Structures in a differentiable manifold and their applications to the tangent bundle

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Abstract. The differential geometry of tangent bundles was studied by several authors, for example: D. E. Blair, E. T. Davies, P. Dombrowski, S. Ianus, A. J. Ledger and K. Yano, V. Oproiu, C. Udriste, Yano and Davies, Yano and Ishihara and among others [15, 23, 24, 25, 26]. It is well known that different structures defined on a manifold M can be lifted to the same type of structures on its tangent bundle. However, when we consider the generalized almost r-contact structure not the same type of structure is obtained on the tangent bundle. The aim of this study is to investigate the prolongations of G-structures immersed in the generalized almost r-contact structure on a manifold M to its tangent bundle $T(M)$. Moreover, An almost contact structure, Lorentzian almost paracontact structure etc. on the tangent bundle $T(M)$ are discussed.

Keywords: complete lift, vertical lift, tangent bundle, G-structure, Hsu-structure, Lorentzian almost paracontact structure, integrability.

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1. Introduction

The study was made based on the general theory of prolongations, the geometric properties of the prolongations of pseudo-group structures and G-structures to tangent bundles [3]. The previous study investigated the prolongation of G-structures to tangent bundles of first and higher orders and showed that the integrability of G-structures is equivalent to the integrability of its prolongations [1, 2]. In 2006 Das and Nivas [7] defined certain structures on tangent bundle and studied prolongation of F-structures to the tangent bundles of order r. The prolongations of different structures like F-structure, golden structure, G-structure has been studied [5, 6, 11, 12].

In the previous papers[10, 13, 22], we have studied the complete and vertical lifts in Quarter-symmetric semi-metric connection, Quarter-symmetric metric connection to tangent bundles and semi-symmetric non-metric connection on a Kähler manifold. Earlier investigators studied the complete and horizontal lifts of golden structure in the tanent bundle [11].

The paper is structured as follows: In Section 2, let us recall the definitions of vertical lift, complete lift, Hsu-structure and generalized almost r-contact structure. Section 3 deals with the study of prolongation of tensor fields and G-structure to the tangent bundle and the integrability of the prolongation of a G-structure. In Section 4, we present results on some classical G-structures defined by tensor fields immersed in the generalized almost r-contact structure to tangent bundle. Moreover, our goal is to see as to what kind of structure is defined on the tangent bundle $T(M)$ when we consider an almost contact structure, Lorentzian almost paracontact structure etc. in the last Section.

2. Preliminaries

Let M be an n-dimensional differentiable manifold of class C^∞ and $T_p(M)$ be the tangent space of M at a point of M. Then the set

$$T(M) = \bigcup_{p \in M} T_p(M)$$

is called the tangent bundle over the manifold M ([4]).

2.1 Vertical lifts

If f is a function in M, we write f^V for the function in $T(M)$ obtained by forming the composition of $\pi : T(M \longrightarrow)M$ and $f : M \longrightarrow R$, so that

$$f^V = fo\pi$$

thus, if a point $\tilde{p}\epsilon\pi^{-1}(U)$ has induced coordinates (x^h, y^h) then

$$f^V(\tilde{p}) = f^V(x, y) = fo\pi(\tilde{p}) = f(p) = f(x),$$

thus, the value of $f^V(\tilde{p})$ is constant along each fibre $T_p(M)$ and equal to the value $f(p)$. We call f^V the vertical lift of the function f ([4]).

2.2 Complete lifts

If f is a function in M, we write f^C for the function in $T(M)$ defined by

$$f^C = i(df)$$

and call f^C the complete lift of the function f. The complete lift f^C of a function f has the local expression

$$f^C = y^i \partial_i f = \partial f$$

with respect to the induced coordinates in $T(M)$, where ∂f denotes $y^i \partial_i f$. Suppose that $X \in \Im_0^1(M)$. We define a vector field X^C in $T(M)$ by

$$X^C f^C = (Xf)^C$$

f being an arbitrary function in M and call X^C the complete lift of X in $T(M)$. The complete lift X^C of X with components x^h in M has components

$$X^C : \begin{bmatrix} x^h \\ \partial x^h \end{bmatrix}$$

with respect to the induced coordinates in $T(M)$.

Suppose that $X \in \Im_0^1(M)$ Then a 1-form ω^C in $T(M)$ defined by

$$\omega^C(X^C) = (\omega(X))^C$$

X being an arbitrary vector field in M. We call ω^C the complete lift of ω.

We use vertical and complete lifts of functions, vector fields, forms and tensor fields in the sense of ([4]).

2.3 Hsu-structure

The base space M is said to possess a Hsu-structure if there exists on M a tensor field F of type (1, 1) such that

$$F^2 = a^r I, \tag{1}$$

where I is the unit tensor field and a is a real or imaginary number ([8, 9]).

2.4 Generalized almost r-contact structure

The manifold M will be said to possess the generalized almost r-contact structure if there exists on M a tensor field F of type (1, 1), $r(C^\infty)$ contravariant vector fields U_p and $r(C^\infty)$ 1-forms ω_p satisfying ([7])

$$F^2 = a^r I + \epsilon \sum_{p=1}^{r} \omega_p \otimes U_p, \tag{2}$$

where

$$\text{(3)} \qquad \begin{aligned} &\text{(i)} \quad FU_p = 0, \\ &\text{(ii)} \quad \omega_p \circ F = 0, \\ &\text{(iii)} \quad \omega_p(U_q) = -\frac{a^r}{\epsilon}\delta_q^p, \end{aligned}$$

where $p, q = 1, 2, \cdots r$ and δ_q^p denote the Kronecker delta while ϵ is non-zero real numbers. The manifold M is called a generalized almost r-contact manifold and manifold with a generalized almost r-contact structure or in short with an $(F, U_p, \omega_p, a, \epsilon)$-structure. The structure is said to be normal if the tensor $S = [F, F] + \epsilon \sum_{p=1}^{r} \omega_p \otimes U_p$ vanishes.

3. Prolongations of G-structure immeresed in generalized almost r-contact structure to tangent bundle

Let there be given a Lie subgroup G of $GL(n, R)$ and a tensor field $\overset{o}{F}$ of type (1,1) in R^n, which is invariant by G. An n-dimensional manifold M is assumed to admit a G-structure P. We take a coordinate neighborhood $\{U, X^h\}$ of M and an n-frame $\{X_{(i)}\}$ in U, which is adapted to the G-structure P. Thus, if we put

$$\overset{o}{F} = \overset{o}{F}{}_i^h \; X_{(h)}\theta^{(i)} \tag{4}$$

in U, $\{\theta^{(i)}\}$ being the coframe dual to $\{X_{(i)}\}$in U and $\overset{o}{F}{}_i^h$ components of $\overset{o}{F}$ in R^n. The local tensor field F, defined by equation (4) in each coordinate neighborhood U, is determined independently of the choice of the adapted frame $\{X_{(i)}\}$ and hence defines globally in M a tensor field denoted by F, which is called the tensor field induced in M from $(\overset{o}{F}, P)$ ([4]).

Some classical G-structures defined by tensor fields immersed in the generalized almost r-contact structure to tangent bundle.

(I) $G = GL(n, C)$. Let $\overset{o}{F}$ be a tensor of type (1,1) in R^n such that $F^2 = a^r I$ and denoted by $GL(n, C)$ the group of all the linear transformations which leave $\overset{o}{F}$ invariant. Then the complete lift $\overset{o}{F}{}^C$ of $\overset{o}{F}$ to $T(R^n)$ is a tensor of type (1,1) satisfying $F^2 = a^r I$ and the tangent group $T(G)$ leaves $\overset{o}{F}{}^C$ invariant. Thus we obtain

$$T(G) = GL(n, C).$$

Therefore, we have the following theorem:

Theorem 3.1. *If P is Hsu-structure (as a G-structure) on manifold M, then the prolongation $\widetilde{P}$ of P to the tangent bundle $T(M)$ is the Hsu-structure determined* *by the complete lift* $\overset{o}{F}{}^C$ *of* F *to* $T(M)$*. The prolongation* $\widetilde{P}$ *of* P *to* $T(M)$ *is the Hsu-structure if and only if* P *is integrable.*

(II) $G = GL(s, C) \times GL(m, R)$. Let $\overset{o}{F}$ be a tensor of type (1,1) and of rank $2s$ in $(2s + m)$ such that $F^3 - a^r F = 0$. If we denote by G the group of all the elements of $GL(n, R)$, which leave $\overset{o}{F}$ invariant, then we easily obtain $T(G) \subset GL(2s, C) \times GL(2m, R) \subset GL(n, R)$. Then the complete lift $\overset{o}{F}{}^C$ of $\overset{o}{F}$ to satisfies $F^3 - a^r F = 0$ and is of $2s$ rank. Thus, we obtain

$$T(G) \subset GL(2s, C) \times GL(2m, R) \subset GL(2n, R).$$

Therefore, we have the following theorem:

Theorem 3.2. *If a manifold* M *admits Hsu-structure* P *(as a G-structure) determined by a tensor field* F *of type (1,1) and of rank* s *everywhere such that* $F^3 - a^r F = 0$*, then on the tangent bundle* $T(M)$*, the prolongation* $\widetilde{P}$ *of* P *to* $T(M)$ *is the Hsu-structure determined by the complete lift* $\overset{o}{F}{}^C$ *of* $\overset{o}{F}$ *to* $T(M)$ *where* $\overset{o}{F}{}^C$ *is of rank* $2s$*. The Hsu-structure* P *is integrable in* M *if and only if the prolongation* $\tilde{P}$ *of* P *to* $T(M)$ *is integrable.*

(III) $G = GL(n, C) \times I$. Let $\overset{o}{F}$ be a tensor of type (1, 1) and of rank $2s$, contravariant vector fields $\overset{o}{U}_p$ and 1-forms $\overset{o}{\omega}_p, p = 1, 2, \cdots r$ in R^{2n+r} such that

$$\overset{o}{F}{}^2 = a^r I + \epsilon \sum_{p=1}^{r} \overset{o}{\omega}_p \otimes \overset{o}{U}_p, \tag{5}$$

where

$$\begin{aligned} &\text{(i)} && \overset{o}{F}\overset{o}{U}_p = 0, \\ &\text{(ii)} && \overset{o}{\omega}_p \circ \overset{o}{F} = 0, \\ &\text{(iii)} && \overset{o}{\omega}_p (\overset{o}{U}_q) = -\frac{a^r}{\epsilon}\delta^p_q, \end{aligned} \tag{6}$$

then if we denote by G the group of all the elements of $GL(2n + r, R)$, which leave $\overset{o}{F}, \overset{o}{U}_p, \overset{o}{\omega}_p, p = 1, 2, \cdots r$ invariant, then we easily obtain

$$G = GL(n, C) \times I \subset GL(2n + r, R),$$

where I denotes the trivial group. The complete lifts $\overset{o}{F}{}^C, \overset{o}{U}_p{}^C, \overset{o}{\omega}_p{}^C$, the vertical lifts $\overset{o}{F}{}^V, \overset{o}{U}_p{}^V, \overset{o}{\omega}_p{}^V, p = 1, 2, \cdots r$ satisfy

$$(\overset{o}{F}{}^C)^2 = a^r I + \epsilon \sum_{p=1}^{r} \left\{ \overset{o}{\omega}_p{}^V \otimes \overset{o}{U}_p{}^C + \overset{o}{\omega}_p{}^C \otimes \overset{o}{U}_p{}^V \right\},$$

$$\overset{oC}{F}\ \overset{oC}{U_p} = 0, \overset{oC}{F}\ \overset{oV}{U_p} = 0, \overset{oC}{\omega_p}\ \overset{oC}{F} = 0, \overset{oV}{\omega_p}\ \overset{oC}{F} = 0,$$

$$(\overset{oC}{\omega_p})(\overset{oC}{U_q}) = 0, (\overset{oV}{\omega_p})(\overset{oV}{U_q}) = 0, (\overset{oC}{\omega_p})(\overset{oV}{U_q}) = -\frac{a^r}{\epsilon}\delta^p_q, (\overset{oV}{\omega_p})(\overset{oC}{U_q}) = -\frac{a^r}{\epsilon}\delta^p_q \tag{7}$$

and the tangent group $T(G)$ of G leaves $\overset{oC}{F}, \overset{oC}{U_p}, \overset{oC}{\omega_p}, p = 1, 2, \cdots r$ invariant. Thus we obtain

$$T(G) \subset GL(2n, C) \times I \subset GL(4n + 2r, R).$$

Therefore, we have the following Theorem:

Theorem 3.3. *If a manifold M of $2n + r$ dimensions admits the generalized almost r-contact structure P (as a G-structure) determined by a tensor field F of type (1, 1), contravariant vector fields U_p and 1-forms $\omega_p, p = 1, 2, \cdots r$ such that*

$$F^2 = a^r I + \epsilon \sum_{p=1}^{r} \omega_p \otimes U_p \tag{8}$$

where

$$\begin{aligned} &\text{(i)} \quad FU_p = 0, \\ &\text{(ii)} \quad \omega_p \circ F = 0, \\ &\text{(iii)} \quad \omega_p(U_q) = -\frac{a^r}{\epsilon}\delta^p_q, \end{aligned} \tag{9}$$

then, on the tangent bundle $T(M)$, the prolongation $\widetilde{P}$ of P is a framed the generalized almost r-contact structure ([18, 14]).

If we put

$$\widetilde{J} = \overset{oC}{F} + \frac{\epsilon}{a^{\frac{r}{2}}} \sum_{p=1}^{r} \left\{ \overset{oV}{\omega_p} \otimes \overset{oC}{U_p} + \overset{oC}{\omega_p} \otimes \overset{oV}{U_p} \right\}, \tag{10}$$

then in view of the equations (7)-(9) it is easily shown that $\widetilde{J^2} = a^r I_{T(M)}$ and that $\widetilde{J}$ is left-invariant by the tangent group $T(G)$. Thus we obtain

$$T(G) \subset GL(n, C) \times I \subset GL(n + r, R) \subset GL(2n + 2r, R).$$

Therefore, we have the following theorem:

Theorem 3.4. *If a manifold M of $2n + r$ dimensions admits the generalized almost r-contact structure P (as a G-structure) determined by $(F, U_p, \omega_p, a, \epsilon)$ given in theorem (3.3), then, on the tangent bundle $T(M)$, the prolongation $\widetilde{P}$ of P is the generalized almost r-contact structure determined by the tensor field*

$$\widetilde{J} = \overset{oC}{F} + \frac{\epsilon}{a^{\frac{r}{2}}} \sum_{p=1}^{r} \left\{ \overset{oV}{\omega_p} \otimes \overset{oC}{U_p} + \overset{oC}{\omega_p} \otimes \overset{oV}{U_p} \right\}. \tag{11}$$

4. Conclusion

In this section, we will discuss to what kind of structure is defined on tangent bundle when we consider an almost contact structure, Lorentzian almost paracontact structure etc.

Putting $r = 1, a = -1$ in equation (1), we have

$$F^2 = -I$$

then, the tensor field F of type (1,1) is called almost complex structure.

Putting $r = 1, a = -1 \in= 1$ in equation (2), we have

$$F^2 = -I + \omega \otimes U, \qquad \omega(U) = 1$$

then, the tensor field F of type (1,1) is called an almost contact structure on M ([21, 19]).

The G-structures defined by tensor fields immersed in an almost contact structure to tangent bundle then Theorem (3.1), Theorem (3.3) and Theorem (3.4) change as Theorem (4.1), Theorem (4.2) and Theorem (4.3) respectively. Thus, we have

Theorem 4.1. *If P is almost complex structure (as a G-structure) on manifold M, then the prolongation $\widetilde{P}$ of P to the tangent bundle $T(M)$ is almost complex structure determined by the complete lift $\overset{o}{F}{}^{C}$ of F to $T(M)$. The prolongation $\widetilde{P}$ of P to $T(M)$ is a complex structure if and only if P is integrable.*

Theorem 4.2. *If a manifold M admits almost contact structure P (as a G-structure) determined by a tensor field F of type (1, 1), contravariant vector field U and 1-form ω such that*

$$F^2 = -I + \omega \otimes U \tag{12}$$

where

$$\text{(i)} \quad FU = 0,$$

$$\text{(ii)} \quad \omega \circ F = 0, \tag{13}$$

$$\text{(iii)} \quad \omega(U) = 1, \tag{14}$$

then on the tangent bundle $T(M)$, the prolongation $\widetilde{P}$ of P is a framed almost contact structure.

Theorem 4.3. *If a manifold M admits almost contact structure P (as a G-structure) determined by (F, U, ω) given in theorem 4.2, then, on the tangent bundle $T(M)$, the prolongation $\widetilde{P}$ of P is almost contact structure determined by the tensor field*

$$\widetilde{J} = \overset{o}{F}{}^{C} - \left\{ \overset{o}{\omega}{}^{V} \otimes \overset{o}{U}{}^{C} + \overset{o}{\omega}{}^{C} \otimes \overset{o}{U}{}^{V} \right\}. \tag{15}$$

Putting $r = 0, \in = 1$ in equation (2), we have $F^2 = I$. The tensor field F of type (1,1) is called an almost product structure.

$$F^2 = I + \omega \otimes U, \qquad \omega(U) = -1.$$

Then, the tensor field F of type (1,1) is called Lorentzian almost paracontact structure on M ([20]).

The G-structures defined by tensor fields immersed in Lorentzian almost paracontact structure to tangent bundle then Theorem (3.1), Theorem (3.3) and Theorem (3.4) change as Theorem (4.4), Theorem (4.5) and Theorem (4.6) respectively. Thus, we have

Theorem 4.4. *If P is almost product structure (as a G-structure) on manifold M, then the prolongation $\widetilde{P}$ of P to the tangent bundle $T(M)$ is almost product structure determined by the complete lift $\overset{o}{F}{}^{C}$ of F to $T(M)$. The prolongation $\widetilde{P}$ of P to $T(M)$ is almost product structure if and only if P is integrable.*

Theorem 4.5. *If a manifold M admits Lorentzian almost paracontact structure P (as a G-structure) determined by a tensor field F of type (1, 1), contravariant vector field U and 1-form ω such that*

$$F^2 = I + \omega \otimes U \tag{16}$$

where

$$\text{(i)} \quad FU = 0,$$

$$\text{(ii)} \quad \omega \circ F = 0, \tag{17}$$

$$\text{(iii)} \quad \omega(U) = -1, \tag{18}$$

then on the tangent bundle $T(M)$, the prolongation $\widetilde{P}$ of P is a framed Lorentzian almost paracontact structure.

Theorem 4.6. *If a manifold M admits Lorentzian almost paracontact structure P (as a G-structure) determined by (F, U, ω) given in theorem 4.5, then, on the tangent bundle $T(M)$, the prolongation $\widetilde{P}$ of P is Lorentzian almost paracontact structure determined by the tensor field*

$$\widetilde{J} = \overset{o}{F}{}^{C} + \overset{o}{\omega}{}^{V} \otimes \overset{o}{U}{}^{C} + \overset{o}{\omega}{}^{C} \otimes \overset{o}{U}{}^{V}. \tag{19}$$

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