Numerical solution of nonlinear oscillatory differential equations using shifted second kind Chebyshev wavelet method

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Abstract. A numerical method for solving Oscillatory differential equations using shifted second kind Chebyshev wavelets is presented. An operational matrix of derivative is introduced and utilized to reduce the oscillatory differential equations and its initial conditions to a set of algebraic equations in the unknown expansion coefficients. Illustrative examples are included to demonstrate the validity and applicability of the method. Numerical solution of Van der Pol oscillator problem and Duffing Equations are obtained by using this method.

Keywords: Chebyshev wavelets, operational matrix of derivative, oscillatory differential equations.

1. Introduction

The study of non linear oscillatory problems is of crucial importance in solid state physics, plasma physics, fluid dynamics, mathematical biology and chemical kinetics. Several techniques such as variational iteration method [1], Adomian decomposition method [2], Homotopy Perturbation Method [3], differential transform method [4] have been used for solving these problems.

Orthogonal functions and polynomial series have received considerable attention in dealing with numerical solution of these problems. The approach is based on reducing the given problem to a system of algebraic equations by approximating the unknown function in terms of truncated orthogonal series. Walsh functions, block pulse functions, Laguerre polynomials, Legendre poly-

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nomials, Chebyshev polynomials and Fourier series are widely used orthogonal functions.

In this paper we will apply wavelet based numerical method for solution of nonlinear oscillatory differential equations. Wavelets have properties like orthogonality and compact support. These properties are quite useful for approximating any function in terms of truncated orthogonal series.

Operational matrix of derivative and integration of wavelets are widely used for numerical solution of differential equations [5] [8] [10], fractional differential equations [6],problems of calculus of variations [9] .The method consists of reducing the oscillatory differential equation into a set of algebraic equations by expanding the candidate function as wavelet basis with unknown coefficients.

The paper is organized as follow In section 2 we describe the second kind Chebyshev Polynomials and their shifted form, Shifted second kind Chebyshev wavelet and its operational matrix of derivative. In Section 3 Mathematical formulation of proposed method is given. In Section 4 some examples are solved using the proposed numerical method. Conclusion is drawn in Section 5.

2. Second kind Chebyshev polynomials and their shifted form

The second kind Chebyshev Polynomials are defined on $[-1, 1]$ by

$$U_n(t) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad t = \cos\theta.$$

These polynomials are orthogonal w.r.t to the weight function $\sqrt{1-t^2}$ on the interval $[-1, 1]$ i.e.

$$\int_{-1}^{1} \sqrt{1-t^2}U_m(t)U_n(t)dt = \begin{cases} 0, & n = m \\ \frac{\pi}{2}, & \text{n} \neq m \end{cases}$$

and also satisfy the following recursive formula[7]

$$\begin{aligned} U_0(t) &= 1, \\ U_1(t) &= 2t, \\ U_{m+1}(t) &= 2tU_m(t) - U_{m-1}(t), \quad m = 1, 2, \ldots \end{aligned}$$

The Shifted second kind Chebyshev polynomials are defined on $[0, 1]$ by $U_n^*(t) = U_n(2t-1)$. These polynomials are orthogonal on $[0, 1]$ w.r.t the weight function $\sqrt{t-t^2}$ i.e.

$$\int_{0}^{1} \sqrt{t-t^2}U_n^*(t)U_m^*(t) = \begin{cases} 0, & n = m \\ \frac{\pi}{8}, & \text{n} \neq m. \end{cases}$$

2.1 Wavelets and shifted second kind Chebyshev wavelets

Wavelet methods have been used extensively for the solution of various problems of science and engineering.Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet[11], [12],[14], [15]. When the dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets

$$\psi_{a,b}(t) = |a|^{\frac{-1}{2}}\psi\Big(\frac{t-b}{a}\Big), \quad a, b \in R, \quad a \neq 0.$$

The second kind Chebyshev wavelet $\psi_{n,m}(t) = \psi(k, n, m, t)$ have four arguments $n = 0, 1, 2, ...2^k - 1$, k can assume any positive integer, m is the degree of the second kind Chebyshev polynomials and t denotes the time. They are defined on the interval [0,1] as

$$\psi_{n,m}(t) = \begin{cases} \frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}} U_m^*(2^k t - n), & \text{if } \frac{n}{2^k} \leq t \leq \frac{n+1}{2^k} \\ 0, & \text{otherwise,} \end{cases}$$

where $m = 0, 1, 2, ..., M$.

A function $y(t)$ defined over $L^2[0,1]$ can be expanded in terms of shifted second kind Chebyshev wavelets ψ_{nm} as:

$$y(t) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_{nm}\psi_{nm}(t), \tag{2.1}$$

where $c_{nm} =< y(t), \psi_{nm}(t) >$ denotes the inner product and is given by $c_{nm} = \int_0^1 \sqrt{t-t^2}y(t)\psi_{nm}(t)$. If the infinite series in (2.1) is truncated, then (2.1) can be written as

$$y(t) = \sum_{n=0}^{2^k-1}\sum_{m=0}^{M} c_{nm}\psi_{nm}(t) = C^T\Psi(t),$$

where C and $\Psi(t)$ are $2^k(M+1) \times 1$ matrices given by

$$C = \{c_{0,0}, c_{0,1}, .., c_{0,M}, ..., c_{2^k-1,0}, c_{2^k-1,1}, .., c_{2^k-1,M}\}^T \quad \text{and}$$
$$\Psi(t) = \{\psi_{0,0}, \psi_{0,1}, .., \psi_{0,M}, .., \psi_{2^k-1,0}, \psi_{2^k-1,1}, .., \psi_{2^k-1,M}\}^T.$$

The first derivative of second kind Chebyshev wavelet vector $\psi(t)$ can be defined as

$$\frac{d\psi}{dt} = D\psi(t),$$

where D is $2^k(M+1)$ square operational matrix of derivative of shifted second kind Chebyshev wavelet and is given by

$$D = \begin{pmatrix} F & 0 & \cdots & 0 \\ 0 & F & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & F \end{pmatrix}$$

where F is $(M+1)$ square matrix whose $(r,s)th$ element is given by

$$F_{r,s} = \begin{cases} 2^{k+2}s, & r \geq 2,\ r > s,\ (r+s)\text{odd} \\ 0, & \text{otherwise.} \end{cases}$$

Generally, the operational matrix of nth order derivative is given by

$$\frac{d^n\psi(t)}{dt} = D^n\psi(t), \qquad n = 1, 2, ...$$

3. Mathematical formulation

In this paper we will consider the following general Oscillatory differential equation

$$u'' + f(t, u, u') = 0 \tag{3.1}$$

with the initial conditions $u(0) = a,\ u'(0) = b$ where u is displacement, t is time and $'$ denotes differentiation w.r.t t.

The method consist of reducing the given problem to a set of algebraic equations by expanding the unknown function $u(t)$ as shifted second kind Chebyshev wavelets with unknown coefficients c_{nm}:

$$u(t) = \sum_{n=0}^{2^k-1} \sum_{m=0}^{M} c_{nm}\psi_{nm}(t) = C^T\Psi(t) \tag{3.2}$$

u' and u'' can also be approximated as

$$u'(t) = C^T D\Psi(t), \quad u''(t) = C^T D^2\Psi(t). \tag{3.3}$$

Substitution of (3.2) and (3.3)in (3.1) reduces the equation (3.1) to the following form

$$C^T D^2\psi(t) + f(t,\ C^T\psi(t),\ C^T D\psi(t)) = 0. \tag{3.4}$$

Now, we will collocate (3.4) at $2^k(M+1) - 2$ Chebyshev points which yields $2^k(M+1) - 2$ algebraic equations in unknown coefficients c_{nm}.

The initial conditions can also be expressed in terms of Chebyshev wavelets as

$$C^T\psi(0) = a, \quad C^T D\psi(0) = b. \tag{3.5}$$

Equations (3.4) and (3.5) give $2^k(M+1)$ algebraic equations which can be solved for the unknown components of the vector C. So the underlying Oscillatory differential equation together with initial conditions is converted into system of algebraic equations via operational matrix of derivatives. This system can be solved for the unknown $C$and an approximate wavelet solution to $u(t)$ given in (3.2) can be obtained.

4. Numerical examples

All numerical computations have been done by taking $M = 2, k = 0$. The matrices D, D^2, C and ψ which are used in calculation are given by

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 8 & 0 \end{pmatrix}, \quad D^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 32 & 0 & 0 \end{pmatrix}, \psi(x) = \sqrt{\frac{2}{\pi}} \begin{pmatrix} 2 \\ 8t-4 \\ 32t^2-32t+6 \end{pmatrix},$$

$$C = \begin{pmatrix} c_{0,0} & c_{0,1} & c_{0,2} \end{pmatrix}^T = \sqrt{\frac{\pi}{2}} \begin{pmatrix} c_0 & c_1 & c_2 \end{pmatrix}^T.$$

Example 1 (Van Der Pol Oscillator problem)**.** Consider the Van Der Pol Oscillator problem $u'' + u' + u + u^2u' = 2\cos t - \cos^3 t$ with initial conditions $u(0) = 0$,$u'(0) = 1$.

The analytic solution of this problem is given by $u(t) = \sin t$.

For numerical solution we approximate $u(t)$ in terms of Shifted second kind Chebyshev wavelet basis as follow

$$u(t) = C^T\Psi(t). \tag{4.1}$$

By using operational matrix of derivative we can approximate $u'(t), u''(t)$ as

$$\begin{aligned} u'(t) &= C^T D\Psi(t), \\ u''(t) &= C^T D^2\Psi(t). \end{aligned} \tag{4.2}$$

Substituting (4.1)and (4.2) into given equation, we have

$$C^TD^2\Psi(t) + C^TD\Psi(t) + C^T\Psi(t) + (C^T\Psi(t))^2(C^TD\Psi(t)) = 2\cos t - \cos^3 t$$

which is equivalent to

$$\begin{aligned} &64c_2 + 8c_1 + (64t-32)c_2 + 2c_0 + (8t-4)c_1 + (32t^2-32t+6)c_2 + (2c_0 \\ &+(8t-4)c_1 + (32t^2-32t+6)c_2)^2(8c_1 + (64t-32)c_2) = 2\cos t - \cos^3 t. \end{aligned} \tag{4.3}$$

Collocating eq.(4.3)in $2^k(M+1)-2$ Chebyshev points(i.e. at $t = \frac{2-\sqrt{2}}{4}$) we get

$$\begin{aligned} &64c_2 + 8c_1 - 22.6304c_2 + (2c_0 - 2.8288c_1 + 2.0011c_2) \\ &+(4c_0^2 + 8.0021c_1^2 4.0044c_2^2 \\ &-11.3152c_0c_1 - 11.3215c_1c_2 + 8.0044c_0c_2)(8c_1 - 22.6304c_2) = 1.0104. \end{aligned} \tag{4.4}$$

The initial conditions can also be expressed in terms of Chebyshev wavelets as

$$\begin{aligned} c_0 - 2c_1 + 3c_2 &= 0, \\ 8c_1 - 32c_2 &= 1. \end{aligned} \tag{4.5}$$

Table 1: Absolute error of exact and approximate solution

t	*Exact Solution*	*Wavelet method*	*Variational iteration method*
0	0	0.000000106	0
.1	0.09983341665	0.09932344712	.09983338930
.2	0.1986693308	0.1972934769	0.1986676148
.3	0.2955202067	0.2939101953	0.2955011070
.4	0.3894183423	0.3891736023	0.3893138342
.5	0.4794255386	0.483083698	0.4790386507
.6	0.5646424734	0.5756404823	0.5635252927
.7	0.6442176872	0.6668439553	0.6415028932
.8	0.7173560909	0.7566941169	0.7115469778
.9	0.7833269096	0.8451909671	0.7720563532
1	0.8414709848	0.932334506	0.8212443348

Equation (4.4) together with (4.5) gives a non linear system of equations which can be solved to find the unknowns c_0, c_1, c_2. By using Newton's iterative formula we get $c_0 = 0.2394273, \quad c_1 = 0.1165418, \quad c_2 = -0.002114549$. So the unknown function u can be found using (4.1). A comparison between the wavelet method and Variational iteration method [16] is shown in Table 1.

Example 2. $u'' - u + u^2 + u'^2 = 1$ with initial conditions $u(0) = 2, u'(0) = 0$ The analytic solution of this problem is $u(t) = 1 + \cos t$. For numerical solution we approximate $u(t)$ in terms of Shifted second kind Chebyshev wavelet basis as follow

$$u(t) = C^T \Psi(t). \tag{4.6}$$

By using operational matrix of derivative we can approximate $u'(t), u''(t)$ as

$$\begin{aligned} u'(t) &= C^T D \Psi(t), \\ u''(t) &= C^T D^2 \Psi(t) \end{aligned} \tag{4.7}$$

by substituting (4.6)and (4.7) into given equation, one can have

$$C^T D^2 \psi(t) - C^T \psi(t) + (C^T \psi(t))^2 + (C^T D \psi(t))^2 = 1$$

which is equivalent to

$$\begin{aligned} &64c_2 - (2c_0 + (8t-4)c_1 + (32t^2 - 32t + 6)c_2) + (2c_0 + (8t-4)c_1 \\ &+(32t^2 - 32t + 6)c_2)^2 + (8c_1 + (64t - 32)c_2)^2 = 1. \end{aligned} \tag{4.8}$$

Collocating eq.(4.8)in $2^k(M+1) - 2$ Chebyshev points (i.e. at $t = \frac{2-\sqrt{2}}{4}$) we get

$$\begin{aligned} &64c_2 - (2c_0 - 2.8288c_1 + 2.0011c_2) + 4c_0^2 + 8.0021c_3^2 \\ &+4.0044c_2^2 - 11.3152c_0c_1 1 - 11.3215c_1c_2 + 8.0044c_0c_2 \\ &+64c_1^2 + 512.1350c_2^2 - 362.0864c_2c_1 = 1. \end{aligned} \tag{4.9}$$

Table 2: Absolute error of exact and approximate solution

t	*Exact Solution*	*Wavelet method*	*Variational iteration method*
0	2.0	2.0	2.000000000
.1	1.995004165	1.995053594	1.994995832
.2	1.980066578	1.980214374	1.979933244
.3	1.955336489	1.955482342	1.954661486
.4	1.921060994	1.920857498	1.918927628
.5	1.877582562	1.87633984	1.872374035
.6	1.825335615	1.82192937	1.814534782
.7	1.764842187	1.757626086	1.744830994
.8	1.696706709	1.68343	1.662565054
.9	1.621609968	1.599341082	1.566913614
1	1.540302306	1.50535936	1.456919365

The initial conditions can also be expressed in terms of Chebyshev wavelets as

$$\begin{aligned} c_0 - 2c_1 + 3c_2 &= 1, \\ c_1 - 4c_2 &= 0. \end{aligned} \tag{4.10}$$

Equation (4.9) together with (4.10)gives a non linear system of equations which can be solved to find the unknowns c_0, c_1, c_2. By using Newton's iterative formula we get $c_0 = 0.9227124, \quad c_1 = -0.06183008, \quad c_2 = -0.01545752$. So the unknown function u can be found using (4.6). A comparison between the wavelet method and variational iteration method [16] is shown in Table 2.

Example 3 (Duffing Equation)**.** Consider one dimensional unforced Duffing oscillator $u'' + \alpha u + \epsilon u^3 = 0$ with initial conditions $u(0) = 1, u'(0) = 0$. Here α controls the size of the stiffness and ϵ controls the amount of nonlinearity in the restoring force.

For $\alpha = 1$ analytic solution of this problem is given by

$$u(t) = \cos t + \epsilon\Big[\frac{1}{32}\cos 3t - \frac{1}{32}\cos t - \frac{3}{8}t\sin t\Big], \epsilon \to 0^+.$$

Now, we will solve this problem numerically on the interval [0,1] for $\epsilon = 10^{-4}$. To find numerical solution approximate $u(t)$ in terms of Shifted second kind Chebyshev wavelet basis as follow

$$u(t) = C^T\Psi(t). \tag{4.11}$$

By using operational matrix of derivative we can approximate $u'(t), u''(t)$ as

$$\begin{aligned} u'(t) &= C^T D\Psi(t), \\ u''(t) &= C^T D^2\Psi(t), \end{aligned} \tag{4.12}$$

Table 3: Absolute error of exact and approximate solution

t	*Exact Solution*	*Numerical Solution*	*Absolute error*
0	1.0	1.0	4.0e-8
.1	0.9950037	0.9950525	0.00004883866
.2	0.9800646	0.9802101	0.0001455221
.3	0.9553321	0.9554728	0.0001407008
.4	0.9210534	0.9208406	0.0002128132
.5	0.8775711	0.8763134	0.001257611
.6	0.8253196	0.8218914	0.00342826
.7	0.7648213	0.7575744	0.00724695
.8	0.6966807	0.6833624	0.01331828
.9	0.6215788	0.5992556	0.02232319
1	0.540266	0.5052538	0.03501217

by substituting (4.11)and (4.12) into given equation, one can have

$$C^T D^2\psi(t) + C^T\psi(t) + 10^{-4}(C^T\psi(T))^3 = 0$$

which is equivalent to

$$\begin{aligned} &64c_2 + 2c_0 + (8t-4)c_1 + (32t^2 - 32t + 6)c_2 \\ &+10^{-4}(2c_0 + (8t-4)c_1 + (32t^2 - 32t + 6)c_2)^3 = 0. \end{aligned} \tag{4.13}$$

Collocating eq.(4.13)in $2^k(M+1) - 2$ Chebyshev points(i.e. at $t = \frac{2-\sqrt{2}}{4}$) we get

$$64c_2+(2c_0-2.8288c_1+2.0011c_2)+(10^{-4})(2c_0-2.8288c_1+2.0011c_2)^3=0. \tag{4.14}$$

The initial conditions can also be expressed in terms of Chebyshev wavelets as

$$\begin{aligned} 2c_0 - 4c_1 + 6c_2 &= 1, \\ c_1 - 4c_2 &= 0. \end{aligned} \tag{4.15}$$

Equation (4.14) together with (4.15) gives a non linear system of equations which can be solved to find the unknowns c_0, c_1, c_2. By using Newton's iterative formula we get $c_0 = 0.4226959$, $c_1 = -0.06184327$, $c_2 = -0.01546082$. So the unknown function u can be found using (4.11). A comparison between the wavelet method and the method discussed in [17] shows that wavelet method can be adapted in a simple way to solve this kind of problems numerically. Table 3 shows the absolute error of wavelet method.

5. Conclusion

A computational method based on Chebyshev wavelets and its operational matrix of derivative is proposed for solving oscillatory differential equations. The main advantage of the proposed method is that it transforms the oscillatory differential equations into non-linear systems of algebraic equations which can be easily solved by Newton's iterative method. Some examples are given to verify and demonstrate the effectiveness of the proposed method. Comparison of the exact solution and wavelet solution shows that the method is highly accurate.

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