$\Delta^*(\mathcal{H})$ operators and the hyperinvariant subspace problem

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Abstract. This paper is devoted to study the relation between an operator T that satisfies the operator equation TK KT* = C for some compact operator K and some rank one operator C, and the hyperinvariant subspace problem.
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1. Introduction

Let $\mathcal{H}$ be an infinite dimensional separable complex Hilbert space and $\mathcal{B}(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$. Recall that a closed linear manifold (subspace) M of $\mathcal{H}$ is said to be a nontrivial invariant subspace for an operator $T \in \mathcal{B}(\mathcal{H})$ if $TM \subseteq M$. If , moreover, $0 \neq M \neq \mathcal{H}$ then M is called a nontrivial invariant subspace. M is said to be a hyperinvariant subspace for T if $SM \subseteq M$ for every operator S that commutes with T.

The motivation of this note is the remarkable result obtained by V. Lomonosov in [9], he proved that any nonscalar operator commutes with a nonzero compact operator has a nontrivial hyperinvariant subspace. H.W. Kim, C. Pearcy, and A.L. Shields had defined a class,$\Delta(\mathcal{H})$, of all operators T in $\mathcal{B}(\mathcal{H})$ with the property that there exists a compact operator K such that $TK - KT$ has rank one. It was shown that every operator in $\Delta(\mathcal{H})$ has a nontrivial hyperinvariant subspace [7]. This class was extensively studied in [6] and [8].

In this paper, we define the class $\Delta^*(\mathcal{H})$ of all operators $T \in \mathcal{B}(\mathcal{H})$ with the property that there exists a compact operator K such that $TK - KT^*$ has rank one. The main purpose is to obtain some results concerning the relation between the class $\Delta^*(\mathcal{H})$ and the hyperinvariant subspace problem. In section 2, we study the basic properties of $\Delta^*(\mathcal{H})$ and give some examples. However, we can not prove that the nonscalar elements in $\Delta^*(\mathcal{H})$ have nontrivial hyperinvariant subspaces. In section 3, we define a subclass $\Delta^*\mathcal{F}(\mathcal{H})$ of $\Delta^*(\mathcal{H})$, $\Delta^*\mathcal{F}(\mathcal{H}) = \{\mathcal{T} \in \mathcal{B}(\mathcal{H})|$ there exists a finite rank operator F such that $TF - FT^*$ has rank one $\}$. We show that every nonscalar element in $\Delta^*\mathcal{F}(\mathcal{H})$ has a nontrivial hyperinvariant subspace. Furthermore, we prove that $T \in \Delta^*\mathcal{F}(\mathcal{H})$ iff T has an eigenvalue provided that $T \neq \lambda I, \lambda \in \mathbb{R}$.

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2. The class $\Delta^*(\mathcal{H})$: basic properties

We introduce a new class of operators, denoted by $\Delta^*(\mathcal{H})$. The aim of this section is to study some basic properties of $\Delta^*(\mathcal{H})$.

Definition 2.1. By $\Delta^*(\mathcal{H})$ we mean the set of all operators T in $\mathcal{B}(\mathcal{H})$ with the property that there exists a compact operator K such that the rank of $TK-KT^*$ is equal to one. That is, $\Delta^*(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H})|$ there exists a compact operator K such that $TK - KT^*$ has rank one $\}$.

Remark 2.2. One can easily see that $\Delta^*(\mathcal{H})$ is not empty. In fact $iI \in \Delta^*(\mathcal{H})$, for let $K = x \otimes y$ be the rank one operator that sends y to x provided that x and y are nonzero unit vectors, then clearly K is compact and $T = iT, TK - KT^* = 2i(x \otimes y)$ is of rank one.

We begin with the following proposition

Proposition 2.3. *1. For any scalar operator λI, $\lambda I \in \Delta^*(\mathcal{H})$ iff $\lambda \notin \mathbb{R}$.*

2. $T \in \Delta^*(\mathcal{H})$ iff $\alpha T + \beta I \in \Delta^*(\mathcal{H})$ for each $\alpha \neq 0, \beta \in \mathbb{R}$

Proof. 1. It is clear that if $\lambda \in \mathbb{R}, \lambda K - K(\lambda I)^* = 0$ for any compact operator K, thus $\lambda I \notin \Delta^*(\mathcal{H})$. If $\lambda \notin \mathbb{R}$ then $\lambda = a + ib$ where $a, b \in \mathbb{R}$ and $b \neq 0$. Thus, if K is an operator of rank one,

$$\lambda K - K(\lambda I)^* = aK + ibK - (aK - ibK) = 2ibK$$

which is also of rank one. Consequently, $\lambda I \in \Delta^*(\mathcal{H})$.

2. Suppose that $T \in \Delta^*(\mathcal{H})$, then there exists a compact operator K such that $TK - KT^*$ has rank one. Let $\alpha, \beta \in \mathbb{R}$ with $\alpha \neq 0$ then $(\alpha T + \beta I)K - K(\alpha T + \beta I)^* = \alpha(TK - KT^*)$ which has rank one. Thus, $\alpha T + \beta I \in \Delta^*(\mathcal{H})$. The converse is trivial, just assume $\alpha = 1, \beta = 0$. □

Remark 2.4. The class$\Delta^*(\mathcal{H})$ is not closed under the usual algebraic operations. For example, it is not a linear manifold since clearly $0 \in \Delta^*(\mathcal{H})$. Now, if $T_1, T_2 \in \Delta^*(\mathcal{H})$, it does not always follow that $T_1 + T_2 \in \Delta^*(\mathcal{H})$ or $T_1 T_2 \in \Delta^*(\mathcal{H}))$. In fact, if $T_1 = iI$ and $T_2 = iI$ then Proposition 2.3 implies that both $T_1, T_2 \in \Delta^*(\mathcal{H})$ while neither $T_1 + T_2 = 0$ nor $T_1 T_2 = I$ belongs to $\Delta^*(\mathcal{H})$.

Now, we go through some examples of operators that belong to $\Delta^*(\mathcal{H})$. We give first the following remark which will be used several times in our work.

Remark 2.5. For any nonzero vectors $f, g \in \mathcal{H}$, for any $T \in \mathcal{B}(\mathcal{H})$,

$$T(f \otimes g) - (f \otimes g)T^* = (Tf \otimes g) - (f \otimes Tg).$$

Proposition 2.6. *Let T be any nonzero operator which is not 1-1, then $T \in \Delta^*(\mathcal{H})$.*

Proof. By assumption, there exist nonzero vectors $x, y \in \mathcal{H}$ such that $Tx \neq 0$ and $Ty = 0$. Let $K = x \otimes y$, then K is of rank one hence compact and

$$TK - KT^* = T(x \otimes y) - (x \otimes y)T^* = Tx \otimes y - x \otimes Ty = Tx \otimes y$$

which has rank one. Thus $T \in \Delta^*(\mathcal{H})$. □

Corollary 2.7. *Any nonzero nilpotent operator belongs to* $\Delta^*(\mathcal{H})$.

Proof. Let T be a nonzero nilpotent operator, then there exists $n \in \mathbf{N}$ such that $T^n = 0$ and $T^{n-1} \neq 0$. Let $f \in \mathcal{H}$ be such that $T^{n-1}f \neq 0$ then $T(T^{n-1}f) = T^n f = 0$ so T is not 1-1. Hence by proposition 6, $T \in \Delta^*(\mathcal{H})$. □

Corollary 2.8. *Any nonzero finite rank operator belongs to* $\Delta^*(\mathcal{H})$.

Proposition 2.9. *Let* T *be a nonzero operator. Assume there exists a polynomial* p *such that* $p(T) \neq 0$ *but* $Tp(T) = 0$ *then* $T \in \Delta^*(\mathcal{H})$.

Proof. By hypothesis, there exist nonzero vectors $f, g \in \mathcal{H}$ such that $p(T)f \neq 0$ and $Tg \neq 0$. Let $K = p(T)f \otimes g$ then

$$TK - KT^* = Tp(T)f \otimes g - p(T)f \otimes Tg = -p(T)f \otimes Tg,$$

is of rank one. Thus, $T \in \Delta^*(\mathcal{H})$. □

Corollary 2.10. *Every nonzero idempotent operator which is not the identity belongs to* $\Delta^*(\mathcal{H})$.

Proof. Let T be an idempotent operator such that $T \neq I$. If $p(x) = x - 1$ is a polynomial then p satisfies the condition of Proposition 3.9, thus $T \in \Delta^*(\mathcal{H})$. □

Recall that an operator T is called algebraic if $p(T) = 0$ for some polynomial p .

Proposition 2.11. *Let* T *be a nonzero algebraic operator. If* T *is non invertible then* $T \in \Delta^*(\mathcal{H})$.

Proof. Let $p(x) = a_n x^n + a_{n1} x^{n1} + \ldots + a_o; a_i \in \mathbb{R}, i = 0, 1, ..., n$ be a polynomial such that $p(T) = 0$. If $a_o \neq 0$ then $a_n T^n + a_{n1} T^{n1} + \ldots + a_1 T = a_o I$, So

$$T(a_n T^{n1} + a_{n1} T^{n2} + \ldots + a_1) = a_o I,$$

which contradicts the non invertibility of T. Let $q(T) = a_n T^{n1} + a_{n1} T^{n2} + \ldots + a_1$, then $Tq(T) = 0$. One can assume that the degree of p is the smallest one among those polynomials that annihilate T. Thus, $q(T) \neq 0$ and therefore proposition 9 implies that $T \in \Delta^*(\mathcal{H})$. □

Proposition 2.12. *If* $T, K, R \in \mathcal{B}(\mathcal{H})$ *with* K *compact and if* p *is a polynomial with real coefficients such that* $p(T)K - Kp(T)^* = R$, *then there exists a compact operator* K' *such that* $TK' - K'T^* = R$.

Proof. Let $p(x) = a_n x^n + a_{n-1}x^{n1} + \ldots + a_o; a_i \in \mathbb{R}, i = 0, 1, ..., n$ and $a_n \neq 0$ Then $p(T)K - Kp(T)^* = a_n(T^n K - K(T^*)^n) + a_{n1}(T^{n-1}K - K(T^*)^{n-1} + \ldots + a_1(TK - KT^*)$.

Now, for any $\ell \in \mathbf{N}$,

$$T^\ell K - K(T^*)^\ell = T(T^{\ell-1}K + T^{\ell-2}KT^* + T^{\ell-3}K(T^*)^2 + \ldots + K(T^*)^{\ell-1})$$
$$- (T^{\ell-1}K + T^{\ell-2}KT^* + T^{\ell-3}K(T^*)^2 + \ldots + K(T^*)^{\ell 1})T^*$$
$$= TK' - K'T^*,$$

where $K' = \sum_{i=0}^{\ell-1} = T^{\ell-1-i}K(T^*)^i$. Clearly K' is compact as the set of compact operators form an ideal in $\mathcal{B}(\mathcal{H})$. Thus, there exist compact operators $K_1, K_2, \ldots, K_n$ such that

$$p(T)K - K(p(T))^* = a_n(TK_n - K_nT^*) + a_{n1}(TK_{n1} - K_{n1}T^*)$$
$$+ \ldots + a_1(TK_1 - K_1T^*)$$
$$= T(a_nK_n + a_{n1}K_{n1} + \ldots + a_1K_1) - (a_nK_n + a_{n1}K_{n1} + \ldots + a_1K_1)T^*.$$

Let $K'\prime = a_nK_n + a_{n1}K_{n1} + \ldots + a_1K_1$, then $K'\prime$ is compact and the result follows. Clearly if $R = 0$ then choose $K' = 0$. □

Corollary 2.13. *Let p be a polynomial with real coefficients and $T \in \mathcal{B}(\mathcal{H})$. If $p(T) \in \Delta^*(\mathcal{H})$ then $T \in \Delta^*(\mathcal{H})$.*

Proof. Since $p(T) \in \Delta^*(\mathcal{H})$, then there exists a compact operator K such that $p(T)K - K(p(T))^* = R$ where R is a rank one operator. The result follows directly by the previous proposition. □

Combine the above corollary with Proposition 2.3, we get

Theorem 2.14. *Let p be a polynomial with real coefficients except for the constant term to be non real and let $T \in \mathcal{B}(\mathcal{H})$. If $p(T) = 0$ then $T \in \Delta^*(\mathcal{H})$.*

Proof. Let $p(x) = a_nx^n + a_{n1}x^{n1} + \ldots + a_o$ be a polynomial with $a_i \in \mathbb{R}, i = 0, 1, ..., n$ and $a_n \neq 0$. If $p(T) = 0$ then $p(T) - a_0I = -a_0I$ which by Proposition 2.3 belongs to $\Delta^*(\mathcal{H})$. But $p(T) - a_0I$ is a polynomial with real coefficients and $p(T) - a_0I \in \Delta^*(\mathcal{H})$. Thus, by Corollary 2.13, $T \in \Delta^*(\mathcal{H})$. □

In what follows in this section, we assume that T is invertible operator.

Proposition 2.15. *Let T be an invertible operator. If $T \in \Delta^*(\mathcal{H})$ then $T^{-1} \in \Delta^*(\mathcal{H})$. follows in this section, we assume that T is invertible operator.*

Proof. Let K be a compact operator such that $TK - KT^*$ has rank one. Now,

$$TK - KT^* = (TKT^*)(T^*)^{-1} - T^{-1}(TKT^*).$$

Since TKT^* is compact, the result follows. □

Theorem 2.16. *Let T be an invertible operator. If there exist nonzero vectors $f, g \in \mathcal{H}$ such that*

$$\sum_{n=0}^{\infty} \|T^n f\| \|T^{-n} g\| < \infty.$$

Then $T \in \Delta^*(\mathcal{H})$.

Proof. Let $K_n = \sum_{k=0}^{n} T^k f \otimes T^{-k} g$, then K_n is of finite rank for each n, if $K = \sum_{k=0}^{\infty} T^k f \otimes T^{-k} g$, then by hypothesis

$$\|K - K_n\| = \| \sum_{k>n} T^k f \otimes T^{-k} g\| \leq \sum_{k>n} \|T^k f\| \|T^{-k} g\| \to 0.$$

Consequently, $K_n \to K$, hence K is compact. Now

$$\begin{aligned} TK - KT^* &= T(\sum_{k=0}^{\infty} T^k f \otimes T^{-k} g) - (\sum_{k=0}^{\infty} T^k f \otimes T^{-k} g)T^* \\ &= \sum_{k=0}^{\infty} T^{k+1} f \otimes T^{-k} g - \sum_{k=0}^{\infty} T^k f \otimes T^{-k+1} g \\ &= Tf \otimes g + T^2 f \otimes T^{-1} g = T^3 f \otimes T^{-2} g + ... \\ &= -f \otimes Tg. \end{aligned}$$

But $Tg \neq 0$ as T is invertible, hence $TK - KT^*$ has rank one, so $T \in \Delta^*(\mathcal{H})$. □

As an illustration we have

Example 2.17. Let $\{e_n\}_{n \in \mathbf{Z}}$ be an orthonormal basis of $\mathcal{H}$. Define an operator B by $Be_n = w_{n+1} e_{n+1}, n \in \mathbf{Z}$, where $w_{n+1} = \frac{1}{2}$ when $n \geq 0$, and $w_{n+1} = 4$ when $n \leq 0$. This operator is called weighted bilateral shift operator with weights w_n. It is well-known that B is invertible operator and $B^{-1} e_n = \frac{1}{w_n} e_{n-1}, n \in \mathcal{Z}$. Note also that $B^{-k} e_n \to 0$ as $k \to \infty$ for each $n \in \mathcal{Z}$ and $B^k e_n \to 0$ as $k \to \infty$ for each $n \in \mathcal{Z}$. Easily one can see that $\sum_{k=0}^{\infty} \|B^K e_n\| \|B^{-k} e_n\| < \infty$ for any fixed $n \in \mathcal{Z}$. Thus, Theorem 2.16 implies that $B \in \Delta^*(\mathcal{H})$.

Compare the following result with Theorem 2.16,

Proposition 2.18. *Let $T \in \mathcal{B}(\mathcal{H})$, M be a nonzero subspace which is invariant under T and $T_1 = T|M$ is invertible. If there exist nonzero vectors $f \in \mathcal{H}$ and $g \in M$ such that $\sum_{k=0}^{\infty} \|T^k f\| \|T_1^{-k} g\| < \infty$ then $T \in \Delta^*(\mathcal{H})$.*

Proof. Let $K = \sum_{k=0}^{\infty} T^k f \otimes T_1^{-k} g$ then the hypothesis implies that the series converges in the operator norm topology, and hence K is a compact operator. Furthermore,

$$\begin{aligned} TK - KT^* &= (K = \sum_{k=0}^{\infty} T^{k+1} f \otimes T_1^{-k} g) - (\sum_{k=0}^{\infty} T^k f \otimes T_1^{-k+1} g) \\ &= -f \otimes T_1 g. \end{aligned}$$

Since T_1 is invertible then $T_1 g \neq 0$ and thus $TK - KT^*$ has rank one, i.e., $T \in \Delta^*(\mathcal{H})$. □

Example 2.19. Let $T \in \mathcal{B}(\mathcal{H})$ be such that $\sigma(T) = \{0, 2\}$. Then $\sigma(T) = \{0\} \cup \{2\}$. Thus Riesz decomposition theorem implies that $\mathcal{H}$ splits into the direct sum of two nontrivial hyperinvaiant subspaces M_1 and M_2 such that

$$\sigma(T|M_1) = \{0\}, \sigma(T|M_2) = \{2\}.$$

Clearly $T = T|M_2$ is invertible and the spectral radius $r(T|M_1) = 0$ while $r(T^{-1}) = \frac{1}{2} < 1$, consequently $\sum_{k=0}^{\infty} \|T^k f\| \|T^{-k} g\| < \infty$ for any nonzero vectors $f \in \mathcal{H}, g \in M_2$. Thus, $T \in \Delta^*(\mathcal{H})$.

3. The class $\Delta^*(\mathcal{H})$ and the hyperinvariant subspace problem

In this section we try to find a relation between the class $\Delta^*(\mathcal{H})$ and the hyperinvariant subspace problem. For this purpose we study a subset of $\Delta^*(\mathcal{H})$, denoted by $\Delta^*\mathcal{F}(\mathcal{H})$ which is defined as follows.

Definition 3.1. By $\Delta^*\mathcal{F}(\mathcal{H})$ we mean the set of all operators T in $\mathcal{B}(\mathcal{H})$ with the property that there exists a finite rank operator F such that the rank of $TF - FT^*$ is equal to one. That is $\Delta^*\mathcal{F}(\mathcal{H}) = \{\mathcal{T} \in \mathcal{B}(\mathcal{H})|$ there exists a finite rank operator F such that $TF - FT^*$ has rank one$\}$.

Remark 3.2. Clearly $\Delta^*\mathcal{F}(\mathcal{H}) \subseteq \Delta^*(\mathcal{H})$, however the converse is not true as we see later. It is easy to verify that $\Delta^*\mathcal{F}(\mathcal{H})$ is nonempty set, for example λI belongs to $\Delta^*\mathcal{F}(\mathcal{H})$ iff $\lambda \notin \mathbb{R}$ (see proof of Proposition 2.3).

The following proposition exhibits several cases for an operator T belongs to $\Delta^*\mathcal{F}(\mathcal{H})$. For the proof just notice that the compact operator found in the corresponding cases appeared in the last section is, in fact, of finite rank.

Proposition 3.3. *Let T be a nonzero operator, then $T \in \Delta^*\mathcal{F}(\mathcal{H})$ if it satisfies any of the following statements.*
1. T is nilpotent.
2. T is of finite rank.
3. $T \neq I$ is idempotent.
4. T is non invertible algebraic operator.
5. T is invertible with $T^{-1} \in \Delta^*\mathcal{F}(\mathcal{H})$.

The following theorem reflects the importance of the class $\Delta^*\mathcal{F}(\mathcal{H})$. For this purpose we need some preliminaries.

Lemma 3.4. *Let $T \in \mathcal{B}(\mathcal{H})$. T has a nontrivial hyperinvariant subspace iff T^* has so.*

Proof. Suppose that M is a nontrivial hyperinvariant subspace for T, let $S \in \{T^*\}'$, the commutant of T^*, i.e., $ST^* = T^*S$, then $TS^* = S^*T$. Hence, $S^*M \subseteq M$ which implies that $SM^{\perp} \subset M^{\perp}$ [2]. Since M is nontrivial and $M \oplus M^{\perp} = \mathcal{H}$ then $M^{\perp}$ is also nontrivial subspace. Consequently $M^{\perp}$ is a nontrivial hyperinvariant subspace for T^*.The converse is obvious since $(T^*)^* = T$. □

Lemma 3.5. *Let M be a nontrivial invariant subspace for an operator T. If M is of finite dimension then T has an eigenvalue.*

Proof. Suppose that an operator T has a finite dimensional nontrivial invariant subspace M, then $T|M$ represents an operator on a finite dimension Hilbert space over the complex numbers, hence by the Fundamental theorem of algebra, it has an eigenvalue. □

It is known that, for operators A and B in $\mathcal{B}(\mathcal{H})$. if $AB - BA$ has rank one, then either $kerA$ or $rngA$ is invariant under B. This fact is due to Laffeye [3]. As a modification to Laffeye's theorem, it has been shown that for operators A, B and $T \in \mathcal{B}(\mathcal{H})$, if $AB - TA$ has rank one, then either $kerA$ is invariant under B or $rngA$ is invariant under T (see [1]). In particular we can prove the following lemma

Lemma 3.6. *Let $A, T \in \mathcal{B}(\mathcal{H})$. If $TA - AT^*$ has rank one, then either $kerA$ is invariant under T^* or $rngA$ is invariant under T.*

Proof. Assume $kerA$ is not invariant under T^*, then $kerA$ must be nontrivial and hence there exists a nonzero vector $x \in \mathcal{H}$ such that $Ax = 0$ and $AT^*x \neq 0$. Since $TA - AT^*$ has rank one, then there exists a nonzero vector $z \in \mathcal{H}$ such that for each $y \in \mathcal{H}, (TA - AT^*)y = c_y z$ where $c_y z$ is a complex number that may depend on y. In particular, $(TA - AT^*)x = -AT^*x \neq 0$, hence $(TA - AT^*)y = -c_y AT^*x$ for each $y \in \mathcal{H}$. Therefore $TAy = AT^*y - c_y AT^*x = AT^*(y - c_y x)$ for each $y \in \mathcal{H}$. Consequently, $TA\mathcal{H} \subseteq AT^*\mathcal{H} \subseteq A\mathcal{H}$. i.e., $rngA$ is invariant under T. □

Now we are ready to give the following theorem. Recall that if F is a finite rank operator then so is F^* [4, p.49].

Theorem 3.7. *Let T be any non-real scalar operator. If $T \in \Delta^*\mathcal{F}(\mathcal{H})$ then T has a nontrivial hyperinvariant subspace.*

Proof. The assumption implies that there exists a finite rank operator F such that $TF - FT^*$ has rank one. Clearly $F \neq 0$. On the other hand, since F is of finite rank, then $kerF \neq 0$ and $rngF$ is a finite dimensional subspace. Thus, by Lemma 3.6, either $kerF$ is a nontrivial invariant subspace for T^*, hence $\overline{rng}(F^*) = (kerF)^{\perp}$ is a nontrivial invariant subspace for T or $rngF$ is a nontrivial invariant subspace for T. Since F^* is also of finite rank, i.e., $rngF^*$ is a finite dimensional subspace, then we get that either $rngF$ or $rngF^*$ is a nontrivial invariant subspace for T. In both cases T has a finite dimensional nontrivial invariant subspace thus by Lemma 5, T has an eigenvalue, say λ. Moreover, since $T \neq \lambda I$ by assumption, $ker(T - \lambda I)$ is a nontrivial invariant subspace for T. Consequently, $M = rng(T^* - \bar{\lambda}I)$ is a nontrivial invariant subspace for T^*. Now, for each $S \in \{T^*\}'$, the commutant of T^*,

$$S(T^* - \bar{\lambda}I)\mathcal{H} = (T^* - \bar{\lambda}I)S\mathcal{H} \subseteq rng(T^* - \bar{\lambda}I),$$

i.e., $S(rng(T^* - \bar{\lambda}I)) \subseteq rng(T^* - \bar{\lambda}I)$ which implies that $SM \subseteq M$. Thus M is a nontrivial invariant subspace for T^* and consequently by Lemma 3.4, T has a nontrivial hyperinvariant subspace. □

Remark 3.8. The proof of Theorem 7 shows that any operator T which belongs to $\Delta^*\mathcal{F}(\mathcal{H})$ has a nontrivial invariant subspace of finite dimension and thus by Lemma 3.5 has an eigenvalue. The following proposition shows that the converse is true provided that T is nonscalar operator.

Proposition 3.9. *Let T be any non-real scalar operator. If T has an eigenvalue, then $T \in \Delta^*\mathcal{F}(\mathcal{H})$.*

Proof. Let $\lambda \in \mathcal{C}$ be such that $Tf = \lambda f$ for some nonzero vector $f \in \mathcal{H}$. Moreover there exists a nonzero vector $g \in \mathcal{H}$ such that $Tg \neq \bar{\lambda}g$ otherwise $T = \bar{\lambda}I$ which contradicts the assumption. Let $F = f \otimes g$ then,

$$\begin{aligned} TF - FT^* &= T(f \otimes g) - (f \otimes g)T^* \\ &= TF \otimes g - f \otimes Tg - \lambda f \otimes g - f \otimes Tg \\ &= f \otimes \bar{\lambda}g - f \otimes Tg - f \otimes (\bar{\lambda}g - Tg). \end{aligned}$$

Since $\bar{\lambda}g \neq Tg$ then $TF - FT^*$ has rank one. Hence $T \in \Delta^*\mathcal{F}(\mathcal{H})$. □

The following theorem characterizes the elements in $\Delta^*\mathcal{F}(\mathcal{H})$.

Theorem 3.10. *Let $T \in \mathcal{B}(\mathcal{H})$. Then $T \in \Delta^*\mathcal{F}(\mathcal{H})$ iff T has an eigenvalue provided that $T \neq \lambda I$ where $\lambda \in \mathbb{R}$.*

Proof. If $T = \lambda I$ then clearly λ has an eigenvalue for T. Thus $T \in \Delta^*\mathcal{F}(\mathcal{H})$ iff $\lambda \notin \mathbb{R}$. Let T be any non-real scalar operator, then Remark 3.8 and Proposition 3.9 imply that $T \in \Delta^*\mathcal{F}(\mathcal{H})$ iff T has an eigenvalue. □

It is worthy to notice that the operator B defined in Example 2.17 does not belong to $\Delta^*\mathcal{F}(\mathcal{H})$ as $\sigma_p(B) = \phi$ ([5, p.229]). However, $B \in \Delta^*(\mathcal{H})$ Example 2.17.

Theorem 3.11. *$T \in \Delta^*\mathcal{F}(\mathcal{H})$ iff T has a finite dimensional nontrivial invariant subspace provided that $T \neq \lambda I, \lambda \in \mathbb{R}$.*

Proof. If $T \in \Delta^*\mathcal{F}(\mathcal{H})$ then Remark 3.8 shows that T has a finite dimensional non invariant subspace. Suppose that T has a finite dimensional nontrivial invariant subspace and $T \neq \lambda I, \lambda \in \mathbb{R}$, then Lemma 3.5 implies that T has an eigenvalue and thus the results follows from Theorem 3.10. □

Remark 3.12. If U is the unilateral shift operator then $U \notin \Delta^*\mathcal{F}(\mathcal{H})$ since $\sigma_p(U) = \phi$ ([5, p.227]). On the other hand, U has a nontrivial invariant subspace, defined as follows.

Fix a positive integer k and consider $M_k =$ the linear manifold generated by $\{e_n\}, n \geq k$, where $\{e_n\}$ is an orthogonarmal basis of $\mathcal{H}$. Since $Ue_n = e_{n+1}$ for all $n \geq 1$ by definition, then $UM_k \subseteq M_k$ for all $k \geq 1$. Thus, M_k, $k \geq 2$ is a nontrivial invariant subspace of U. Clearly, M_k is of infinite dimension for all $k \geq 2$.

The following example shows that $\Delta^*\mathcal{F}(\mathcal{H})$ is not selfadjoint class.

Example 3.13. Let U be the unilateral shift operator then $\sigma_p(U) = \phi$ while $\sigma_p(U^*) \neq \phi$ as $U^*e_1 = 0$ ([5, p.227]). Consequently, Theorem 3.10 implies that $U^* \in \Delta^*\mathcal{F}(\mathcal{H})$ while $U \notin \Delta^*\mathcal{F}(\mathcal{H})$.

On the other hand, for any nonscalar operator T, if either T or T^* belongs to $\Delta^*\mathcal{F}(\mathcal{H})$, then Lemma 3.4 and Theorem 3.10 imply that T has a nontrivial hyperinvariant subspace.

In what follows, more results about $\Delta^*\mathcal{F}(\mathcal{H})$ are obtained.

Proposition 3.14. *Let $T \in \Delta^*\mathcal{F}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{H})$ satisfies $VT = SV$ for some 1-1 operator V, then $S \in \Delta^*\mathcal{F}(\mathcal{H})$.*

Proof. Let F be a finite rank operator such that $TF - FT^*$ has rank one, then there exist nonzero vectors $f, g \in \mathcal{H}$ such that $TF - FT^* = f \otimes g$. Now, $V(TF - FT^*)V^* = V(f \otimes g)V^*$, so $VTFV^* - VFT^*V^* = Vf \otimes Vg$, thus

$$S(VFV^*) - (VFV^*)S^* = Vf \otimes Vg.$$

Since V is 1-1 then $Vf \neq 0$ and $Vg \neq 0$ hence $Vf \otimes Vg$ is of rank one. Consequently, $S \in \Delta^*\mathcal{F}(\mathcal{H})$ as VFV^* is of finite rank. □

Corollary 3.15. *Let $T \in \Delta^*\mathcal{F}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{H})$. If T is similar to S, then $S \in \Delta^*\mathcal{F}(\mathcal{H})$.*

Recall that the compression spectrum of an operator T, denoted by $\Gamma(T)$, is the set of all complex numbers λ such that $\overline{rng(T - \lambda I)} \neq \mathcal{H}$, and $\sigma(T) = \Gamma(T) \cup \sigma_{ap}(T)$ ([5, p.41]).

Proposition 3.16. *Let $T \in \mathcal{B}(\mathcal{H})$. If $\Gamma(T) \neq \phi$ then $T^* \in \Delta^*\mathcal{F}(\mathcal{H})$.*

Proof. Let $\lambda \in \Gamma(T)$ then $\overline{rng(T - \lambda I)} \neq \mathcal{H}$. Since $\overline{rng(T - \lambda I)} \oplus ker(T^* - \overline{\lambda} I) = \mathcal{H}$ then $ker(T^* - \overline{\lambda} I) \neq 0$, i.e., $\overline{\lambda} \in \sigma_p(T^*)$ and by Proposition 3.9, $T^* \in \Delta^*\mathcal{F}(\mathcal{H})$. □

Proposition 3.17. *Let $T \in \mathcal{B}'(\mathcal{H})$. If there exists $\lambda \in \sigma(T)$ such that $rng(T - \lambda I)$ is closed then either $T \in \Delta^*\mathcal{F}(\mathcal{H})$ or $T^* \in \Delta^*\mathcal{F}(\mathcal{H})$.*

Proof. Let $\lambda \in \sigma(T)$ satisfy $rng(T-\lambda I)$ is closed. Since $\sigma(T) = \Gamma(T) \cup \sigma_{ap}(T)$, then either $\lambda \in \Gamma(T)$ which implies by Proposition 3.16, that $T^* \in \Delta^*\mathcal{F}(\mathcal{H})$ or $\lambda \in \sigma_{ap}(T)$, thus, $\lambda \in \sigma_p(T)$. Consequently, $T \in \Delta^*\mathcal{F}(\mathcal{H})$ Proposition 3.9. □

Corollary 3.18. *Let T be a non invertible operator. If $rngT$ is closed then either $T \in \Delta^*\mathcal{F}(\mathcal{H})$ or $T^* \in \Delta^*\mathcal{F}(\mathcal{H})$.*

Proof. Since T is not invertible, then $0 \in \sigma(T)$ and the result follows directly from Proposition 3.17. □

Corollary 3.19. *Let T be a normal operator that satisfies the conditions in Proposition 3.17, then $T \in \Delta^*\mathcal{F}(\mathcal{H})$.*

Proof. Since $\sigma(T) = \sigma_{ap}(T)$ then $ker(T-\lambda I) \neq 0$, i.e., $\lambda \in \sigma_p(T)$. Hence $T \in \Delta^*\mathcal{F}(\mathcal{H})$ Proposition 3.9. □

Recall that an operator T is isometric if $ST = I$ for some $S \in \mathcal{B}(\mathcal{H})$, equivalently, $\|Tx\| = \|x\|$ for each $x \in \mathcal{H}$ ([2, p.143]).

Corollary 3.20. *Let T be an isometric which is not unitary, then either $T \in \Delta^*\mathcal{F}(\mathcal{H})$ or $T^* \in \Delta^*\mathcal{F}(\mathcal{H})$.*

Proof. Clearly T is 1-1 by the definition. Since any isomerty operator is unitary iff it is invertible, then T is not invertible. It is easy to check that $rngT$ is closed. Thus, by Corollary 3.18, either T or T^* belongs to $\Delta^*\mathcal{F}(\mathcal{H})$. □

Recall that an operator T is called a Fredholm operator if $rngT$ is closed and both $kerT$ and$(rngT)^{\perp}$ are finite dimensional ([5, p.96]). For example it is easy to check that the unilateral shift operator is Fredholm operator.

Corollary 3.21. *Let T be a non invertible Fredholm operator, then either $T \in \Delta^*\mathcal{F}(\mathcal{H})$ or $T^* \in \Delta^*\mathcal{F}(\mathcal{H})$.*

Proof. Just apply Corollary 3.18. □

The following result related to the size of $\Delta^*\mathcal{F}(\mathcal{H})$.

Proposition 3.22. *The class $\Delta^*\mathcal{F}(\mathcal{H})$ is dense in $\mathcal{B}(\mathcal{H})$ in the norm operator topology.*

Proof. Since the set of all operators with nonempty point spectrum is dense in $\mathcal{B}(\mathcal{H})$, the result is an immediate consequence of Theorem 3.10. □

Corollary 3.23. *The class $\Delta^*\mathcal{F}(\mathcal{H})$ is dense in $\mathcal{B}(\mathcal{H})$ in the norm operator topology and hence dense in both the operator strong and weak topology.*

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