{\sim}{\mathbf{R}}_3 \circ \underset{\sim}{\mathbf{R}}_2}(y,\ w) &= \max_z \left\{ \min \left\{ \mu_{\underset{\sim}{\mathbf{R}}_2}(y,z), \mu_{\underset{\sim}{\mathbf{R}}_3}(z,w) \right\} \right\}. \end{aligned}$$

Now,

$$\begin{aligned} &\mu_{\underset{\sim}{\mathbf{R}}_3 \circ (\underset{\sim}{\mathbf{R}}_2 \circ \underset{\sim}{\mathbf{R}}_1)}(x,w) = \max_z \left\{ \min \left\{ \mu_{\underset{\sim}{\mathbf{R}}_2 \circ \underset{\sim}{\mathbf{R}}_1}(x,z), \mu_{\underset{\sim}{\mathbf{R}}_3}(z,w) \right\} \right\} \\ &= \max_z \left\{ \min \left\{ \max_y \left\{ \min \left\{ \mu_{\underset{\sim}{\mathbf{R}}_1}(x,y),\ \mu_{\underset{\sim}{\mathbf{R}}_2}(y,z) \right\} \right\}, \mu_{\underset{\sim}{\mathbf{R}}_3}(z,w) \right\} \right\} \\ &= \max_z \left\{ \min \left\{ \min \left\{ \mu_{\underset{\sim}{\mathbf{R}}_1}(x,y_k), \mu_{\underset{\sim}{\mathbf{R}}_2}(y_k,z) \right\}, \mu_{\underset{\sim}{\mathbf{R}}_3}(z,w) \right\} \right\}, \end{aligned}$$

for some $y_k \in Y$

$$= \min \left\{ \min \left\{ \mu_{\underset{\sim}{\mathbf{R}}_1}(x,y_k),\ \mu_{\underset{\sim}{\mathbf{R}}_2}(y_k,z_m) \right\}, \mu_{\underset{\sim}{\mathbf{R}}_3}(z_m,w) \right\},$$

for some $z_m \in Z$.

Now, suppose that:

$$\mu_{\underset{\sim}{\mathbf{R}}_1}(x,y_k) = a, \qquad \mu_{\underset{\sim}{\mathbf{R}}_2}(y_k,z_m) = b, \qquad \text{and} \qquad \mu_{\underset{\sim}{\mathbf{R}}_3}(z_m,w) = c.$$

Then, we have:

$$\mu_{\underset{\sim}{\mathbf{R}}_3 \circ (\underset{\sim}{\mathbf{R}}_2 \circ \underset{\sim}{\mathbf{R}}_1)}(x,w) = \min\{\min\{a,\ b\}, c\} = \min\{a,\ b,\ c\}. \tag{1}$$

Also,

$$\begin{aligned} \mu_{(\underset{\sim}{\mathbf{R}}_3 \circ \underset{\sim}{\mathbf{R}}_2) \circ \underset{\sim}{\mathbf{R}}_1}(x,w) &= \max_y \left\{ \min \left\{ \mu_{\underset{\sim}{\mathbf{R}}_1}(x,y), \mu_{\underset{\sim}{\mathbf{R}}_3 \circ \underset{\sim}{\mathbf{R}}_2}(y,w) \right\} \right\} \\ &= \max_y \left\{ \min \left\{ \mu_{\underset{\sim}{\mathbf{R}}_1}(x,y), \max_z \left\{ min\{ \mu_{\underset{\sim}{\mathbf{R}}_2}(y,z),\ \mu_{\underset{\sim}{\mathbf{R}}_3}(z,w) \} \right\} \right\} \right\} \\ &= \max_y \left\{ \min \left\{ \mu_{\underset{\sim}{\mathbf{R}}_1}(x,y),\ \min \left\{ \mu_{\underset{\sim}{\mathbf{R}}_2}(y,z_m),\ \mu_{\underset{\sim}{\mathbf{R}}_3}(z_m,w) \right\} \right\} \right\}, \end{aligned}$$

for some $z_m \in Z$ and

$$= \min\left\{\mu_{\underset{\sim}{\mathbf{R}_1}}(x, y_k\ ),\ \min\left\{\mu_{\underset{\sim}{\mathbf{R}_2}}(y_k\ , z_m),\ \mu_{\underset{\sim}{\mathbf{R}_3}}(z_m, w)\right\}\right\}\ ,$$

for some $y_k \in Y$.

Now, suppose that:

$$\mu_{\underset{\sim}{\mathbf{R}_1}}(x, y_k\ ) = a, \quad \mu_{\underset{\sim}{\mathbf{R}_2}}(y_k\ , z_m) = b\ , \ \text{ and } \ \mu_{\underset{\sim}{\mathbf{R}_3}}(z_m, w) = c.$$

Then, we have

$$\begin{aligned} \mu_{(\underset{\sim}{\mathbf{R}_3}\ \circ \underset{\sim}{\mathbf{R}_2})\circ \underset{\sim}{\mathbf{R}_1}}(x, w) &= \min\{a, \min\{b,\ c\}\} \\ &= \min\{a,\ b,\ c\ \} \end{aligned} \tag{2}$$

From (1) and (2), we have

$$\mu_{(\underset{\sim}{\mathbf{R}_3}\ \circ \underset{\sim}{\mathbf{R}_2})\circ \underset{\sim}{\mathbf{R}_1}}(x, w) \ = \ \mu_{(\underset{\sim}{\mathbf{R}_3}\ \circ \underset{\sim}{\mathbf{R}_2})\circ \underset{\sim}{\mathbf{R}_1}}(x, w)\ \forall(x\ , w)\in X \times W,$$

and therefore:

$$\mathbf{R}_3 \circ (\underset{\sim}{\mathbf{R}_2} \circ \underset{\sim}{\mathbf{R}_1}) = (\underset{\sim}{\mathbf{R}_3}\ \circ \underset{\sim}{\mathbf{R}_2}) \circ \underset{\sim}{\mathbf{R}_1},$$

which completes the proof. □

Conclusion

In this paper, we study the associativity of Max-Min composition of three fuzzy relations. First, we present some concepts and definitions related to the Max-Min composition of fuzzy relations. The definitions cover fuzzy sets, equality of two fuzzy sets, fuzzy relations, and Max-Min composition of two fuzzy relations. After that, we prove the associativity of Max-Min composition of three fuzzy relations.

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Accepted: 23.11.2018