Free commutative $B$-algebras

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Abstract. In this paper, the notion free commutative $B$-algebras was introduced. In order to characterize free commutative $B$-algebras and obtain its properties, the concept of external direct product of $B$-algebras was also introduced.

Keywords: $B$-algebras, direct product, external direct product, internal direct product, free commutative $B$-algebras, $B$-homomorphism, basis and combination.

1. Introduction

In [8], Neggers, J. and Kim, H. S. introduced the notion of $B$-algebras. A $B$-algebra is a nonempty set $X$ with a constant 0 and a binary operation “$*$” satisfying the following axioms for all $x, y, z \in X$: (B1) $x * x = 0$, (B2) $x * 0 = x$, (B3) $(x * y) * z = x * (z * (0 * y))$. $B$-algebra can be derived from a given group and a group can be derived from a $B$-algebra. From this relationship, parallel concepts and results were formulated and established. A $B$-algebra $(X; *, 0)$ is said to be commutative if $a * (0 * b) = b * (0 * a)$ for any $a, b \in X$. They also introduced the concepts of subalgebras and normality in [9]. A nonempty subset $A$ of $X$ is said to be a subalgebra of $X$ if $a * b \in A$ for all $a, b \in A$. It is said to be normal in $X$ if for any $x, y, a, b \in X$ with $x * y, a \in A$ implies $(x * a) * (y * b) \in A$.

In [10], Walendziak, A. characterized normality in $B$-algebras. The concept of $B$-homomorphism was introduced by Neggers, J., and Kim, H. S. [9].
A map $\phi : X \to Y$, where $(X; \ast_X, 0_X)$ and $(Y; \ast_Y, 0_Y)$ are B-algebras, is called a B-homomorphism if $\phi(x \ast_X y) = \phi(x) \ast_Y \phi(y)$ for any $x, y \in X$. If $\phi$ is onto (respectively, one-to-one), then $\phi$ is called a B-epimorphism (respectively, B-monomorphism). Moreover, if $\phi$ is a bijection, then $\phi$ is called a B-isomorphism. The kernel of $\phi$, denoted by $\text{Ker} \, \phi$, is defined to be the set \{x \in X : \phi(x) = 0\}. The following properties for a B-algebra are used in this paper: for any $x, y, z \in X$, (P1) $x \ast y = 0$ implies $x = y$, (P2) $0\ast (0\ast x) = x$, (P3) $x \ast (y \ast z) = (x \ast (0 \ast z)) \ast y$, and (P4) $x \ast y = 0 \ast (y \ast x)$. If $X$ is commutative, then (P5) $(x \ast y) \ast z = (x \ast z) \ast y$ and (P6) $x \ast (x \ast y) = y$ [8, 11, 6]. In [4], Gonzaga, N. C. and Vilela, J. P. introduced cyclic B-algebras as well as established the laws of exponents for B-algebras. In [1], Endam, J. C. and Teves, R. C. established some properties of cyclic B-algebras. In [7], Lincong, J. A. and Endam, J. C. introduced the concept of direct product of B-algebras and established its structure. From these parallel concepts and results, one may wonder if every concept in groups has its counterpart for B-algebras and if every result is applicable to B-algebras. In this paper, we introduced the concept of free commutative B-algebra. In order to investigate and characterize this type of B-algebra, we also introduced the external direct product of B-algebras. Throughout this paper, $X$ means a B-algebra $(X; \ast, 0)$.

2. Preliminaries

This section presents some concepts and results needed in this paper. We start with some examples of a B-algebra.

**Example 2.1.** [1] Let $X = \{e, a, b, c\}$ be a set whose binary operation $\ast$ is shown in the following Cayley Table.

<table>
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<th>e</th>
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By routine calculations, $X$ is a commutative B-algebra. $X$ is called the Klein B-algebra $K_4$.

**Example 2.2.** [7] Consider the set of integers $\mathbb{Z}$ and the usual subtraction of integers “−”. Then $(\mathbb{Z}; -, 0)$ is a B-algebra. We denote $(\mathbb{Z}; -, 0)$ by $^*\mathbb{Z}$. Since for every $x, y \in \mathbb{Z}$, $x - (0 - y) = x + y = y + x = y - (0 - x)$, $^*\mathbb{Z}$ is commutative.

In [2], the intersection of a family of subalgebras of a B-algebra $X$ is a subalgebra of $X$.

**Definition 2.3** ([4]). Let $A \subseteq X$ be nonempty and $\{N_\alpha : \alpha \in I\}$ be a collection of subalgebras of $X$ with $A \subseteq N_\alpha$ for all $\alpha \in I$. Then $\bigcap_{\alpha \in I} N_\alpha$ is called the subalgebra $X$ generated by $A$, denoted by $\langle A \rangle_B$. If $A$ is finite, then we say
that \( \langle A \rangle_B \) is finitely generated. If \( A = \{a\} \), then \( \langle A \rangle_B = \langle a \rangle_B \) is called the cyclic subalgebra of \( X \) generated by \( a \). If \( X = \langle S \rangle_B \), then \( S \) is called a set of generators for \( X \).

**Remark 2.4** ([4]). Let \( S \) be a subset of \( X \). Then \( \langle S \rangle_B \) is the smallest subalgebra of \( X \) containing \( S \). If either \( S = \emptyset \) or \( S = \{0\} \), then \( \langle S \rangle_B = \{0\} \). If \( S \) is a subalgebra of \( X \), then \( \langle S \rangle_B = S \). In particular, \( \langle X \rangle_B = X \).

Let \( x \in X \). From [8], Neggers, J. and Kim, H. S. defined \( x^n = x^{n-1} * (0 * x) \) \((n \geq 1)\) and \( x^0 = 0 \). Note that \( x^1 = x^0 * (0 * x) = 0 * (0 * x) = x \) by (P2).

**Definition 2.11**. Let \( X \) be a subalgebra of \( X \). If \( x \in X \), then \( -x = 0 * x \) and \( x^{-n} = (-x)^n \) for each \( n \geq 1 \). In [1], Endam, J. C. and Teves, R. C. defined \( x^m = 0 * x^{-m} \) for \( m \leq -1 \).

By (B3), \((x * y) * z = x * (z * (0 * y))\). For convenience, we directly write \((x * y) * z = x * (z * y^{-1})\) at times. Similarly, by (P3) \( x * (y * z) = (x * z^{-1}) * y \).

**Definition 2.5** ([3]). The order of \( a \in X \), denoted by \(|a|_B\), is the smallest positive integer \( k \) such that \( a^k = 0 \).

\( a \in X \) is said to be of finite order if \(|a|_B < +\infty \); otherwise, it is of infinite order.

**Theorem 2.6** ([3]). Let \( a \in X \). Then \(|a|_B = |\langle a \rangle_B|\). In particular, if \(|a|_B = n < +\infty \), then \( \langle a \rangle_B = \{0, a, a^2, ..., a^{n-1}\} \).

**Theorem 2.7** ([4]). If \( \emptyset \neq A \subseteq X \), then \( \langle A \rangle_B \) consists of all finite products \( \cdots ((a_1^{m_1} * a_2^{m_2} * a_3^{m_3} * ... * a_t^{m_{t-1}}) * a_t^{m_t}) \) where \( a_i \in A \) and \( n_j \in \mathbb{Z} \) for all \( i, j = 1, 2, ..., t \), with \( t < \infty \).

**Corollary 2.8** ([4]). Let \( x \in X \) and \( m, n \in \mathbb{Z} \). Then:

(i) \( x^{-n} = (-x)^n = -(x^n) = 0 * x^n \);

(ii) \( x^m \cdot x^n = x^{m+n} \).

**Theorem 2.9** ([8]). Let \( x \in X \). Then \( (x^m)^n = x^{mn} \) for any \( m, n \in \mathbb{Z} \).

**Corollary 2.10** ([4]). If \( a_i, b_j \in X \) and \( m_i, n_j \in \mathbb{Z} \) for \( i = 1, 2, ..., r \) and \( j = 1, 2, ..., s, r, s < +\infty \), then

\[
\cdots ((a_1^{m_1} * a_2^{m_2} * a_3^{m_3} * ... * a_r^{m_r}) * \cdots * b_s^{0s})
\]

\[
= \cdots ((\cdots ((a_1^{m_1} * a_2^{m_2} * a_3^{m_3} * ... * a_r^{m_r}) * b_s^{0s}) * b_{s-1}^{-ns-1}) * \cdots * b_{-1}^{-ns-1}) * b_1^{0s}).
\]

**Definition 2.11** ([2]). Let \( H \) and \( K \) be subalgebras of \( X \). Define the subset \( HK \) of \( X \) to be the set \( HK = \{ x \in X : x = h * (0 * k), h \in H, k \in K \} \).

**Theorem 2.12.** Let \( H \) and \( K \) be subalgebras of \( X \). Then

(i) \( HK \) is a subalgebra of \( X \) if and only if \( HK = \langle H \cup K \rangle_B \);
A subalgebra

Let $X$ for all

Let $Y$ for any

Every infinite cyclic $B$-algebra is isomorphic to the $B$-algebra $N$.

Theorem 2.13 ([10]). A subalgebra $N$ of $X$ is normal if and only if $x(x+y) \in N$ for any $x \in X, y \in N$.

Remark 2.14 ([5]). Let $N_1$ and $N_2$ be normal subalgebras of $X$ such that $N_1 \cap N_2 = \{0\}$. Then $x(0+y) = y(0+x)$ for all $x \in N_1$ and $y \in N_2$.

Theorem 2.15 ([3]). Every infinite cyclic $B$-algebra is isomorphic to the $B$-algebra $\mathbb{Z}$.

Proposition 2.16 ([9]). Let $\varphi : X \to Y$ be a $B$-homomorphism. Then $\varphi$ is one-to-one if and only if $\ker \varphi = \{0_X\}$.

Definition 2.17 ([5]). Let $\{X_i : i \in I\}$ be a family of $B$-algebras. The direct product of this family, denoted by $\prod_{i \in I} X_i$, is the set of all functions $f : I \to \bigcup_{i \in I} X_i$ given by $f(i) \in X_i$ for each $i \in I$ and whose operation is componentwise.

Theorem 2.18 ([5]). If $\{X_i : i \in I\}$ is a family of $B$-algebras, then

(i) $\prod_{i \in I} X_i$ is a $B$-algebra;

(ii) $X_i$ is commutative if and only if $\prod_{i \in I} X_i$ is commutative.

Theorem 2.19 ([5]). Let $\{f_i : X_i \to Y_i : i \in I\}$ be a family of $B$-homomorphisms and let $f = \prod f_i$ be the map $\prod_{i \in I} X_i \to \prod_{i \in I} Y_i$, given by $\{a_i\} \mapsto \{f_i(a_i)\}$. Then $f$ is a $B$-homomorphism such that $f(\prod_{i \in I} X_i) = \prod_{i \in I} Y_i$ and $\ker f = \prod_{i \in I} \ker f_i$. Consequently, $f$ is a $B$-monomorphism [respectively $B$-epimorphism] if and only if each $f_i$ is.

Definition 2.20 ([5]). Let $J_1$ and $J_2$ be normal subalgebras of $X$. Then $X$ is called the internal product of $J_1$ and $J_2$ if every $a \in A$ can be uniquely expressed as $a = a_1(0+a_2)$, where $a_1 \in J_1$ and $a_2 \in J_2$.

Theorem 2.21 ([5]). Let $J_1$ and $J_2$ be normal subalgebras of $X$. Then $X$ is an internal direct product of $J_1$ and $J_2$ if and only if $A = J_1 \cup J_2$ and $J_1 \cap J_2 = \{0\}$.

3. Some properties of $B$-algebras

If $N$ is a subalgebra of a commutative $B$-algebra $X$, then by (P6), $x(x+y) = y \in N$ for any $x \in X, y \in N$. By Theorem 2.13, it follows that any subalgebra is normal in a commutative $B$-algebra. The following Lemma follows from (P4).

Lemma 3.1. For all $x, y \in X$, $(x+y)^{-1} = y * x$.

Proposition 3.2. Let $x, y \in X$. Then for all $n, m \in \mathbb{Z}$, $(x+y)^n y^m = x y^{n+m}$.
Proof. Let \( x, y \in X \). Then by (B3) and Corollary 2.8(ii), (i) \((y^m \cdot 0) \cdot y^n = x \cdot (y^m \cdot y^n) = x \cdot y^{m+n} = x \cdot y^{m+n} \). □

Proposition 3.3. Let \( X \) be a commutative \( B \)-algebra, \( x_1, x_2, \ldots, x_k \in X \) and \( n_1, n_2, \ldots, n_k \in \mathbb{Z} \). Then

(i) \([(x_1^{n_1} \ast x_2^{n_2}) \ast x_3^{n_3}] \ast \ldots \ast x_k^{n_k}] \ast x_1^m = \ldots [(x_1^{n_1} \ast x_2^{n_2}) \ast x_3^{n_3}] \ast \ldots \ast x_k^{n_k} \ast x_1^m ;

(ii) \([(x_1^{n_1} \ast x_2^{n_2}) \ast x_3^{n_3}] \ast \ldots \ast x_k^{n_k}] \ast x_1^m = \ldots [(x_1^{n_1} \ast x_2^{n_2}) \ast x_3^{n_3}] \ast \ldots \ast x_k^{n_k} \ast x_1^m ;

for all \( i = 2, \ldots, k \).

Proof. (i) By (P5) and Corollary 2.8(ii),

\[
[(x_1^{n_1} \ast x_2^{n_2}) \ast x_3^{n_3}] \ast \ldots \ast x_k^{n_k}] \ast x_1^m = \ldots [(x_1^{n_1} \ast x_2^{n_2}) \ast x_3^{n_3}] \ast \ldots \ast x_1^m \ast x_k^{n_k} ;
\]

\[
\vdots
\]

\[
= \ldots [(x_1^{n_1} \ast x_2^{n_2}) \ast x_3^{n_3}] \ast \ldots \ast x_k^{n_k-1} \ast x_k^m \ast x_k^{n_k} ;
\]

(ii) If \( k = 2 \) (implying that \( i = 2 \)), then the conclusion follows from Proposition 3.2. We now consider \( k > 2 \). By Proposition 3.2 and (P5),

\[
[(x_1^{n_1} \ast x_2^{n_2}) \ast x_3^{n_3}] \ast \ldots \ast x_k^{n_k}] \ast x_1^m = \ldots [(x_1^{n_1} \ast x_2^{n_2}) \ast x_3^{n_3}] \ast \ldots \ast x_1^m \ast x_k^{n_k} ;
\]

\[
\vdots
\]

\[
= \ldots [(x_1^{n_1} \ast x_2^{n_2}) \ast x_3^{n_3}] \ast \ldots \ast x_k^{n_k-1} \ast x_k^m ;
\]

This completes the proof. □

Proposition 3.4. Let \( X \) be a commutative \( B \)-algebra and \( x_1, x_2, \ldots, x_k \in X \). Then for all \( n_1, m_1 \in \mathbb{Z} \),

\[
[(x_1^{n_1} \ast x_2^{n_2}) \ast x_3^{n_3}] \ast \ldots \ast x_k^{n_k}] \ast [(x_1^{m_1} \ast x_2^{m_2}) \ast x_3^{m_3}] \ast \ldots \ast x_k^{m_k} = \ldots [(x_1^{n_1-m_1} \ast x_2^{n_2-m_2}) \ast x_3^{n_3-m_3}] \ast \ldots \ast x_k^{n_k-m_k} .
\]

Proof. Let \( X \) be a commutative \( B \)-algebra, \( x_1, x_2, \ldots, x_k \in X \) and \( n_1, m_1 \in \mathbb{Z} \). By Proposition 3.3 and Corollary 2.10,

\[
[(x_1^{n_1} \ast x_2^{n_2}) \ast x_3^{n_3}] \ast \ldots \ast x_k^{n_k}] \ast [(x_1^{m_1} \ast x_2^{m_2}) \ast x_3^{m_3}] \ast \ldots \ast x_k^{m_k} = \ldots [(x_1^{n_1-m_1} \ast x_2^{n_2-m_2}) \ast x_3^{n_3-m_3}] \ast \ldots \ast x_k^{n_k-m_k} .
\]

This completes the proof. □
4. Direct product and free commutative B-algebra

Proposition 4.1. Let \( \{X_i : i \in I\} \) be a family of B-algebras and \( \{u_i\} \in \prod_{i \in I} X_i \). Then \( \{u_i\}^n = \{u_i^n\} \) for all \( n \in \mathbb{Z} \).

Proof. Let \( \{X_i : i \in I\} \) be a family of B-algebras. By Theorem 2.18(i), \( (\prod_{i \in I} X_i; \otimes, \{0\}) \) is a B-algebra. Let \( \{u_i\} \in \prod_{i \in I} X_i \) and \( n \in \mathbb{Z} \).

Case 1. \( n = 0 \). Then \( \{u_i\}^0 = \{0\} = \{0^n\} \).

Case 2. \( n > 0 \). Now,
\[
\{u_i\}^n = \{u_i\}^{n-1} \otimes (\{0\} \otimes \{u_i\})
\]
\[
= \left[ \{u_i\}^{n-2} \otimes (\{0\} \otimes \{u_i\}) \right] \otimes \{0 \ast u_i\}
\]
\[
= \left( \{u_i\}^{n-2} \otimes \{0 \ast u_i\} \right) \otimes \{0 \ast u_i\}
\]
\[
= \left[ \ldots \left[ \left( \{u_i\} \otimes \{0 \ast u_i\} \right) \otimes \{0 \ast u_i\} \right] \otimes \{0 \ast u_i\} \right] \otimes \{0 \ast u_i\}
\]
\[
= \left[ \ldots \left( \{u_i \ast (0 \ast u_i)\} \otimes \{0 \ast u_i\} \right] \otimes \{0 \ast u_i\} \right] \otimes \{0 \ast u_i\}
\]
\[
= \left[ \ldots \left( \{u_i^2\} \otimes \{0 \ast u_i\} \right] \otimes \{0 \ast u_i\} \right] \otimes \{0 \ast u_i\}
\]
\[
= \left[ \ldots \left( u_i^{n-1} \otimes \{0 \ast u_i\} \right] \otimes \{0 \ast u_i\} \right] \otimes \{0 \ast u_i\}
\]
\[
= \{u_i^{n-1} \otimes \{0 \ast u_i\} \}
\]
\[
= \{u_i^n \}.
\]

Case 3. \( n < 0 \). Then \(-n > 0\). By Case 2 and Corollary 2.8(i), \( \{u_i\}^n = \{0\} \otimes \{u_i\}^{-n} = \{0 \otimes \{u_i\}^{-n}\} = \{0 \ast u_i^{-n}\} = \{u_i^n\} \).

Definition 4.2. The external direct product of a family of B-algebras \( \{(X_i; \ast_i, 0_i) : i \in I\} \), denoted by \( \prod_{i \in I} X_i \), is the set of all \( f \in \prod_{i \in I} X_i \) such that \( f(i) = 0 \), for all but a finite number of \( i \in I \). If all of the B-algebras \( X_i \) are commutative, \( \prod_{i \in I} X_i \) is usually called the external direct sum and is denoted by \( \sum_{i \in I} X_i \).

If \( I \) is finite, then the external direct product coincides with direct product. The following remark follows from Definition 4.2 and definition of subalgebra.

Remark 4.3. If \( \{X_i : i \in I\} \) is a family of B-algebras, then \( \prod_{i \in I} X_i \) is a subalgebra of \( \prod_{i \in I} X_i \).

Consider the B-algebra \( \ast \mathbb{Z} = (\mathbb{Z}; -, 0) \). For clarity, we emphasize that \( \ast \mathbb{Z} \) denotes the B-algebra and \( \mathbb{Z} \) the set of integers. To avoid confusion, we also emphasize that \( \prod \ast \mathbb{Z} \) (respectively \( \sum \ast \mathbb{Z} \)) would denote the B-algebra and \( \prod \mathbb{Z} \) (respectively \( \sum \mathbb{Z} \)) the set of elements. The following results are proved similarly as in Theorems 2.18(ii) and 2.19.
Remark 4.4. Let \{\(A_i; \ast, 0\) : \(i \in I\)\} be a family of B-algebras. Then \(A_i\) is commutative if and only if \(\prod_{i \in I} w A_i\) is commutative.

Remark 4.5. Let \(\{f_i : X_i \to Y_i : i \in I\}\) be a family of B-homomorphisms and let \(f = \prod f_i\) be the map \(\prod f_i X_i \to \prod f_i Y_i\), given by \(\{a_i\} \mapsto \{f_i(a_i)\}\). Then \(f\) is a B-homomorphism such that \(f(\prod f_i X_i) = \prod f_i Y_i\) and \(\text{Ker } f = \prod f_i \text{Ker } f_i\). Consequently, \(f\) is a B-monomorphism [respectively B-epimorphism] if and only if each \(f_i\) is.

The following remark follows from Theorems 2.12 and 2.21.

Remark 4.6. Let \(H\) and \(K\) be normal subalgebras of \(X\). Then \(X\) is the internal direct product of \(H\) and \(K\) if and only if \(X = \langle H \cup K \rangle_B\) and \(H \cap K = \{0\}\).

Theorem 4.7. Let \(\{N_i : i \in I\}\) be a family of normal subalgebras of \(X\) such that \(X = \langle \bigcup_{i \in I} N_i \rangle_B\) and \(N_k \cap \langle \bigcup_{i \neq k} N_i \rangle_B = \{0\}\) for each \(k \in I\). Then \(X \cong \prod_{i \in I} w N_i\).

Proof. Let \(x \in N_{i_s}\) and \(y \in N_{i_t}\) with \(s \neq t\). By Remark 2.14, \(x \ast y^{-1} = x \ast (0 \ast y) = y \ast (0 \ast x) = y \ast x^{-1}\). By (B3), Remark 2.14 and (P4) we also have \(x^{-1} \ast y = (0 \ast x) \ast y = 0 \ast (y \ast (0 \ast x)) = 0 \ast (x \ast (0 \ast y)) = (0 \ast y) \ast x = y^{-1} \ast x\).

Let \(\{u_i\} \subseteq \prod_{i \in I} w N_i\). Then there exists \(I' = \{i_1, i_2, \ldots, i_n\} \subseteq I\) such that \(u_i = 0\) for all \(i \in I \setminus I'\) and \(u_{i_1}, u_{i_2}, \ldots, u_{i_n} \neq 0\). For each \(j = 1, 2, \ldots, n\), there is a normal subalgebra \(N_{i_j}\) such that \(u_{i_j} \in N_{i_j}\). By hypothesis, it follows that \(u_{i_j} \not\in N_{i_k}\) for all \(j \neq k\). Define \(f : \prod_{i \in I} w N_i \to X\) by \(f(\{u_i\}) = (u_{i_1} \ast u_{i_2}^{-1} \ast \cdots \ast u_{i_{n-1}}^{-1} \ast u_{i_n}^{-1}) \ast u_{i_1}^{-1}\). By (B3), (P3), and Corollary 2.8(i), notice that for \(a \in N_r, b \in N_s\) and \(c \in N_t\), where \(r, s, t \in I\) are distinct, we have \((a \ast b^{-1}) \ast c^{-1} = (b \ast a^{-1}) \ast c^{-1} = b \ast (c^{-1} \ast (0 \ast a^{-1})) = b \ast (c^{-1} \ast a) = b \ast (a^{-1} \ast c) = (b \ast c^{-1}) \ast a^{-1}\). Similarly, \((a \ast c^{-1}) \ast b^{-1} = (c \ast b^{-1}) \ast a^{-1} = (c \ast a^{-1}) \ast b^{-1}\). This could be extended to any finite number of elements. This implies that the arrangement of the elements does not matter. It follows that \(f\) is well-defined.

Let \(\{a_i\}, \{b_i\} \subseteq \prod_{i \in I} w N_i\). Then there exist \(I', I'' \subseteq I\) with \(|I'|, |I''| < \infty\) such that \(a_i, b_j = 0\) for all \(i \in I \setminus I'\) and \(j \in I \setminus I''\). Since \(|I'|, |I''| < \infty\), \(|I' \cup I''| = n < \infty\). Note that the operation \(\ast\) is the same in every \(N_i\). Thus,

\[
\begin{align*}
 f(\{a_i\} \oplus \{b_i\}) &= f(\{a_i \ast b_i\}) \\
 &= \left[\cdots \left((a_{i_1} \ast b_{i_1}) \ast (a_{i_2} \ast b_{i_2})^{-1}\right) \ast (a_{i_3} \ast b_{i_3})^{-1}\right] \ast \\
 &\quad \cdots \left((a_{i_{n-1}} \ast b_{i_{n-1}})^{-1}\right) \ast (a_{i_n} \ast b_{i_n})^{-1},
\end{align*}
\]

where \(i_j \in I' \cup I''\) for all \(j = 1, 2, \ldots, n\). Consider \((a_{i_1} \ast b_{i_1}) \ast (a_{i_2} \ast b_{i_2})^{-1}\). By Lemma 3.1, (B3) and (P3),

\[
(a_{i_1} \ast b_{i_1}) \ast (a_{i_2} \ast b_{i_2})^{-1} = (a_{i_1} \ast b_{i_1}) \ast (b_{i_2} \ast a_{i_2}) = a_{i_1} \ast ((b_{i_2} \ast a_{i_2}) \ast b_{i_1}^{-1}).
\]
\[
= a_{i_1} \ast (b_{i_1} \ast (b_{i_2} \ast a_{i_2})^{-1})
\]
\[
= a_{i_1} \ast (b_{i_1} \ast (a_{i_2} \ast b_{i_2}))
\]
\[
= a_{i_1} \ast ((b_{i_2} \ast b_{i_2}^{-1}) \ast a_{i_2})
\]
\[
= (a_{i_1} \ast a_{i_2}^{-1}) \ast (b_{i_1} \ast b_{i_2}^{-1}).
\]

Similarly, \([a_{i_1} \ast b_{i_1}] \ast (a_{i_2} \ast b_{i_2})^{-1}] \ast (a_{i_3} \ast b_{i_3})^{-1} = ((a_{i_1} \ast a_{i_2}^{-1}) \ast (a_{i_3} \ast b_{i_3})^{-1}] \ast \ldots \ast (a_{i_{n-1}} \ast b_{i_{n-1}}) \ast (a_{i_n} \ast b_{i_n})^{-1} = (((a_{i_1} \ast a_{i_2}^{-1}) \ast a_{i_3}^{-1}) \ast \ldots \ast a_{i_{n-1}}^{-1} \ast (a_{i_1} \ast b_{i_2}^{-1}) \ast b_{i_3}^{-1} \ast \ldots \ast b_{i_n}^{-1}) \ast f(f\{a_i\}) \ast f(f\{b_j\}).

Hence, \(f(\{a_i\} \ast \{b_j\}) = f(\{a_i\}) \ast f(\{b_j\})\). Accordingly, \(f\) is a \(B\)-homomorphism.

Let \(y \in X = \langle \bigcup_{i \in I} N_i \rangle_B\). Then \(y = (\ldots((x_1^{m_1} \ast x_2^{m_2} \ast x_3^{m_3}) \ast \ldots \ast x_n^{m_n}) \ast x_n^{m_n}) \ast \ldots \ast x_n^{m_n}) \ast \ldots \ast x_n^{m_n}) \ast x_n^{m_n})\) for some \(x_j \in \bigcup_{i \in I} N_i\), \(n_j \in \mathbb{Z}\). For each \(j = 1, 2, \ldots, n\), there exists \(N_{i_j}\) such that \(x_j \in N_{i_j}\). Since \(N_{i_j}\) is a subalgebra, \(a_j = x_j^{m_j} \in N_{i_j}\) for each \(j\). Hence, \(a_j^{-1} = x_j^{-m_j} \in N_{i_j}\) for each \(j\). By hypothesis, it follows that \(a_j \notin N_{i_k}\) for all \(j \neq k\). Let \(\{u_i\} \in \prod_{i \in I} N_i\) be such that \(u_i = 0\) for all \(i \neq i_j = 1, 2, \ldots, n\), \(u_{i_1} = a_1\) and \(u_{i_k} = a_k^{-1}, k = 2, \ldots, n\). Since \(m < \infty\), \(\{u_i\} \in \prod_{i \in I} u N_i\) and

\[
f(\{u_i\}) = (\ldots((a_1 \ast (a_2^{-1})^{-1}) \ast a_3^{-1}) \ast \ldots \ast (a_n^{-1})^{-1} \ast (a_1^{-1})^{-1}
\]
\[
= (\ldots((a_1 \ast a_2) \ast a_3) \ast \ldots \ast a_{n-1}) \ast a_n
\]
\[
= (\ldots((x_1^{m_1} \ast x_2^{m_2}) \ast x_3^{m_3}) \ast \ldots \ast x_n^{m_n}) \ast x_n
\]
\[
= y.
\]

Hence, \(f\) is onto.

Let \(\{u_i\} \in \prod_{i \in I} u N_i\). Then there exists \(I' = \{i_1, i_2, \ldots, i_n\} \subseteq I\) such that \(u_i = 0\) for all \(i \in I \setminus I'\) and \(u_{i_1}, u_{i_2}, \ldots, u_{i_n} \neq 0\). Suppose \(\{u_i\} \in \text{Ker } f\). Then \(f(\{u_i\}) = 0\). Thus, \((\ldots((u_1 \ast \ldots \ast u_{i_2}^{-1} u_{i_3}^{-1}) \ast \ldots \ast u_{i_{n-1}}^{-1}) \ast u_{i_n}^{-1} = 0\). By (P1), it follows that \((\ldots((u_1 \ast \ldots \ast u_{i_2}^{-1} u_{i_3}^{-1}) \ast \ldots \ast u_{i_{n-1}}^{-1} \ast u_{i_n}^{-1} = u_{i_n}^{-1})\). Since \(u_{i_n} \in N_{i_n}\), \(u_{i_n}^{-1} = 0 \ast u_{i_n} \in N_{i_n}\). Hence, \(u_{i_n}^{-1} \in N_{i_n} \cap \left\{ \bigcup_{i \notin i_n} \bigcup_{i \notin N_{i_n}} \right\}_B\). By hypothesis, it follows that \(u_{i_n}^{-1} = 0\). Accordingly, \(u_{i_n} = (u_{i_n}^{-1})^{-1} = 0^{-1} = 0\). Repeating the process, we have \(u_{i_j} = 0\) for all \(j\). Thus, \(u_i = 0\) for all \(i \in I\). Hence, \(\{u_i\} = \{0\}\). By Proposition 2.16, \(f\) is one-to-one. Therefore, \(f\) is a \(B\)-isomorphism.

In view of Remark 4.6 and Theorem 4.7, we shall extend the notion of internal direct product to an arbitrary family of normal subalgebras.

**Definition 4.8.** The \(B\)-algebra \(X\) satisfying the conditions of Theorem 4.7 is called the internal direct product of the family \(\{N_i : i \in I\}\) (or the internal direct sum if \(X\) is commutative).

**Example 4.9.** Consider the Klein \(B\)-algebra \(K_4\) described in Example 2.1 and its subalgebras \(\langle a \rangle_B = \{c, a\}\) and \(\langle b \rangle_B = \{e, b\}\). Since \(K_4\) is commutative, \(\langle a \rangle_B\) and \(\langle b \rangle_B\) are normal in \(K_4\). Notice that \(\langle a \rangle_B \cap \langle b \rangle_B = \{e\}\). Since \(e, a, b, c = \ldots\)
Consider the $B$-algebra $\sim u$ there are $w$ $f_w$ $w$ $w_f$ $w_w$ $w_w$. Continuing this process, we have $w$ $f_w$ $w$ $w_w$ $w_w$. Example 4.10. Consider the $B$-algebra $\ast Z = (\mathbb{Z}, -, 0)$ and $X = \sum \ast Z$ (external direct sum) where the copies of $\ast Z$ is indexed by an arbitrary set $I$. For each $i \in I$, let $N_i = \{\{u_j\} \in \sum \mathbb{Z} : u_i \in \mathbb{Z}, u_j = 0$ for all $j \neq i\}$. Clearly, $N_i$ is a subalgebra of $X$ for all $i \in I$. Since $\ast Z$ is commutative, $X$ is also commutative, by Remark 4.4. Hence, $N_i$ is normal in $X$ for all $i$.

Let $\{u_i\} \in \sum \mathbb{Z}$. Then $u_i \in \mathbb{Z}$ and $u_i = 0$ for all but a finite number of $i \in I$. Thus, there exists $t \in \mathbb{Z}^+$ and $k_1, k_2, ..., k_t \in I$ such that $u_{k_j} \neq 0$ and $u_i = 0$ for all $i \neq k_j$ for all $j = 1, 2, ..., t$.

Consider $k_1 \in I$. Then for each $i \in I$, we can write $u_i = w'_{1,i} - w_{1,i}$ where $w'_{1,i} = u_i$ and $w_{1,i} = 0$ for all $i \neq k_1$, $w'_{1,k_1} = 0$ and $w_{1,k_1} = -u_{g_1}$. Thus, $\{u_i\} = \{w'_{1,i} - w_{1,i}\} = \{w'_{1,i}\} \ast \{w_{1,i}\}$. Consider $k_2 \in I$. Then for each $i \in I$, we can write $w'_{1,i} = w'_{2,i} - w_{2,i}$ where $w'_{2,i} = w_{2,i}$ and $w_{2,i} = 0$ for all $i \neq k_2$, $w'_{2,k_2} = 0$ and $w_{2,k_2} = -w'_{1,k_2} = -u_{k_2}$. Thus,

$\{u_i\} = \{w'_{1,i}\} \ast \{w_{1,i}\} = \{w'_{2,i} - w_{2,i}\} \ast \{w_{1,i}\} = (\{w'_{2,i}\} \ast \{w_{2,i}\}) \ast \{w_{1,i}\}$.

Continuing this process, we have

$\{u_i\} = (\{w'_{2,i}\} \ast \{w_{2,i}\}) \ast \{w_{1,i}\} = (((\{w'_{3,i}\} \ast \{w_{3,i}\}) \ast \{w_{2,i}\}) \ast \{w_{1,i}\})$

$\vdots$

$= \ldots (((\{w'_{k-1,i}\} \ast \{w_{k-1,i}\}) \ast \{w_{k-2,i}\}) \ast \ldots \ast \{w_{2,i}\}) \ast \{w_{1,i}\}$

where $w_{j,i} = 0$ for all $i \neq k_j$ and $w_{j,k_j} = -u_{k_j}$ for all $j = 1, 2, ..., t - 1$. Since there are $t$ $k_j$s, it follows that $w'_{k-1,i} = 0$ for all $i \neq k_t$ and $w'_{k-1,k_t} = u_{k_t}$. Let $w'_{k-1,i} = w_{i,i}$. Thus, $\{u_i\} = \ldots (((\{w'_{k-1,i}\} \ast \{w_{k-1,i}\}) \ast \{w_{k-2,i}\}) \ast \ldots \ast \{w_{2,i}\}) \ast \{w_{1,i}\}$ where $w_{j,i} = 0$ for all $i \neq k_j$, $w_{j,k_j} = -u_{k_j}$, $j = 1, 2, ..., t - 1$, and $w_{i,i} = 0$ for all $i \neq k_t$ and $w_{i,k_t} = u_{k_t}$. Hence, $\{w_{j,i}\} \in N_{k,j}$ for $j = 1, 2, ..., t$. Accordingly, $\{u_i\} \in (\cup_{i \in I} N_i)_B$. Thus, $\sum \mathbb{Z} \subseteq (\cup_{i \in I} N_i)_B$. Hence, $\sum \mathbb{Z} = (\cup_{i \in I} N_i)_B$.

Suppose there exists $\{u_i\} \in N_k \cap \left(\cup_{q \neq k} N_q\right)_B$ such that $\{u_i\} \neq \{0\}$. Then $\{u_i\} \in N_k$, $\{u_i\} \in \left(\cup_{q \neq k} N_q\right)_B$, and there exists $r \in I$ such that $u_r \neq 0$. Thus, $u_i = 0$ for all $i \neq k$ and $u_k \in \mathbb{Z}$, $u_k \neq 0$. Also,

$\{u_i\} = (\ldots (((\{w_{1,i}\}^{n_1} \ast \{w_{2,i}\}^{n_2} \ast \{w_{3,i}\}^{n_3} \ast \ldots \ast \{w_{t-1,i}\}^{n_{t-1}}) \ast \{w_{t,i}\}^{n_t}) \ast \{w_{t-1,i}\}^{n_{t-1}}) \ast \ldots \ast \{w_{1,i}\}^{n_1}$

for some $\{w_{s,i}\} \in \cup_{q \neq k} N_q$, $s = 1, 2, ..., t$. Hence, for each $s$, there exists $N_{q_s}$ such that $\{w_{s,i}\} \in N_{q_s}$, $q_s \neq k$. It follows that $w_{s,q_k} = 0$. By Proposition 4.1, $\{w_{s,i}\}^{n_s} = \{w_{s,i}\}^{n_s}$. Thus, $w_{s,q_k}^{n_s} = 0^{n_s} = 0$ for all $s = 1, 2, ..., t$.

Consider $\{w_{1,i}\}^{n_1} \ast \{w_{2,i}\}^{n_2}$. We have $w_{1,q_k}^{n_1} = 0$ and $w_{2,q_k}^{n_2} = 0$. It follows that by Proposition 4.1 and definition of $\ast$, $\{w_{1,i}\}^{n_1} \ast \{w_{2,i}\}^{n_2} = \{w_{1,i}\}^{n_1} \ast \{w_{2,i}\}^{n_2} = \{w_{1,i}\}^{n_1} \ast \{w_{2,i}\}^{n_2}$. Therefore, $K_4$ is the internal direct product of $\langle a \rangle_B$ and $\langle b \rangle_B$. 

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\{w_{2,k}n^2\} \text{ where } w_{1,qk}n^1 \ast w_{2,qk}n^2 = 0 \ast 0 = 0, \text{ by (B1). Continuing in this manner, it follows that}
\[
(...(w_{1,qk}n^1 \ast w_{2,qk}n^2) \ast w_{3,qk}n^3) \ast ... \ast w_{l-1,qk}n^{l-1}) \ast w_{l,qk}n^l = 0
\]
But \(u_k \neq 0\) and \(u_k = (...(w_{1,qk}n^1 \ast w_{2,qk}n^2) \ast w_{3,qk}n^3) \ast ... \ast w_{l-1,qk}n^{l-1}) \ast w_{l,qk}n^l, a contradiction. Hence, \(\{u_i\} = \{0\}\). Therefore, \(X\) is the internal direct direct sum of the family \(N_i = \{u_j\} \in \sum \mathbb{Z}: u_i \in \mathbb{Z}, u_j = 0\text{ for all } j \neq i\).

**Definition 4.11.** Let \(X\) be a \(B\)-algebra and \(S \subseteq X\). A combination of elements of \(S\) is a finite product \([...[(x_1^{n_1} \ast x_2^{n_2}) \ast x_3^{n_3}] \ast ... \ast x_{k-1}^{n_{k-1}}] \ast x_k^{n_k}\) where \(x_i \in S\) and \(n_i \in \mathbb{Z}\) for all \(i = 1, 2, ..., k\), with \(k < \infty\).

**Definition 4.12.** Let \((X; \ast, 0)\) be a \(B\)-algebra. A subset \(S \subseteq X\) is said to be a basis for \(X\) if \(X = \langle S \rangle_B\) and for distinct \(x_1, x_2, ..., x_k \in S\) and \(n_i \in \mathbb{Z}\), \([...[(x_1^{n_1} \ast x_2^{n_2}) \ast x_3^{n_3}] \ast ... \ast x_{k-1}^{n_{k-1}}] \ast x_k^{n_k} = 0\) implies \(n_i = 0\) for all \(i = 1, 2, ..., k\).

**Definition 4.13.** Let \(X\) be a commutative \(B\)-algebra. \(X\) is said to be a free commutative \(B\)-algebra if it has a nonempty basis.

**Example 4.14.** Consider the \(B\)-algebra \(*\mathbb{Z} = (\mathbb{Z}, -, 0)\) and \(\{1\} \subseteq \mathbb{Z}\). Note that \(*\mathbb{Z} = \langle 1 \rangle_B\). Also, for all \(n \in \mathbb{Z}\), \(1^n = 0\) if and only if \(n = 0\). Thus, \(\{1\}\) is a basis for \(\mathbb{Z}\). Therefore, \(*\mathbb{Z} = (\mathbb{Z}; -, 0)\) is a free commutative \(B\)-algebra.

**Example 4.15.** By Remark 2.4, \(\langle \emptyset \rangle_B = \{0\}\). Thus, \(\{0\}\) is not a free commutative \(B\)-algebra.

**Lemma 4.16.** Let \((F; \ast, 0)\) be a free commutative \(B\)-algebra with basis \(S\). Then \(0 \not\in S\).

**Proof.** Suppose \(0 \in S\). Note that \(0^1 = 0\). Since \(1 \neq 0\), we get a contradiction to the definition of a basis. Hence, \(0 \not\in S\).

**Theorem 4.17.** Let \((F; \ast, 0)\) be a free commutative \(B\)-algebra with basis \(S\). Then the representation of every element of \(F\) as a combination of elements of \(S\) is unique.

**Proof.** Let \((F; \ast, 0)\) be a free commutative \(B\)-algebra with basis \(S\) and \(u \in F\), \(u \neq 0\). Then \(u = [...[(x_1^{n_1} \ast x_2^{n_2}) \ast x_3^{n_3}] \ast ... \ast x_{k-1}^{n_{k-1}}] \ast x_k^{n_k}\) for some \(x_i \in S\), \(n_i \in \mathbb{Z}\) and \(x_i \neq 0\) for all \(i\), by Lemma 4.16. Suppose \(x_i = x_j\) for some \(i \neq j\). Without loss of generality, we assume that \(i < j\). By Proposition 3.3, we may assume that \(x_i \neq x_j\) for all \(i \neq j\). Suppose
\[
u = [...[(y_1^{m_1} \ast y_2^{m_2}) \ast y_3^{m_3}] \ast ... \ast y_{l-1}^{m_{l-1}}] \ast y_l^{m_l}\]
Thus, where assume that \( x = \[ u \] \) terms
\[
= \text{Case 1: } x_i = y_i \text{ for all } i, j = 1, 2, ..., k. \text{ By Proposition 3.4,}
\]
\[
0 = u * u = \text{Case 2. } x_i \neq y_i \text{ for some } i = 1, 2, ..., k. \text{ By Corollary 2.10, Proposition 3.3}
\]
\[
0 = u * u = \text{Proposition 3.3 and (P5) for } i = 1, \text{ and for all } i, m_i = n_i \text{ for all } i.
\]

By Corollary 2.10, Proposition 3.3 and (P5) for \( i = 2, 3, ..., k, \)
\[
0 = u * u = \text{Thus, } n_j - m_j = 0, \text{ that is, } n_j = m_j \text{ for all } j = 2, 3, ..., k \text{ and } n_1, m_1 = 0. \text{ By}
\]
\[
0 = u = \text{Corollary 2.10, Proposition 3.3 and (P5) for } i = 2, 3, ..., k, \text{ hence,}
\]
\[
u = \text{Thus, } n_j - m_j = 0, \text{ that is, } n_j = m_j \text{ for all } j \neq i \text{ and } n_i, m_i = 0, \text{ that is,}
\]
\[
= \text{Hence,}
\]
\[
= \text{Consequently, } x_i^n = y_i^0 = 0 = y_i = x_i^n. \text{ Hence,}
\]
\[
u = \text{and}
\]
\[
u = \text{and}
\]

for some \( x_i, y_j \in S, n_i, m_j \in \mathbb{Z}, i = 1, 2, ..., k, \) \( j = 1, 2, ..., t. \) Again, we may assume that \( x_p \neq x_q \) and \( y_p \neq y_q \) for all \( p \neq q. \) If \( k > t, \) then we may write
\[
\begin{align*}
u = \text{and}
\end{align*}
\]
with \( x_j = y_j \) and \( n_j = m_j \) for all \( j \neq i \). By (B2), 0 plays no role in each equality. Accordingly, we may disregard each \( i \text{th} \) term. If follows that if

\[
\begin{aligned}
\mathcal{B} &= \{ x_{1}^{n_{1}} * x_{2}^{n_{2}} * x_{3}^{n_{3}} * \ldots * x_{k-1}^{n_{k-1}} * x_{k}^{n_{k}} \} \\
\mathcal{B} &= \{ y_{1}^{m_{1}} * y_{2}^{m_{2}} * y_{3}^{m_{3}} * \ldots * y_{k-1}^{m_{k-1}} * y_{k}^{m_{k}} \},
\end{aligned}
\]

then \( x_j = y_j \) and \( n_j = m_j \) for all \( j \). This proves the uniqueness of the representation. \( \square \)

The following theorem is a characterization of a free commutative \( B \)-algebra.

**Theorem 4.18.** Let \( (F; *, 0) \) be a commutative \( B \)-algebra. Then the following are equivalent:

(i) \( F \) has a nonempty basis;

(ii) \( F \) is the internal direct sum of a family of infinite cyclic subalgebras;

(iii) \( F \) is (isomorphic to) a direct sum of copies of the \( B \)-algebra \( *Z \).

**Proof.** (i) \( \Rightarrow \) (ii) Suppose \( F \) has a nonempty basis \( S \). Let \( x \in S \) and \( n \in \mathbb{Z} \). By the definition of a basis, it follows that if \( n = 0 \), then \( n = 0 \). Thus, \( |x|_B = +\infty \).

Since \( F \) is commutative, \( \langle x \rangle_B \) is normal in \( F \). Since for each \( x \in S \), \( \langle x \rangle_B \subseteq \langle S \rangle_B = F \), it is clear that \( F = \bigcup_{x \in S} \langle x \rangle_B \).

Suppose there exists \( z \in S \) such that \( \langle z \rangle_B \cap \bigcup_{x \in S} \langle x \rangle_B \neq \{0\} \). Then there exists \( w \in \langle z \rangle_B \cap \bigcup_{x \in S} \langle x \rangle_B \) such that \( w \neq 0 \). Thus, \( w \in \langle z \rangle_B \) and \( w \in \bigcup_{x \in S} \langle x \rangle_B \).

Hence, \( w = z^n \) for some \( n \in \mathbb{Z} \) and \( w = \ldots \ldots \) for some distinct \( y_1, y_2, \ldots, y_k \in \bigcup_{x \in S} \langle x \rangle_B \). Now, for each \( i \), there exists \( x_i \in S \setminus \langle z \rangle \) such that \( y_i = \langle x_i \rangle \). Hence, \( y_i = x_i^m \) for some \( m_i \in \mathbb{Z} \).

By Theorem 2.9,

\[
\begin{aligned}
z^n &= w \\
&= \ldots \ldots \langle y_1^{n_1} * y_2^{n_2} * y_3^{n_3} * \ldots * y_{k-1}^{n_{k-1}} * y_k^{n_k} \\
&= \ldots \langle (x_1^{m_1} * x_2^{m_2} * x_3^{m_3} * \ldots * x_{k-1}^{m_{k-1}} * x_k^{m_k} \rangle \rangle \\
&= \ldots \langle (x_1^{m_1} * x_2^{m_2} * x_3^{m_3} * \ldots * x_{k-1}^{m_{k-1}} * x_k^{m_k} \rangle \rangle \\
&= \ldots \langle x_i^{m_i} \rangle \\
&= \ldots \langle x_i^{m_i} \rangle
\end{aligned}
\]

Thus, \( 0 = z^n * z^n = \ldots \ldots \)
with $z \neq x_1, x_2, \ldots, x_k$. If the $x_i$'s are distinct, it follows that $m_i n_i, n = 0$ for all $i$ since $S$ is a basis. If $x_p = x_t$ for some $p \neq t$, then by Proposition 3.3, we have

$$0 = \ldots \sum f_{m_k n_k} \cdot z^n = \ldots \sum \left( f_{m_1 n_1} \cdot x_2 \cdot \ldots \cdot x_i \cdot \ldots \cdot x_{m_t n_t} \right) \cdot z^n.$$

Thus, $m_i n_i = 0$ for all $i \neq p, t$ and $m_p n_p + m_t n_t = 0$. In both cases, $n = 0$. Thus, $w = z^n = z^0 = 0$, a contradiction. Hence, $(z) \cap \left( \bigcup_{x \neq y} \langle x \rangle \right)_B = \{0\}$. It follows that $F = \sum_{x \in S} \langle x \rangle_B$.

(ii) $\Rightarrow$ (iii) Let $S \subseteq F$ such that $F = \sum_{x \in S} \langle x \rangle_B$, where $\langle x \rangle_B$ is infinite for each $x \in S$. By Theorem 2.15, it follows that $\langle x \rangle_B \cong \ast \mathbb{Z}$. Thus, for each $x \in S$, there exists a B-isomorphism $f_x : \langle x \rangle_B \rightarrow \ast \mathbb{Z}$. Let $f = \prod_{x \in S} f_x$ be the map $\sum_{x \in S} \langle x \rangle_B \rightarrow \ast \mathbb{Z}$ given by $f(\{u_x\}) = \{f_x(u_x)\}$ and where the copies of $\ast \mathbb{Z}$ are indexed by $S$. By Remark 4.5, $f$ is a one-to-one and onto B-homomorphism, since each $f_x$ is. Hence, $F = \sum_{x \in S} \langle x \rangle_B \cong \sum \ast \mathbb{Z}$.

(iii) $\Rightarrow$ (i) Suppose $F \cong \sum \ast \mathbb{Z}$ where the copies of $\ast \mathbb{Z}$ are indexed by a nonempty set $S$. Then for all $\{u_x\} \in \sum \mathbb{Z}$, $x \in S$, $u_x = 0$ for all but a finite number of $x \in S$. Note that $\sum \ast \mathbb{Z} = (\sum \mathbb{Z}; *, \{0\})$ is a B-algebra where $\{u_x\} \cdot \{v_x\} = \{u_x - v_x\} \cdot \{u_x\} \in \sum \mathbb{Z}$.

For each $y \in S$, let $\alpha_y = \{u_x\} \in \sum \mathbb{Z}$ such that $u_x = 0$ for all $x \neq y$ and $u_y = 1$. Let $A = \{\alpha_y : y \in S\}$. Since $S \neq \emptyset$, it follows that $\emptyset \neq A \subseteq \sum \mathbb{Z}$. We show that $A$ is a basis for $\sum \ast \mathbb{Z}$.

Let $a \in \sum \mathbb{Z}$. Then $a = \{u_x\} \in \sum \mathbb{Z}$ and $u_x = 0$ for all but a finite $x \in S$. Thus, there exists $k \in \mathbb{Z}^+$ and $y_1, y_2, \ldots, y_k \in S$ such that $u_{y_i} \neq 0$ and $u_x = 0$ for all $x \neq y_i$ for all $i = 1, 2, \ldots, k$.

Consider $y_1 \in S$. Then for each $x \in S$, we can write $u_x = w'_{1,x} - w_{1,x}$ where $w'_{1,x} = u_x$ and $w_{1,x} = 0$ for all $x \neq y_1$, $w'_{1,y_1} = 0$ and $w_{1,y_1} = -u_{y_1}$. Thus, $a = \{u_x\} = \{w'_{1,x} - w_{1,x}\} = \{w'_{1,x}\} \cdot \{w_{1,x}\}$. Consider $y_2 \in S$. Then for each $x \in S$, we can write $w'_{1,x} = w'_{2,x} - w_{2,x}$ where $w'_{2,x} = w'_{1,x} = u_x$ and $w_{2,x} = 0$ for all $x \neq y_2$, $w'_{2,y_2} = 0$ and $w_{2,y_2} = -u_{y_2}$. Thus, $a = \{w'_{1,x}\} \cdot \{w_{1,x}\} = \{w'_{2,x} - w_{2,x}\} \cdot \{w_{1,x}\} = \{w'_{2,x}\} \cdot \{w_{2,x}\} \cdot \{w_{1,x}\}$.

Continuing this process, we have

$$a = \{w'_{2,x}\} \cdot \{w_{2,x}\} \cdot \{w_{1,x}\} = \{w'_{3,x}\} \cdot \{w_{3,x}\} \cdot \{w_{2,x}\} \cdot \{w_{1,x}\}$$

$$\ldots$$

$$= \cdots \left( \{w'_{k-1,x}\} \cdot \{w_{k-1,x}\} \right) \cdot \{w_{k-2,x}\} \cdot \ldots \cdot \{w_{k-2,x}\} \cdot \{w_{k-1,x}\} \cdot \{w_{1,x}\}$$

where $w_{i,x} = 0$ for all $x \neq y_i$ and $w_{i,y_i} = -u_{y_i}$ for all $i = 1, 2, \ldots, k - 1$. Since there are $k$ $g_i$'s, it follows that $w'_{k-1,x} = 0$ for all $x \neq y_k$ and $w_{k-1,k} = u_{y_k}$. Let
$w'_{k-1,x} = w_{k,x}$. Thus, $a = [\ldots((\{w_{k,x}\} \ast \{w_{k-1,x}\}) \ast \{w_{k-2,x}\}) \ast \ldots \ast \{w_{2,x}\}] \ast \{w_{1,x}\}$

where $w_{i,x} = 0$ for all $x \neq y_i$, $w_{i,y_i} = -u_{y_i}$, $i = 1, 2, \ldots, k - 1$, and $w_{k,x} = 0$ for all $x \neq y_k$ and $w_{k,y_k} = u_{y_k}$.

Since $1^n = n$ and $0^n = 0$ for all $n \in \mathbb{Z}$, it follows that $w_{i,x} = 0 = 0^{u_{y_i}}$ for all $x \neq y_i$, $w_{i,y_i} = -u_{y_i} = 1^{-u_{y_i}}$, $i = 1, 2, \ldots, k - 1$ and $w_{k,x} = 0 = 0^{u_{y_k}}$ for all $x \neq y_k$ and $w_{k,y_k} = u_{y_k}$.

For each $i$, let $\alpha_{y_i} = \{v_{i,x}\}$. Then $v_{i,x} = 0$ for all $x \neq y_i$ and $v_{i,y_i} = 1$, $i = 1, 2, \ldots, k$. Thus, $(v_{i,x})^{-u_{y_i}} = 0^{-u_{y_i}} = 0$ for all $x \neq y_i$ and $(v_{y_i})^{-u_{y_i}} = 1^{-u_{y_i}} = -u_{y_i}$, $i = 1, 2, \ldots, k - 1$ and $(v_{k,x})^{u_{y_k}} = 0^{u_{y_k}} = 0$ for all $x \neq y_k$ and $(v_{k,y_k})^{u_{y_k}} = 1^{u_{y_k}} = u_{y_k}$. Hence, $w_{i,x} = (v_{i,x})^{-u_{y_i}}$ for all $i = 1, 2, \ldots, k - 1$ and $w_{k,x} = (v_{k,x})^{u_{y_k}}$. Thus, by Proposition 4.1, $\{w_{i,x}\} = \{(v_{i,x})^{-u_{y_i}}\} = \{v_{i,x}\}^{-u_{y_i}} = (\alpha_{y_i})^{-u_{y_i}}$ for all $i = 1, 2, \ldots, k - 1$ and similarly, $\{w_{k,x}\} = (\alpha_{y_k})^{u_{y_k}}$. Thus, $\alpha_{y_1}, \alpha_{y_2}, \ldots, \alpha_{y_k}$, be distinct elements of $A$ with

$$a = [\ldots((\{w_{k,x}\} \ast \{w_{k-1,x}\}) \ast \{w_{k-2,x}\}) \ast \ldots \ast \{w_{2,x}\}] \ast \{w_{1,x}\}$$

$$= [\ldots((\alpha_{y_1})^{u_{y_1}} \ast (\alpha_{y_2})^{-u_{y_2}}) \ast \ldots \ast (\alpha_{y_k})^{-u_{y_k}}] \ast (\alpha_{y_k})^{u_{y_k}}$$

where $u_{y_i} \in \mathbb{Z}$. By Theorem 2.7, it follows that $a \in \langle A \rangle _B$. Accordingly, $\sum \mathbb{Z} \subseteq \langle A \rangle _B$. Since $A \subseteq \sum \mathbb{Z}$, $\langle A \rangle _B \subseteq \sum \mathbb{Z}$. Thus, $\sum \mathbb{Z} = \langle A \rangle _B$.

Let $\alpha_{y_1}, \alpha_{y_2}, \ldots, \alpha_{y_k}$, be distinct elements of $A$ with

$$[\ldots((\alpha_{y_1})^{n_1} \ast (\alpha_{y_2})^{n_2} \ast (\alpha_{y_3})^{n_3}) \ast \ldots \ast (\alpha_{y_{k-1}})^{n_{k-1}}] \ast (\alpha_{y_k})^{n_k} = \{0\},$$

$y_i \in S$, $n_i \in \mathbb{Z}$, $i = 1, 2, \ldots, k$. By definition of elements of $A$, it follows that $y_1, y_2, \ldots, y_k$ are also distinct elements of $A$.

For each $i$, $\alpha_{y_i} = \{w_{i,x}\}$ where $w_{i,x} = 0$ for all $x \neq y_i$ and $w_{i,y_i} = 1$.

Now, by Proposition 4.1, $(\alpha_{y_1})^{n_1} \ast (\alpha_{y_2})^{n_2} = \{w_{1,x}\}^{n_1} \ast \{w_{2,x}\}^{n_2} = \{(w_{1,x})^{n_1}\} \ast \{(w_{2,x})^{n_2}\} = \{(w_{1,x})^{n_1} - (w_{2,x})^{n_2}\}$. For all $x \neq y_1, y_2$, $(w_{1,x})^{n_1} - (w_{2,x})^{n_2} = 0^{n_1} - 0^{n_2} = 0$. For $x = y_1$, $(w_{1,y_1})^{n_1} - (w_{2,y_1})^{n_2} = 1^{n_1} - 0^{n_2} = n_1 - 0 = n_1$. For $x = y_2$, $(w_{1,y_2})^{n_1} - (w_{2,y_2})^{n_2} = 0^{n_1} - 1^{n_2} = 0 - n_2 = -n_2$.

Let $(\alpha_{y_1})^{n_1} \ast (\alpha_{y_2})^{n_2} = \{v_{1,x}\}$. Then for all $x \neq y_1, y_2$, $v_{x} = 0$, $v_{y_1} = n_1$ and $v_{y_2} = -n_2$. By Proposition 4.1, $(\alpha_{y_1})^{n_1} \ast (\alpha_{y_2})^{n_2} \ast (\alpha_{y_3})^{n_3} = \{v_{x}\} \ast \{(w_{x,y})^{n_3}\} = \{v_{x} - (w_{x,y})^{n_3}\}$. For all $x \neq y_1, y_2, y_3$, $v_{x} - (w_{x,y})^{n_3} = 0 - 0^{n_3} = 0$. For $x = y_1$, $v_{x} - (w_{x,y})^{n_3} = n_1 - 0^{n_3} = n_1 - 0 = n_1$. For $x = y_2$, $v_{x} - (w_{x,y})^{n_3} = -n_2 - 0^{n_3} = -n_2 - 0 = -n_2$. For $x = y_3$, $v_{x} - (w_{x,y})^{n_3} = 0 - 1^{n_3} = 0 - n_3 = -n_3$.

Continuing this process, we obtain

$$[\ldots((\alpha_{y_1})^{n_1} \ast (\alpha_{y_2})^{n_2} \ast (\alpha_{y_3})^{n_3}) \ast \ldots \ast (\alpha_{y_{k-1}})^{n_{k-1}}] \ast (\alpha_{y_k})^{n_k} = \{z_{x}\},$$

where $z_{x} = 0$ for all $x \neq y_1, y_2, \ldots, y_k$, $z_{y_i} = n_1$ and $z_{y_i} = -n_i$ for all $i = 2, 3, \ldots, k$. By hypothesis, $\{z_{x}\} = \{0\}$. Hence, $0 = z_{y_i} = n_1$, $0 = z_{y_1} = n_1$. Accordingly, $n_i = 0$ for all $i = 1, 2, \ldots, k$.

Therefore, $A$ is a basis for $\sum \mathbb{Z}$. Since $\sum = \sum \mathbb{Z}$, it follows that $F$ also has a nonempty basis. \qed
Theorem 4.19. Let \((F; * , 0)\) be a free commutative \(B\)-algebra. Then there exists a nonempty set \(S\) and a function \(\iota : S \to F\) with the following property: given a commutative \(B\)-algebra \(Y\) and a function \(f : S \to Y\), there exists a unique \(B\)-homomorphism \(g : F \to Y\) such that \(g \iota = f\).

**Proof.** Let \(S \neq \emptyset\) be basis for \(F\) and consider the inclusion map \(\iota : S \to F\) given by \(\iota(x) = x\) for all \(x \in S\). Let \((Y; * ', 0')\) be a commutative \(B\)-algebra and \(f : S \to Y\). Let \(u \in F\). Then \(u = \ldots[[x_1^{n_1} * x_2^{n_2} * x_3^{n_3}] * \ldots * x_k^{n_k}]\) for some \(x_i \in S\), \(n_i \in \mathbb{Z}\). Define \(g : F \to Y\) by

\[
g(u) = g(\ldots[[x_1^{n_1} * x_2^{n_2} * x_3^{n_3}] * \ldots * x_k^{n_k}]) = \ldots[[f(x_1)^{n_1} * f(x_2)^{n_2} * f(x_3)^{n_3}] * \ldots * f(x_{k-1})^{n_{k-1}}] * f(x_k)^{n_k}
\]

for all \(u = \ldots[[x_1^{n_1} * x_2^{n_2} * x_3^{n_3}] * \ldots * x_{k-1}^{n_{k-1}}] * x_k^{n_k} \in F\).

Let \(u, v \in F\). Then

\[
u = \ldots[[x_1^{n_1} * x_2^{n_2} * x_3^{n_3}] * \ldots * x_k^{n_k}]\quad \text{and} \quad v = \ldots[[y_1^{m_1} * y_2^{m_2} * y_3^{m_3} * \ldots * y_{t-1}^{m_{t-1}}] * y_t^{m_t}]
\]

for some \(x_i, y_j \in S\), \(n_i, m_j \in \mathbb{Z}\). Suppose \(u = v\). By Theorem 4.17, \(k = t\), \(x_i = y_i\), and \(n_i = m_i\) for all \(i\). By the well-definedness of \(f, f(x_i) = f(y_i)\) for all \(i\). Accordingly,

\[
g(u) = \ldots[[f(x_1)^{n_1} * f(x_2)^{n_2} * f(x_3)^{n_3}] * \ldots * f(x_{k-1})^{n_{k-1}}] * f(x_k)^{n_k} = \ldots[[f(y_1)^{m_1} * f(y_2)^{m_2} * f(y_3)^{m_3}] * \ldots * f(y_{k-1})^{m_{k-1}}] * f(y_k)^{m_k} = g(v).
\]

Thus, \(g\) is well-defined and for all \(x \in S\), \(g(\iota(x)) = g(x) = f(x)\). By Corollary 2.10,

\[
g(u * v) = g(\ldots[[x_1^{n_1} * x_2^{n_2} * x_3^{n_3}] * \ldots * x_k^{n_k}] * (\ldots[[y_1^{m_1} * y_2^{m_2} * y_3^{m_3}] * \ldots * y_{t}^{m_{t}}]))
\]

\[
= g(\ldots[[\ldots[[x_1^{n_1} * x_2^{n_2} * x_3^{n_3}] * \ldots * x_k^{n_k}] * \ldots * y_{t-1}^{m_{t-1}}] * \ldots * y_2^{m_2}] * y_1^{m_1})
\]

\[
= \ldots[[\ldots[[f(x_1)^{n_1} * f(x_2)^{n_2} * f(x_3)^{n_3}] * \ldots * f(x_{k-1})^{n_{k-1}}] * f(x_k)^{n_k}] * \ldots * f(y_{k-1})^{m_{k-1}}] * f(y_k)^{m_k}
\]

\[
= g(u) * g(v).
\]

Hence, \(g\) is a \(B\)-homomorphism.
Suppose there exists \( g' : F \to Y \) such that \( g' \iota = f \). Since \( S \) generates \( F \), \( g' \) is completely determined by its action on \( S \). Let \( x \in S \). Then
\[
g'(x) = g' \iota(x) = g' \iota(x) = f(x) = g(x) = g(\iota(x)) = g(x).
\]
Hence, \( g = g' \). Accordingly, \( g \) is unique.

References


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